10.1515/acsc-2016-0032

Archives of Control Sciences
Volume 26(LXII), 2016
No. 4, *pages 577–595*

Identification of stable elementary bilinear time-series model

LUKASZ MALINSKI

The paper presents new approach to estimation of the coefficients of an elementary bilinear time series model (EB). Until now, a lot of authors have considered different identifiability conditions for EB models which implicated different identifiability ranges for the model coefficient. However, all of these ranges have a common feature namely they are significantly narrower than the stability range of the EB model. This paper proposes a simple but efficient solution which makes an estimation of the EB model coefficient possible within its entire stability range.

Key words: bilinear model, time-series, identification.

1. Introduction

The most commonly used time–series models are the linear ones. They are extensively described in the literature [10], [22]. Effective identification algorithms for linear times-series models are well-known and frequently used in industry, economic and academic research. However, the major problem is that real world phenomena are mostly nonlinear. Therefore, linear models provide a barely approximation of process behavior. As a result one of the challenges that scientists try to master is to extend the set of linear models with nonlinear ones. There are many propositions of nonlinear models to be found along with identification procedures dedicated for them (see [34], [38], and [36]).

Bilinear model, introduced by Granger and Andersen in 1978 [14], is one of the simplest nonlinear time–series models. The most general form of this kind of model is Bilinear Autoregressive Moving Average Model (BARMA). Theory on statistical properties of BARMA model was developed by Subba Rao [35], Pham [33], [32], Gooijger and Heauts [13], Mohler and Tang [31]. Further analysis of this model, concerning its stability, stationarity and invertibility, was done by Mathews and Lee [29], Bibi [1], [2] and Kristensen [23], and finally the estimation algorithms were proposed by Subba Rao [35] (based on Newton-Raphson algorithm), Chellapilla and Rao [12] (using evolutionary al-

L. Malinski is with Division of Industrial Informatics, Silesian University of Technology, Krasinskiego 8, 40-019 Katowice, Poland. E-mail: Lukasz.Malinski@polsl.pl.

This work was partially financed by NCBiR grant no. POIG.02.03.01-24-099/13 GCONiI Upper Silesian Center for Computational Science and Engineering.

Received 03.02.2016.

gorithms) and Hili [18] (based on Hellinger distance). Most of the researchers performed their considerations in relation to the simplified bilinear structures. Numerous identification procedures and analysis of their weaknesses have been presented by Guegan and Pham [17], Mathews and Moon [30], Mathews and Lee [29], Brunner and Hess [11], Wu and Hung [40] and more recently by Wang [39] and Hristova [19].

The elementary bilinear (EB) time–series model is the simplest bilinear structure, consisting of two terms only: a white noise term and a bilinear term. Although the EB model, compared to BARMA model is significantly less complex, it may be applied, as an element of a hybrid linear-bilinear (L-EB) prediction model, (Bielinska [6], [3]) or as a part of a non-linear minimum-variance controller (Bielinska and Zielinski [7]). Besides, understanding the relation between its properties and its identifiability is the key to perform proper identification of the more complex bilinear time–series models.

Properties of the EB model and process originated from it were addressed by Martins [27], [28], Malinski and Bielinska [25], and Malinski and Figwer [26]. Numerous estimation procedures for an invertible EB model were presented by Kim, Billard and Basawa [21], Bouzaachane, Harti and Benghabrit [8], [9]. Moreover, Iwaueze and Johnson [20] studied the problem of misclassification of the diagonal EB model and the linear MA model, and they showed the possible consequences.

The majority of estimation procedures presented in the literature concern a limited range of possible values of the stable EB model coefficient only. Therefore, there is a significant range of useful EB models which are stable, but up to now impossible to be identified due to limits of estimation algorithms. This means that there is a need for some new estimation approach which extends the range of identifiable EB models to their entire stability range. The new estimation approach which seems to be the desired solution, is presented in this paper.

Content of Section 2 of this paper covers main statistical properties of the EB model and process originated from it. Further, in Section 3 the constraints and difficulties of the estimation of EB model coefficient are discussed. The new (proposed by the author), modified approach which breaks the constraints is provided in Section 4. Results of simulations are presented in Section 5 and finally, conclusions and remarks can bee found in Section 6.

2. Elementary bilinear model and process

The elementary bilinear time–series model is defined by the following formula [5]:

$$y_i = \beta e_{i-k} y_{i-l} + e_i; \tag{1}$$

where y_i is the model output, β is a constant model coefficient and e_i is a random identically distributed white noise called stimulation or innovation signal. The further analysis of EB model and process originated from it is made by use of the following assumptions about e_i:

$$E\{e_i\} = 0; \tag{2}$$

$$E\{e_i e_{i+m}\} = \begin{cases} \lambda^2 & : m = 0 \\ 0 & : m \neq 0 \end{cases} . \tag{3}$$

In addition we assume that e_i has a Gaussian distribution.

Parameters k and l are called structure parameters and relation between them implies one of the possible structures of the EB model:

- *super–diagonal* structure for $k < l$,
- *diagonal* structure for $k = l$,
- *sub–diagonal* structure for $k > l$.

The process obtained from stimulation of (1) by a random signal e_i will be called elementary stochastic bilinear process (or EB process). The particular realization of EB process consisting of finite N samples in discrete time will be treated as time–series and called EB time–series.

The stability condition (4) for the EB model and process (in Bounded Input Bounded Output sense) is obtained by calculations of convergence of function series acquired by recursive computation of the model output [5]

$$\beta^2\lambda^2 < 1; \tag{4}$$

where λ^2 is the variance of the stimulation e_i. Therefore, for e_i with the fixed variance λ^2, we can define a range of β, within which the EB model is stable:

$$\beta \in \left(-\frac{1}{|\lambda|}, \frac{1}{|\lambda|} \right). \tag{5}$$

It is also worth to mention that the stability condition (4) is structure (k, l) independent.

The second important condition for the EB model is the invertibility condition. The inverted model to (1) is described by:

$$e_i = y_i - \beta e_{i-k} y_{i-l}. \tag{6}$$

Thus, substituting $\alpha = -\beta$ we can rewrite (6) as follows:

$$e_i = \alpha e_{i-k} y_{i-l} + y_i. \tag{7}$$

Both (1) and (7) are EB models and also $|\alpha| = |\beta|$. Although, y_i does not satisfy assumptions (2) and (3) in [5] is shown that if y_i is limited (it comes from the stable EB model), the stability condition for (7) can be defined as:

$$E\{\alpha^2 y_i^2\} < 1 \Rightarrow \alpha^2 E\{(y_i^2\}) < 1, \tag{8}$$

the same as provided by Granger and Andersen in [15]. Therefore, while the stability range for (1) is (5), the stability range for the inverted model (7) is:

$$\alpha \in \left(-\frac{1}{|\sqrt{E\{y_i^2\}}|}, \frac{1}{|\sqrt{E\{y_i^2\}}|} \right). \tag{9}$$

Because, $E\{e_i^2\} \leqslant E\{(e_i + \beta e_{i-k} y_{i-l})^2\}$, thus $E\{e_i^2\} \leqslant E\{y_i^2\}$, the stability range of the inverted model (9) is narrower than the stability range (5) of the original EB model. So the stable EB model with the sufficient high value of coefficient β have its inverted model unstable.

The model is invertible if it is possible to estimate the input signal e_i by knowing output signal y_i and true value of model coefficient β. It is possible if the inverted model is stable but not necessary otherwise.

Therefore, basing on the stability condition of (7) we can formulate the general range of the invertible EB models:

$$\beta \in \left(-\frac{1}{|\sqrt{E\{y_i^2\}}|}, \frac{1}{|E\sqrt{\{y_i^2\}}|} \right). \tag{10}$$

Going further, we should define the invertibility condition with regard to the EB model structure, because unlike the stability condition, it is structure dependent. As provided in [4] and [25], the variance of the EB model output may be expressed by means of the model coefficient β and the stimulation signal variance λ:

- for the super–diagonal structure [5]:

$$E\{y_i^2\} = \frac{\lambda^2}{1-\beta^2\lambda^2}; \tag{11}$$

- for the diagonal structure [5]:

$$E\{y_i^2\} = \beta^2 \frac{3\lambda^4}{1-\beta^2\lambda^2} + \lambda^2; \tag{12}$$

- for the sub–diagonal structure [25]:

$$E\{y_i^2\} = \beta^{2(k-l+1)}\lambda^{2(k-l+1)+2} \frac{3\lambda^4}{1-\beta^2\lambda^2} + \sum_{m=0}^{k-l} \beta^{2m}\lambda^{2m+2}. \tag{13}$$

By introducing (11), (12) and (13) into (8) and performing the elementary mathematical operations, the following invertibility conditions can be obtained:

- for the super–diagonal structure [5]:

$$\beta^2\lambda^2 < 0.5; \tag{14}$$

- for the diagonal structure [5]:

$$\beta^2\lambda^2 < 0.36; \tag{15}$$

- for the sub–diagonal structure [25]:

$$\beta^2\lambda^2 < \sqrt[k-l+2]{0.5}. \tag{16}$$

In order to check how output of the inverted model corresponds to the original stimulation signal, two EB time–series originated from super–diagonal EB model were generated. Both simulations were performed using stimulation signal of the variance $\lambda^2 = 1$:

- the first one for $\beta = 0.4$ (which lies within both (5) and (10)), so the model is stable and invertible,
- the second one for $\beta = 0.9$ (which is within (5) but outside (10)), thus the model is stable yet not invertible.

Next, the estimates $\hat{e}_i$ were computed using (6) and true β value. Both plots are presented in Fig. 1.

As we can see, for time–series originated from invertible model all estimates $\hat{e}_i$ are equal to e_i, while time–series obtained using non–invertible model, the explosion of estimates occurs about $i = 120$. This explosion is the result of the inverted model instability. This phenomenon has impact on the model identifiability and is more precisely described in the next section.

3. Identifiability

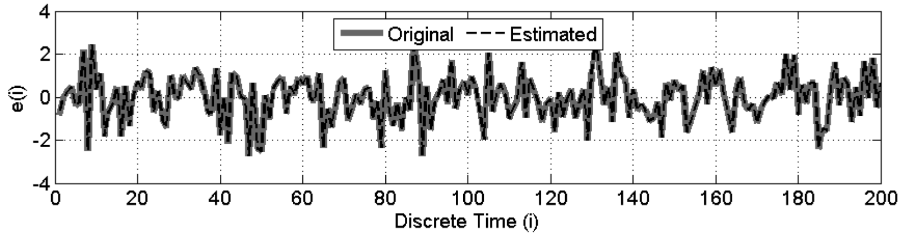
The problem of invertibility of the EB model is the identifiability of this model. For the purpose of this paper we define the identifiability as the possibility to obtain the statistically accurate estimate of model coefficient during the identification.

The most common approach to estimation of the EB model coefficient is based on minimization of the mean square error *MSE*:

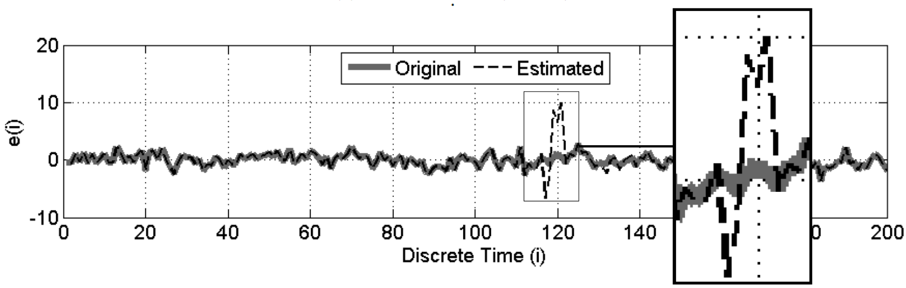
$$MSE = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i^2; \tag{17}$$

where ε_i is a one step prediction error computed as follows:

$$\varepsilon_i = y_i - \hat{\beta}\varepsilon_{i-k}y_{i-l}. \tag{18}$$



(a) Invertible model ($\beta = 0.4$)



(b) Not invertible model ($\beta = 0.9$)

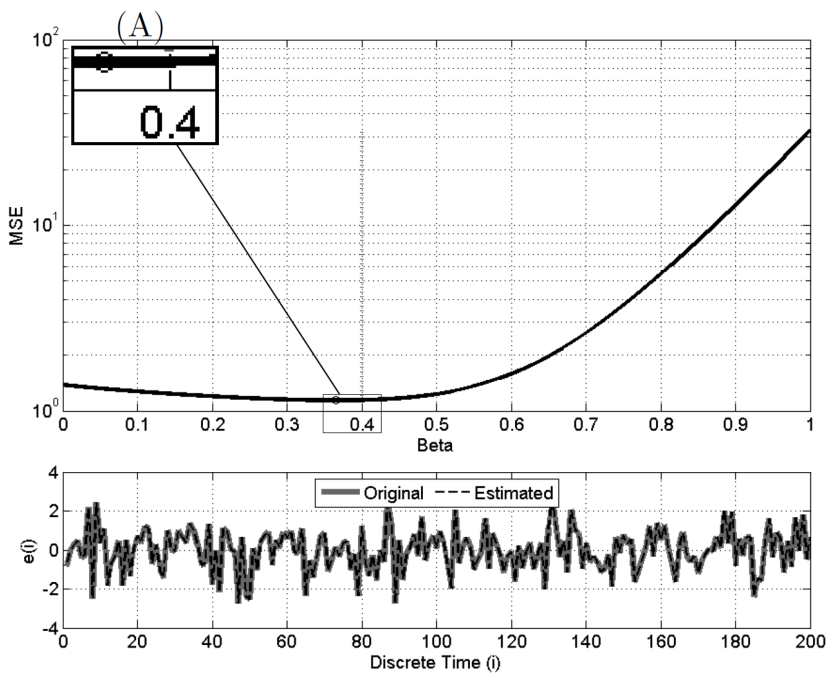
Figure 1: Estimated simulation signals

Equation (18) is similar to (6) which means that ε can be treated as the estimate $\hat{e}_i$ of a stimulation signal e_i. The $\hat{\beta}$ is the estimate of model coefficient β.

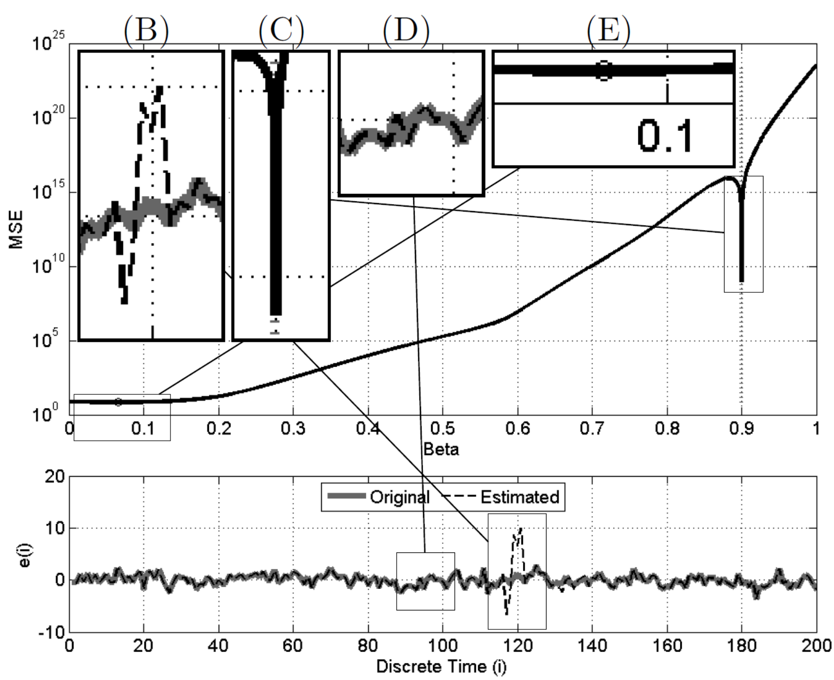
If $\hat{\beta} = \beta$ these estimates, further noted as $\hat{e}_i$, should be convergent to the original e_i values and also their variance should be close to minimal. Randomness of the EB process, and fact that in practice we always operate on the finite time–series, result in some random displacement of the global minimum of MSE cost function from the point $\hat{\beta} = \beta$. Moreover, lower values of β make $\beta e_{i-k} y_{i-l}$ part of the model having less impact of the model output y_i than e_i. This results in increased variance of the MSE global minimum displacement.

Now, if the concerned EB model is non–invertible, the incidental explosions of $\hat{e}_i$ may occur as it was presented in previous section. Lets consider how this phenomenon might influence the shape of the MSE function.

The same set of EB time–series (Fig. 1) is once more taken under consideration. However, this time $\hat{e}_i$ were also computed for $\hat{\beta} \neq \beta$ and for these, the values of MSE cost function were obtained and presented in upper plots of Fig. 2. The vertical dashed line represents the placement of the original β. In the bottom plots both: original e_i and $\hat{e}_i$ estimated for $\hat{\beta} = \beta$ are presented. In this way the convergence of $\hat{e}_i$ to e_i for the original EB model coefficient can be observed.



(a) Invertible model ($\beta = 0.4$)



(b) Not invertible model ($\beta = 0.9$)

Figure 2: MSE function

Analyzing Fig. 2a, we can see that $\hat{e}_i$ is convergent to e_i and the placement of the global minimum (represented by small circle on the MSE plot) is close to the original β (zoom A).

A very different conclusion can be drawn from Fig. 2b, where the explosion at $i = 120$ (zoom B) caused the significant rise of the MSE cost function even though, $\hat{\beta} = \beta$. As a result the minimum located at $\hat{\beta} = 0.9$ (zoom C) is not a global minimum of the MSE cost function. The actual global minimum of MSE for this particular case has been displaced towards very low values (zoom E). Therefore the EB model coefficient acquired as argument of global minimum of this particular cost function will not even be close to original one. The presented situation explains why the invertibility condition is generally considered as the identifiability condition (see [16], [37], [39], [5]). Moreover, many authors report that even narrower range than (9) should be considered if accurate estimation results for EB model coefficient (see [9], [11], [27], [40]) might be obtained.

4. Proposed solution and its impact on identification

Let once more have a look at Fig.2b. We can see that explosion of $\hat{e}_i$ (zoom B) quickly fades out and remaining $\hat{e}_i$ are convergent to e_i (zoom D). This implies that lack of convergence of $\hat{e}_i$ caused by instability of the inverted model does not accumulate over time, thus it can be corrected by a simple bound imposed on $\hat{e}_i$ values. Therefore, author propose to use the Saturated Mean Square Error (SMSE) cost function instead of MSE (17) as the cost function for estimation:

$$SMSE = \frac{1}{N}\sum_{i=1}^{N}\hat{e}_i^2 \tag{19}$$

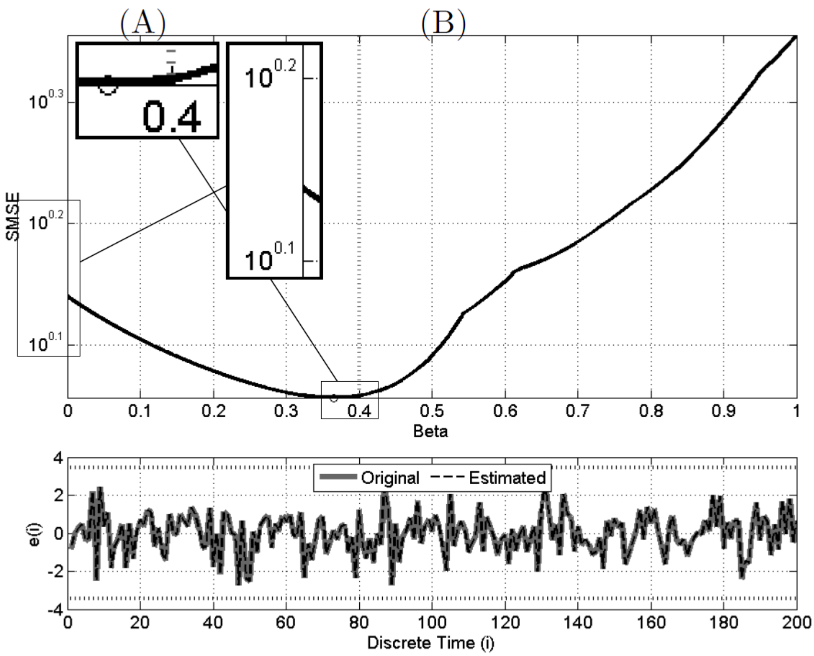
In SMSE ε_i is replaced by $\hat{e}_i$ which are estimates of e_i computed by the following formula:

$$\hat{e}_i = \begin{cases} w & : \quad \varepsilon_i \geqslant w \\ \varepsilon_i & : -w < \varepsilon_i < w \\ -w & : \quad \varepsilon_i \leqslant w \end{cases} . \tag{20}$$

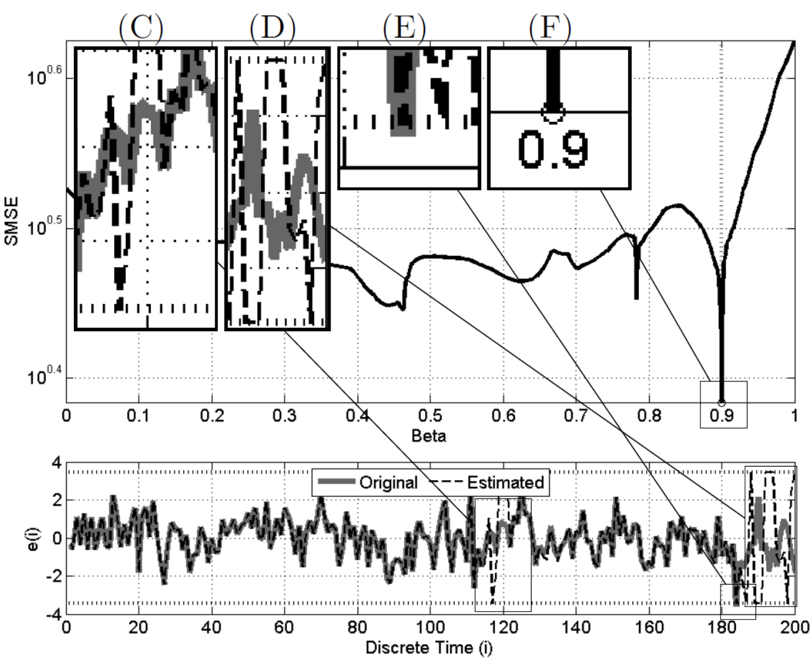
The ε_i is computed using (21).

$$\varepsilon_i = y_i - \hat{\beta}\hat{e}_{i-k}y_{i-l}. \tag{21}$$

The bounding level w is equal to $3S_e$, where S_e is the standard deviation of e_i. Normally S_e is unknown before the estimation, but it is possible to evaluate it in recursive estimation runs [24]. In this paper we simply assume that S_e is known and focus on impact of the proposed bounding level $w = 3S_e$ on estimation using the SMSE cost function. The bounding level w is established basing on assumption of the Gaussian distribution of e_i. By this way no more than 0.3% of samples should be incorrectly truncated.



(a) Invertible model ($\beta = 0.4$)



(b) Not invertible model ($\beta = 0.9$)

Figure 3: SMSE function

Lets check what impact have the SMSE function on the estimation of coefficient of invertible and non–invertible EB model. The shape of the SMSE cost function for already analysed time–series is presented in Fig. 3. For invertible example ($\beta = 0.4$), the global minimum has not not been displaced (zoom A). The shape of the SMSE function for this particular time–series has changed a little but what is the most important, the scale of the cost function values was significantly reduced (zoom B).

Results obtained for the non–invertible model are even more interesting. The explosion at $i = 120$ is now restrained within the limit w (zoom C). This way the variance of $\hat{e}_i$ is significantly decreased for $\hat{\beta} = \beta$. As a result, the global minimum of the cost function is very close to $\hat{\beta} = 0.9$ (zoom F) which is it correct placement. Although, the second explosion occurs about $i = 190$ (zoom F), which is most probably caused by a very subtle error arisen from applying the limit w (zoom E), it seems to have virtually non negative impact on the global minimum placement of the SMSE cost function.

In sequel the results of the simulations similar to those presented above, are shown and discussed.

5. Simulation results

The aim of the research provided in this section was to simulate a large number ($R = 1000$) of independent super–diagonal ($k = 1, l = 2$) bilinear time–series (each of length $N = 1000$ samples). The stimulation signal was a Gaussian noise of zero mean value and variance $\lambda^2 = 1$, thus, stability range (5) for simulated models is as follows:

$$\beta \in (-1, 1), \tag{22}$$

The invertibility range (10) is defined as:

$$\beta \in \left(-\sqrt{0.5}, \sqrt{0.5}\right). \tag{23}$$

Simulations were performed for the positive stability range only, starting from $\beta = 0.01$ and ending at $\beta = 0.99$ with the fixed step $\Delta\beta = 0.01$. For each β value, R independent time–series were generated and passed for estimation of super–diagonal EB models with the known structure ($k = 1, l = 2$).

Identification of the EB model was performed for each β (original coefficient of the simulated EB time–series) separately. As a result for each pair: β, r, estimates $\hat{\beta}_{r,\beta}$ ($r = 1, 2, ...R$) were obtained.

The estimation was performed twice, each time by minimisation of a different cost function (MSE or SMSE). In both cases the same optimisation algorithm was performed:

1. The initial search for global minimum was performed by a simple grid-search within the positive stability range $\hat{\beta} \in (0, 1)$ with fixed step $\Delta\hat{\beta}_{fst} = 0.0001$.

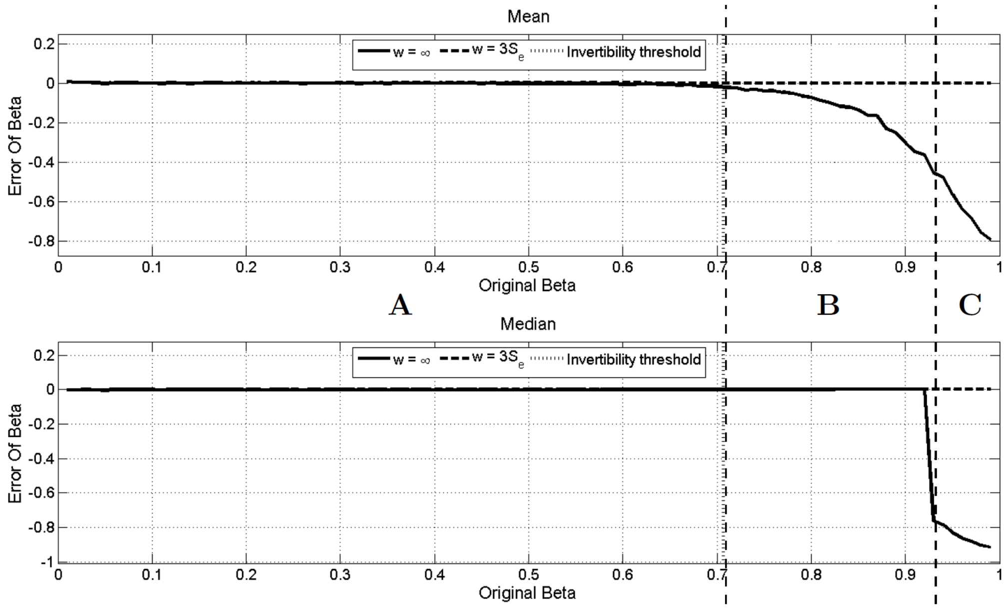


Figure 4: Position Measures

2. $\hat{\beta}_{min}$ corresponding to the minimal value of the cost function from step 1 is used as the center of the second range for search. The second range is $\hat{\beta} \in [\hat{\beta}_{min} - \Delta\hat{\beta}_{fst}, \hat{\beta}_{min} + \Delta\hat{\beta}_{fst}]$ and the second fixed step is $\Delta\hat{\beta}_{sec} = 0.00001$. The $\hat{\beta}_{min}$ obtained in the second step is assumed to be a global minimum of the specified cost function and a final estimation result.

In order to distinguish the minima $\hat{\beta}_{min}$ of MSE and SMSE functions, the following notations will be used respectively: $\hat{\beta}^{MSE}_{\beta,r}$ and $\hat{\beta}^{SMSE}_{\beta,r}$. The estimation result using function F is considered as proper, if $|\beta - \hat{\beta}^{F}_{\beta,r}| < \eta_{max}$. Therefore, for further analysis the following estimation error measures are introduced:

$$\eta^{MSE}_{\beta,r} = \hat{\beta}^{MSE}_{\beta,r} - \beta; \tag{24}$$

and for the SMSE cost function:

$$\eta^{SMSE}_{\beta,r} = \hat{\beta}^{SMSE}_{\beta,r} - \beta. \tag{25}$$

Due to changing statistical properties of identification results along with increasing β, it is no easy to determine a proper η_{max} value. Therefore, a statistical analysis for both $\eta^{SMSE}_{\beta,r}$ and $\eta^{MSE}_{\beta,r}$ was performed.

First, simple position measures (mean and median) were computed and are presented in Fig.4. Next, simple dispersion measures (standard deviation and interquartile range)

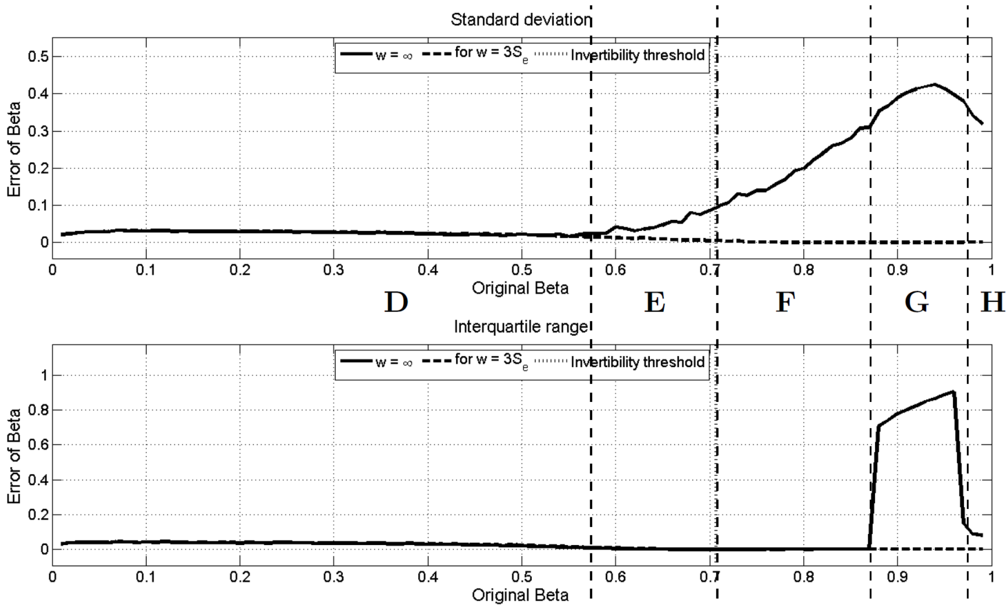


Figure 5: Dispersion Measures

were obtained and are presented in Fig.5. The analysis of the position and dispersion measures provide the following observations (marked in figures by corresponding capital letters):

A. Both the MSE and the SMSE cost functions provide similar, close to zero mean and median values of $\eta^{F}_{\beta,r}$ within the range of invertible model coefficient values (23).

B. There is a range of β values within which the median of $\eta^{MSE}_{\beta,r}$ remain close to zero but the mean value of $\eta^{MSE}_{\beta,r}$ begin to drift towards $-\infty$. It suggest that the outliers begin to occur in estimation results if the original β is beyond (23) and if MSE cost function is in use. Such behavior is not observed for the SMSE cost function.

C. There is the range of β where both median and mean value of $\eta^{MSE}_{\beta,r}$ are drifting towards $-\infty$. This means that incorrect estimates are not longer outliers and begin to dominate the identification results. Such behavior is not observed for the SMSE cost function.

D. Most of dispersion measures for the MSE and the SMSE cost functions are close to each other within the range of invertible model (23). The non–zero values of those measures are caused by randomness occurring in data and seem to fade out along the incising values of β.

E. There is a range of still invertible models (within (23)) for which standard deviation of $\eta_{\beta,r}^{MSE}$ begin to drift towards $+\infty$ while standard deviation of $\eta_{\beta,r}^{SMSE}$ is fading out towards zero. There is no such behavior observed for interquartile range which suggest that this drift is caused by outliers (for the MSE only).

F. There is a subrange of non–invertible models, within which the standard deviation of $\eta_{\beta,r}^{MSE}$ strongly drift towards $+\infty$ while standard deviation of $\eta_{\beta,r}^{SMSE}$ is close to zero. The interquartile range for both $\eta_{\beta,r}^{MSE}$ and $\eta_{\beta,r}^{SMSE}$ is very close to zero which implies that the increasing number of outliers occurs along with the increasing β when the MSE cost function is in use.

G. There is the range of non–invertible modes within which both dispersion measures (standard deviation and interquartile range) for which $\eta_{\beta,r}^{MSE}$ values are far from zero. In this range the incorrect estimates obtained using the MSE cost function are clearly dominative within the data set. If the SMSE cost function is used, no such behavior is observed.

H. At the very end of stability range (22) both dispersion measures of $\eta_{\beta,r}^{MSE}$ seem to fade out towards zero. This observation might not be accidental and it is probably caused by the strong impact of $\beta e_{i-k} y_{i-l}$ in contrast to e_i which results in increased estimation accuracy for both cost functions.

In addition to simple statistics analyzed above, the histograms of $\hat{\beta}_{\beta,r}^{MSE}$ and $\hat{\beta}_{\beta,r}^{SMSE}$ are shown in Fig.6 and Fig.7 respectively. The histograms were normalized so the intensity of gray-scale corresponds to estimated probability of occurrence. Moreover, as we are more interested in showing the incorrect results the scale was truncated on probability equal to 0.05 so all estimated probabilities greater or equal to 0.05 are represented by black points. Thanks to that even the most rare thus the most interesting results can be observed.

The specific observations made by visual analysis of the histogram are enumerated below (the letters corresponds to areas depicted in figures):

I. There is a range of invertible EB models within which the distribution of identification results $\hat{\beta}_{\beta,r}^{MSE}$ is relatively wide but no outliers can be observed.

J. There is the range of rather high β values within which the outliers present in estimation results become visible. Their number successively increases along with increase of original β values.

K. The final range of β located near the very end of the stability range (22) contains a significant or even dominative number of incorrect identification results (the MSE cost function).

L. While using the SMSE cost function, the range of invertible EB models seems to produce the very similar results to those obtained for the MSE cost function.

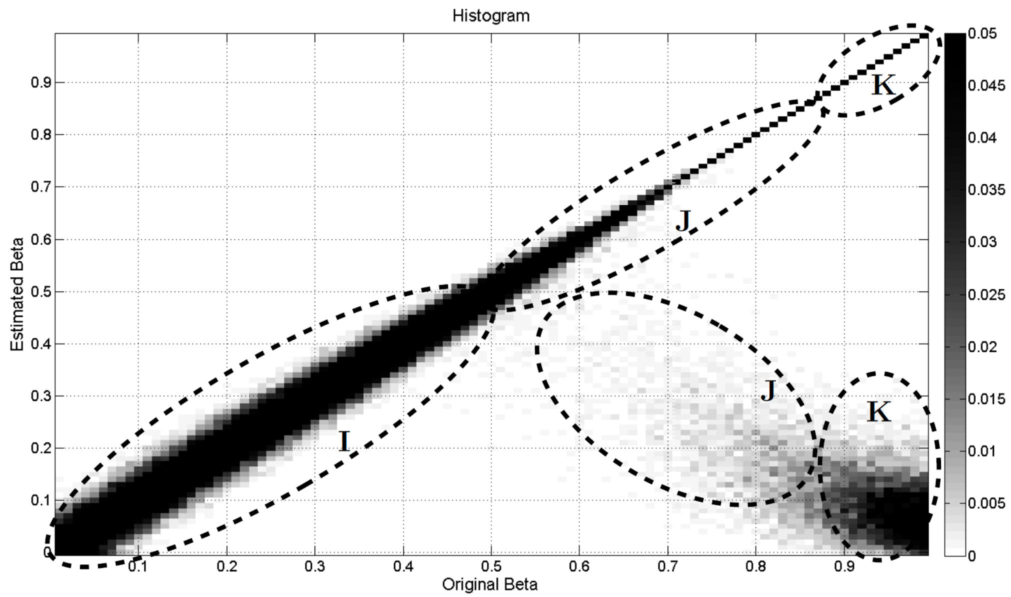


Figure 6: Histogram using MSE cost function ($w = \infty$)

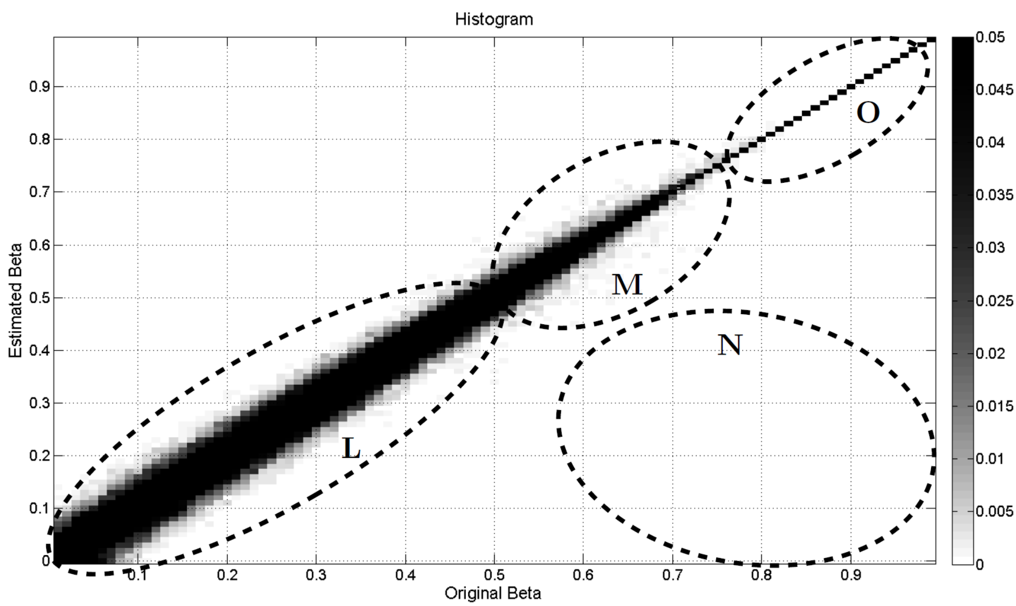


Figure 7: Histogram using SMSE cost function ($w = 3S_e$)

M. For the SMSE cost function in range of the higher β values do not contain any explicit outliers.

N. No characteristic (for the MSE cost function) outliers were found among the results obtained.

O. Only very accurate estimation results can be found in the very end of considered range of original β values.

Summarising the results and the analysis presented above the following conclusions can be drawn:

- Like in typical estimation approaches, using the MSE function is justified mostly within the range of invertible EB models. However, the increase in a standard deviation of $\eta_{\beta,r}^{MSE}$ for higher original β values (within invertiblity range) make estimation using the MSE accurate only in limited range. This was also reported by number of authors before.

- Although the proper estimation of the non invertible models (using the MSE cost function) is possible, it is very unreliable. This claim is supported by observation of numerous outlying estimation results obtained during simulations within this range.

- Use of the SMSE cost function significantly changes the distribution of estimation results $\eta_{\beta,r}^{SMSE}$ by eliminating the outliers. As a result the correct and convergent estimation results can be observed within the entire positive stability range of β.

- The statistical properties of EB model for positive and negative β values are the same so it can be assumed that correct estimation of EB model coefficient within the negative stability range is also possible.

- Distributions of the estimation results obtained at the very end of the stability range (SMSE cost function) are extremely narrow. Therefore, more sophisticated statistical using for e.g. Wilcoxon test, may not provide with credible conclusions.

6. Summary

In the paper the problem of estimation of coefficient of the elementary bilinear time–series model was addressed. Firstly, it was shown that the main issue of the correct estimation of the stable EB model coefficients is the model invertiblity. Next, the new simple solution to this problem was proposed in Section 4 by recommendation of using the SMSE cost function instead of commonly used MSE. Finally, with support of large number of simulations the credibility of the proposed solution was presented in Section 5, where the clear superiority of the SMSE over the MSE is highlighted.

The increase of SMSE computational complexity (in comparison to MSE) is insignificant, yet it seems to provide with far more accurate estimation results. Although, SMSE introduce the additional parameter (bounding level *w*) which correct value is unknown before estimation, it can be easily obtained by using recursive approach proposed in [24].

However, as long as the the SMSE cost function seems to solve the main estimation issue (invertiblity of the model), it should not be treated as the final and universal solution.

The first problem that needs emphasis is the fact that the analysis presented above was performed with assumption of the Gaussian distribution of a stimulation signal. Therefore, the solution for a different types of distribution of a stimulation signal requires different and dedicated rules for evaluation of bounding level *w*.

There is also the second problem not addressed in this paper. If we look at Fig. 3b, we can see that although, the global minimum of the SMSE cost function is in the correct place, the shape of the entire cost function is very complex and consists of multiple extrema. This is a significant problem for the estimation algorithms based on gradient approach. Therefore, a design of dedicated stochastic optimization algorithm should be provided in a future works in order to complete the solution for estimation of EB model coefficients.

References

[1] A. BIBI: A note of the stability and causality of general time-dependent bilinear models. *Statistics and Probability Letters*, **73** (2005).

[2] A. BIBI and A. OYET: Estimation of some bilinear time series models with time varying coefficients. *Stochastic Analysis and Applications*, **22** (2005).

[3] E. BIELINSKA: Identification of a mixed linear-bilinear diagonal time series model. *Systems Science*, **31** (2005).

[4] E. BIELINSKA: *Bilinear time series in signal analysis*. Zeszyty Naukowe Politechniki Slźskiej (in Polish), 2007.

[5] E. BIELINSKA: Bilinear time series models in signal analysis. In *Zeszyty Naukowe Politechniki Slaskiej* (in Polish), 2007.

[6] |sc E. Bielinska: Bilinear time series in signal analysis. In *New Approaches in Automation and Robotics*. I-Tech Education and Publishing, 2008.

[7] E. BIELINSKA and B. ZIELINSKI: Bilinear models in adaptive control. *Int. J. of Adaptive Control and Signal Processing*, **14** 2000.

[8] K. BOUZAACHANE, M. HARTI and Y. BENGHEBRIT: First-order superdiagonal bilinear time series for tracking software reliability. http://interstat.statjournals.net /YEAR/2006/articles/0602002.pdf, (2006).

[9] K. BOUZAACHANE, M. HARTI and Y. BENGHEBRIT: Parameter estimation for pure diagonal bilinear time series: An algorithm for maximum likelihood procedure. http://interstat.statjournals.net /YEAR/2006/articles/0602002.pdf, (2007).

[10] G. BOX and G. JENKINS: *Time Series Analysis: forecasting and control*. Holden-day Inc, 1976.

[11] A. BRUNNER and G. HESS: Potential problems in estimating bilinear time-series models. *J. of Economic Dynamics and Control*, **19** (1995), 663-681.

[12] K. CHELLAPILLA and S. RAO: Optimization of bilinear time series models using fast evolutionary programming. *IEEE Signal Processing Letters*, **5** (1998), 39-42.

[13] J. GOOIJGER and R. HEUTS: Higher order moments of bilinear time series processes with symmetrically distributed errors. In *Proc. of the Second Int. Tempere Conf. in Statistics*, (1987), 467-478.

[14] C. GRANGER and A. ANDERSEN: *An introduction to bilinear time series models*. Vandenhoeck and Ruprecht, 1978.

[15] C. GRANGER and A. ANDERSEN: On the invertibility of time series models. *Stochastic Processes and their Applications*, **8** (1978), 87-92.

[16] D. GRAUPE: *Identification and adaptive filtering*. Robert E. Krieger Publishing Company, 1994.

[17] D. GUEGAN and D. T. PHAM: A note on the estimation of the parameters of the diagonal bilinear model by method of least squares. *Scandinavian J. of Statistics*, **16** (1989), 129-136.

[18] O. HILI: Hellinger distance estimation of general bilinear time series models. *Statistical Methodology*, **5** (2008), 119-128.

[19] D. HRISTOVA: Maximum likelihood estimation of a unit root bilinear model with application to prices. *Studies in Nonlinear Dynamics and Econometrics*, **9** (2005), 56-70.

[20] S. IWAUEZE and J. OHAKWE: Penalties for misclassification of first order bilinear and linear moving average time series processes. *Interstat J. of Statistics*, (2009).

[21] W. KIM, L. BILLARD and I. BASWA: Estimation for first-order diagonal bilinear time series model. *J. of Time Series Analysis*, **11** (1990), 215-229.

[22] G. KIRCHGÄSSNER and J. WOLTERS: *Introduction to Modern Time Series Analysis*. Springer Berlin Heidelberg, 2007.

[23] D. KRISTENSEN: On stationarity and ergodicity of the bilinear model with applications to garch models. *J. of Time Series Analysis*, **30** (2005), 125-144.

[24] L. MALINSKI: The evaluation of saturation level for smse cost function in identification of elementary bilinear time-series model. In *Proc. of the 17th Int. Conf. on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, (2012).

[25] L. MALINSKI and E. BIELINSKA: Statistical analysis of minimum prediction error variance in the identification of a simple bilinear time-series model. In *Advances in System Science*. Academic Publishing House EXIT, 2010.

[26] L. MALINSKI and J. FIGWER: On stationarity of bilinear time–series. In *Proc. to 16th Int. Conf. on Methods and Models in Automation and Robotics*, Miedzyzdroje, Poland, (2013).

[27] C. MARTINS: A note on the autocorrelations related to bilinear model with non-independent shocks. *Statistics and Probability Letters*, **36** (1997), 245-250.

[28] C. MARTINS: A note on third-order moment structure of a bilinear model with non-independent shocks. *Portugaliae Mathematica*, **56** (1999), 115-125.

[29] J. MATHEWS and J. LEE: Techniques for bilinear time series analysis. In *27-th Asilomar Conf. on Systems and Computers*, (1993), 1016-1020.

[30] J. MATHEWS and T. MOON: Parameter estimation for a bilinear time series model. In *Int. Conf. on Acoustics, Speech and Signal Processing*, (1991), 3513-3516.

[31] R. MOHLER and Z. TANG: On bilinear time-series modelling and estimation. In *Proc. of the 27th Conf. on Decision and Control*, Austin, Texas, (1988), 953-954.

[32] T. D. PHAM: Bilinear markovian representation and bilinear modelss. *Stochastic Processes and Their Applications*, **20** (1985), 295-306.

[33] T. D. PHAM: The mixing property of bilinear and generalised random coefficient autoregressive models. *Stochastic Processes and Their Applications*, **23** (1986), 291-300.

[34] M. PRIESTLEY: *Nonlinear and Non-stationary Time Series Analysis*. Academic Press Ltd., 1988.

[35] T. SUBBA RAO: On the theory of bilinear time series models. *J. of the Royal Statistical Society B*, **44** (1981), 244-255.

[36] M. SMALL: *pplied Nonlinear Time Series Analysis*. World Scientific Publishing Co. Pte. Ltd., 2005.

[37] C. THERRIEN: *Discrete random signals and statistical signal processing*. Prentice Hall International Editions, 1992.

[38] H. TONG: *Non-linear time series*. Clarendon Press, 1993.

[39] H. WANG: Parameter estimation and subset detection for separable lower triangular bilinear models. *J. of Time Series Analysis*, **26** (2004), 743-757.

[40] B. WU and S. HUNG: A fuzzy identification procedure for nonlinear time-series: With example on arch and bilinear models. *Fuzzy Sets and Systems*, **108** (1999), 275–287.