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Shape sensitivity of optimal control for the Stokes problem*

by

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Abstract: In this article, we study the shape sensitivity of optimal control for the steady Stokes problem. The main goal is to obtain a robust representation for the derivatives of optimal solution with respect to smooth deformation of the flow domain. We introduce in this paper a rigorous proof of existence of the material derivative in the sense of *Piola*, as well as the shape derivative for the solution of the optimality system. We apply these results to derive the formulae for the shape gradient of the cost functional; under some regularity conditions the shape gradient is given according to the structure theorem by a function supported on the moving boundary, then the numerical methods for shape optimization can be applied in order to solve the associated optimization problems.

Keywords: Stokes equation, optimal control problems, coupled partial differential equations, shape optimization, shape gradient

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1. Introduction

The shape sensitivity analysis has been introduced in several concepts for different models, for the elasticity system (Sokolowski and Zolesio, 1992; Allaire, 2007), piezoelectric system (Leugering et al., 2011), and for inverse problems (Ammari, 2002; Caubet and Dambrine, 2012). Our interest in shape optimization for the problems of fluid mechanics has been motivated by vast applications in automotive and aerodynamic domains, where a natural question is to minimize a work or drag of some body moving in a fluid by choosing an appropriate shape, those problems having been investigated by Boisgerault (2000), Moubachir and Zolésio (2006) in the incompressible fluids, while the compressible case has been studied by Sokolowski and Plotnikov (2010)

In this paper we study the Stokes flow, which represents a type of fluid flow, where advective inertial forces are small compared with viscous forces. There are several works devoted to this model, see, for example, Rösch and Vexler (2006) in the context of optimal control and Abdelwahab and Hassine (2009), Guillarme and Hassine (2008), Amstutz (2005), and Pironneau (1984) in shape and topological optimization. Our approach is a combination of optimal control and shape optimization, more precisely, we study the sensitivity of the optimality system with respect to a perturbed domain. Unlike the cases of Laplacian, heat and hyperbolic equations, which are treated in Sokolowski and Zolesio (1992), for fluids we have an essential constraint, which characterizes the compressibility, and this constraint must be preserved during all the steps, which lead to our main results of sensitivity, material derivative, shape derivative and shape gradient of the cost functional.

This paper is organized as follows: in Section 2, we introduce a distributed optimal control problem for the Stokes flow and we derive the necessary and sufficient optimality conditions. Here the cost functional is of quadratic type, which is widely used in control theory. The results may extend to other types of functionals.

In Section 3, we formulate the shape optimization problem, associated to optimality system, we move a part of the domain, which supports the control function. The technique of boundary variations, employed in this paper is the *Adjugate Matrix Method*, see Plotnikov and Sokolowski (2012). By this method, we can formulate our problem in a fixed domain, named the *reference domain*. Note that the results can be adapted to other general situations as the *Speed Method*, see Sokołowski and Zolesio (1992).

The *Piola* material derivatives for the system are given in Section 4. The key is the implicit function theorem in Sobolev spaces and the *Piola* transformation, which is used to preserve the incompressibility conditions after the change of variables in the optimality system. The *Piola* material derivatives imply under some regularity conditions the existence of the shape derivatives in moving domain. System, which represents the shape derivatives, is formulated in Section 5.

In Section 6, we introduce a second level adjoint system from the Lagrangian formulation. By combining the materiel derivatives and the adjoint state, we can give the shape gradient formula of the shape functional, this formula is given in volume expression and interface representation, see Sokołowski and Zolesio (1992).

The last section is devoted to some numerical examples in a simplified geometry. The computations will be done in two stages, the first is for the optimal control, the second is for the optimal position and shape for the control region.

2. Optimal control problem

In this section we briefly review the optimal control theory for the Stokes model, as this will provide the ground for the setup of our shape optimization problem. For a general presentation of the optimal control theory, the interested reader can refer to Fursikov (1999), Lasiecka (2002), Lions (1971), and Tröltzsch (2000).

Let us consider a domain Ω in $\mathbb{R}^3$, with C^1 boundary Γ. The stationary boundary value problem for Stokes equation can be formulated as follows,

$$\left\{ \begin{array}{rcll} -\mu\Delta u + \nabla p & = & z\chi_\omega & \text{in} \quad \Omega \\ div\, u & = & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \Gamma \end{array} \right. \tag{2.1}$$

where u and p are the velocity and the pressure of the fluid, respectively, and μ is the dynamic viscosity of the fluid. Our first goal is to control the system (2.1) by the source term $z \in L^2(\omega) \subset L^2(\Omega)$; the control function $z = z\chi_\omega$ is defined in a subdomain $\omega \subset \Omega$, as in the representation of Fig. 1, and χ_ω denotes the characteristic function of $\omega \subset \Omega$. The cost functional is given by

$$\mathcal{J}(u, z) =: \frac{1}{2}\int_\Omega \|u - u_d\|^2\, dx + \frac{\alpha}{2}\int_\omega \|z\|^2\, dx \tag{2.2}$$

where $u_d \in L^2(\Omega)$ is a desired state, $\alpha > 0$, and $\|\cdot\|$ denotes the the Euclidean norm in $\mathbb{R}^3$. The natural spaces adapted to the state equation (2.1) (see Galdi, 1994; Boyer and Fabrie, 2006) are:

$$\mathcal{H}_0^1(\Omega) := \{v \in [H_0^1(\Omega)]^3,\ div\, v = 0\}$$

and

$$L_0^2(\Omega) := \{q \in L^2(\Omega),\ \int_\Omega q\, dx = 0\}.$$

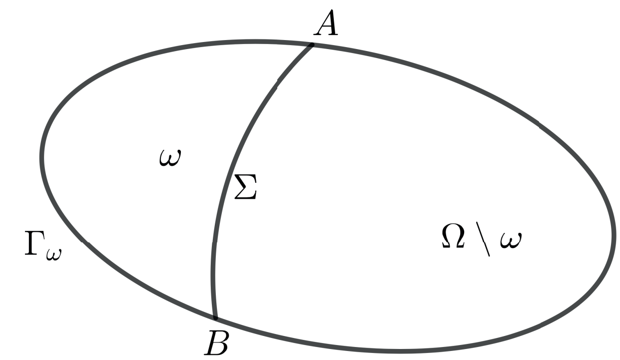

Figure 1. The optimal control problem

Let us note, first, that if we multiply the first equation of the system (2.1) by a smooth function $v \in \mathcal{H}_0^1(\Omega)$ and integrate on Ω, the pressure disappears and the problem is reduced to finding a $u \in \mathcal{H}_0^1(\Omega)$ solution to

$$\int_\Omega \mu \nabla u : \nabla v \, dx = \int_\omega zv \, dx, \quad \forall v \in \mathcal{H}_0^1(\Omega). \tag{2.3}$$

Here, $\nabla u := (\partial_i u_j)_{1\leq i,j\leq 3}$ and $C : D = \sum_{i,j} c_{ij}.d_{ij}$ for $C = (c_{ij})$, $D = (d_{ij})$ in $\mathcal{M}_3(\mathbb{R})$. By the Lax-Milgram theorem, one can show that (2.3) admits a unique solution u and then, by De-Rham's theorem (see Galdi, 1994), there exists $p \in L^2(\Omega)$ unique up to an additive constant, such that

$$-\mu\Delta u + \nabla p = z\chi_\omega, \quad \text{in} \quad \Omega$$

so that for the state equation, we have the existence and uniqueness in the space $\mathcal{H}_0^1(\Omega) \times L_0^2(\Omega)$. On the other hand, the control problem (2.1)-(2.2) is linear quadratic and strictly convex, so that there exists a unique minimizer $\overline{z} \mapsto (\overline{u}(\overline{z}), \overline{z})$ to the cost functional $\overline{z} \mapsto \mathcal{J}(\overline{u}(\overline{z}), \overline{z})$ and this minimizer is characterized by the Euler-Lagrange equation:

$$\mathcal{J}'(\overline{u}, \overline{z})(v) = 0, \ \forall v \in L^2(\omega).$$

We observe that in view of the state equation, the derivative of the linear mapping $z \mapsto u$ coincides with the same mapping. Such a derivative is eliminated by an appropriate adjoint equation.

In order to formulate the optimality system, we introduce the adjoint equation

$$\left\{ \begin{array}{rcll} -\mu\Delta w + \nabla q &=& u - u_d & \text{in} \quad \Omega \\ div\, w &=& 0 & \text{in} \quad \Omega \\ w &=& 0 & \text{on} \quad \Gamma \end{array} \right. . \tag{2.4}$$

Lemma 1 *The necessary and sufficient condition for the optimality of the unique optimal control $\overline{z}$ with the associated state $\overline{u}$ and the associated adjoint state $\overline{w}$, respectively, is the variational equation*

$$(\overline{w}+\alpha\overline{z}, v) = 0, \ \ \forall v \in L^2(\omega). \tag{2.5}$$

Thus, the optimal control can be expressed by the adjoint state, $\overline{z} = -\frac{1}{\alpha}\overline{w}\chi_\omega$. Let $\beta := \dfrac{1}{\alpha}$, therefore, the reduced optimality system is

$$\left\{\begin{array}{rcll} -\mu\Delta\overline{u}+\nabla\overline{p} &=& -\beta\overline{w}\chi_\omega & in \ \ \Omega \\ div\,\overline{u} &=& 0 & in \ \ \Omega \\ -\mu\Delta\overline{w}+\nabla\,\overline{q} &=& \overline{u}-u_d & in \ \ \Omega \\ div\,\overline{w} &=& 0 & in \ \ \Omega \\ \overline{u}=\overline{w} &=& 0 & on \ \ \Gamma \end{array}\right. . \tag{2.6}$$

Proof. Proof is standard, see, e.g., Tröltzsch (2000), Chapter 2 or the monograph by Lions (1971). ∎

Remark 1 *The optimality system* (2.6) *is called the coupled system, since it contains the state equation coupled with the adjoint state equation. Let us recall that the optimality system is written in the strong form. An alternative approach would be the weak form of the coupled system written as two integral identities. The existence and uniqueness of solutions to the coupled system is obvious (Lions, 1971).*

Remark 2 *We are interested in the shape optimization for the optimal control problem. The optimal control $\overline{z}$ is determined through the adjoint state $\overline{w}$, which is characterized by the coupled system (2.6). The optimal value of the cost defines the new shape functional, denoted by $\mathcal{J}(\Omega) := \mathcal{J}(\overline{u},\overline{z})$. In order to improve the value of such a shape functional with respect to the source shape and location, the sensitivity analysis of the optimal control with respect to the boundary variation of ω is performed. Therefore, the shape gradient for the mapping $\omega \mapsto \mathcal{J}(\Omega)$ is obtained by the shape sensitivity analysis of the optimality system (2.6) as well as of the optimal cost $\mathcal{J}(\overline{u},\overline{z})$ with respect to the boundary variations of the subdomain $\omega \subset \Omega$.*

3. Shape perturbation of the optimality system

3.1. Boundary value problems in perturbed domains Ω_t

We use the velocity method for the purposes of the shape sensitivity analysis (Sokolowski and Zolesio, 1992), thus we use the notation $\Omega_t = T_t(\Omega)$, and $\omega_t = T_t(\omega)$ for variable domains with an appropriate mapping T_t. The shape sensitivity of the optimality system with the divergence free fields has been studied in the framework of the *Adjugate Matrix Method* (AMM), see Plotnikov and

Sokolowski (2012) for all details in the case of the more complex compressible Navier-Stokes equations. To this end, we describe AMM with the assumptions which lead us to the main results of this paper. The first step is to generate a perturbation around the domain of control ω as it is shown in Fig. 2. Let ζ be a C^2 vector field such that

$$\langle \zeta, n\rangle = 0, \ \ \forall x \in \Gamma \setminus \Gamma_\omega \tag{3.7}$$

and introduce the mapping T_t, $(t \geq 0)$, from Ω to Ω_t, defined by

$$x \mapsto T_t(x) = x + t\zeta(x).$$

In the rest of this paper, we adopt the notations u, p, w and q instead of $\overline{u}$, $\overline{p}$, $\overline{w}$ and $\overline{q}$, respectively. Upon setting $\Omega_t := T_t(\Omega)$ and $\Gamma_t = \partial\Omega_t$, the system in the perturbed domain Ω_t is given by

$$\left\{\begin{array}{rclll} -\Delta u_t + \nabla p_t & = & -\beta w_t \chi_{\omega_t} & \text{in} & \Omega_t \\ div\, u_t & = & 0 & \text{in} & \Omega_t \\ -\Delta w_t + \nabla q_t & = & u_t - u_d & \text{in} & \Omega_t \\ div\, w_t & = & 0 & \text{in} & \Omega_t \\ u_t = w_t & = & 0 & \text{on} & \Gamma_t \end{array}\right. \tag{3.8}$$

and the associated domain functional is defined by the optimal value of the cost for the control problem

$$\mathcal{J}(\Omega_t) = \frac{1}{2}\int_{\Omega_t} (u_t - u_d)^2\, dx + \frac{\beta}{2}\int_{\omega_t} w_t^2\, dx.$$

The shape derivative of our domain functional $\mathcal{J}$ in the direction of the velocity field ζ at the domain Ω is given by the limit, if the limit exists,

$$d\mathcal{J}(\Omega)[\zeta] := \lim_{t\downarrow 0} \frac{\mathcal{J}(\Omega_t) - \mathcal{J}(\Omega)}{t}.$$

From the assumption (3.7), the vector field ζ is tangential to $\Gamma \setminus \Gamma_\omega$, then we move only the part $\overline{\Gamma}_\omega$ of the boundary Γ. After perturbation of ω, suppose that the boundary Γ_{ω_t} always stays connected smoothly to the rest of the boundary $\Gamma_t \setminus \Gamma_{\omega_t}$, see Fig. 2. According to the structure theorem of the shape gradient in two spatial dimensions, see Sokolowski and Zolesio (1992) and Fremiot (2000), there exist two real numbers α_A, α_B and a distribution g_{Γ_ω}, defined on Γ_ω, such that

$$d\mathcal{J}(\Omega)[\zeta] = \langle g_{\Gamma_\omega}, \zeta_n\rangle + \alpha_A\langle \zeta(A), \tau_\Sigma\rangle + \alpha_B\langle \zeta(B), \tau_\Sigma\rangle \tag{3.9}$$

where n is the normal vector field on Γ_ω, $\zeta_n := \langle \zeta, n\rangle$, and τ_Σ is the tangential vector on $\overline{\Sigma}$.

Remark 3 *In three spatial dimensions the structure theorem is similar (Fremiot, 2000; Laurain, 2006). However, there are: one normal vector and two tangential vectors on the boundary.*

We limit ourselves to the case when the interface $\overline{\Sigma}$ remains fixed so that the real numbers are zero in the expression (3.9) of the shape gradient. This can be ensured by the following assumption:

H-1: The vector field ζ is in $C^2(\mathbb{R}^2)$ such that $\operatorname{Supp}\zeta \cap \Sigma = \emptyset$.

Note that with a slight modification, our study will also include the case when $\Gamma \cap \Gamma_\omega = \emptyset$.

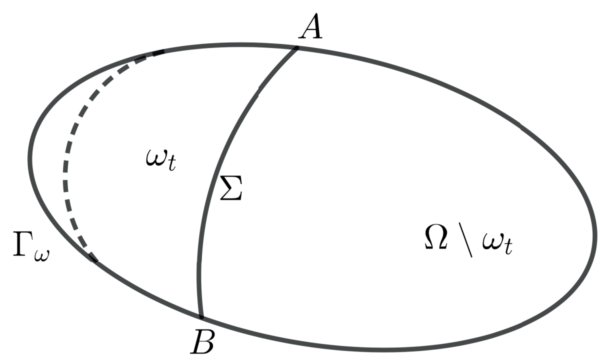

Figure 2. The shape optimization problem

3.2. Transport of the control problem to the fixed domain

By a change of variable, one can introduce the system (3.8) in a fixed domain Ω, named the reference domain. Denote by $\mathbb{M}(t)$ the Jacobi matrix of the mapping T_t and by $g(t)$ the determinant of $\mathbb{M}(t)$,

$$\mathbb{M}(t) = \mathbb{I} + tD\zeta(x), \quad g(t) = \det(\mathbb{I} + tD\zeta(x)),$$

then the adjugate matrix of $\mathbb{M}(t)$ is defined by

$$\mathcal{N}(t) = g(t)\mathbb{M}(t)^{-1}, \ \text{ with } \det\mathcal{N}(t) = g^3(t)\det(\mathbb{M}(t)^{-1}) = g^2(t).$$

Now, introduce the transformation

$$\begin{aligned} \xi_t :& \mathcal{H}_0^1(\Omega_t) \longrightarrow \mathcal{H}_0^1(\Omega) \\ & \theta_t \longmapsto \xi_t(\theta_t) := \mathcal{N}(t)\theta_t(x + t\zeta(x)) \end{aligned} \tag{3.10}$$

which is called the *Piola* transformation.

Lemma 2 *The Piola transformation (3.10) is an isomorphism between $\mathcal{H}_0^1(\Omega_t)$ and $\mathcal{H}_0^1(\Omega)$.*

Proof. See Moubachir and Zolesio (2006).

The essential property of this transformation is that it preserves the divergence free vector fields $div\, u = 0$, which is an important condition that characterizes the incompressibility of the fluid. Indeed, denote by

$$u^t(x) := \xi_t(u_t) = \mathcal{N}(t)u_t(x+t\zeta(x)),\;\; p^t(x) := p_t(x+t\zeta(x)),\;\; x\in\Omega \quad (3.11)$$

and

$$w^t(x) := \xi_t(w_t) = \mathcal{N}(t)w_t(x+t\zeta(x)),\;\; q^t(x) := q_t(x+t\zeta(x)),\;\; x\in\Omega \quad (3.12)$$

so that for the test functions v and v_t, we have

$$\begin{aligned}\int_{\Omega_t} div\, u_t\, v_t &= -\int_{\Omega_t} u_t \nabla v_t + \int_{\Gamma_t} u_t\, v_t.n\\ &= -\int_{\Omega_t} (g(t)\mathbb{M}(t)u_t\circ T_t)\,.\,\big(\mathbb{M}(t)^{-T}\nabla v\circ T_t\big)\\ &= -\int_{\Omega} u^t\,\nabla v^t = \int_{\Omega} div\, u^t v^t.\end{aligned}$$

For the sake of simplicity we consider the strong solutions of the coupled equations, which require the appropriate regularity of the data. The alternative approach uses the weak solutions obtained by the variational formulations of the coupled equations. With the change of variables (3.11), (3.12), the coupled system can be formulated in the reference domain, more precisely, we have

Lemma 3 *Let (u_t, p_t, w_t, q_t) be the solution of (3.8), then (u^t, p^t, w^t, q^t) defined by (3.11) and (3.12) satisfies*

$$\left\{\begin{array}{rcll} -\mu\, div(\mathcal{A}(t)\nabla(\mathcal{N}(t)^{-1}u^t) + \mathcal{N}(t)^T\nabla p^t &=& -\beta g(t)\mathcal{N}(t)^{-1}w^t\chi_\omega & in\quad \Omega\\ div\, u^t &=& 0 & in\quad \Omega\\ -\mu\, div(\mathcal{A}(t)\nabla(\mathcal{N}(t)^{-1}w^t)) + \mathcal{N}(t)^T\nabla q^t &=& g(t)(\mathcal{N}(t)^{-1}u^t - u_d) & in\quad \Omega\\ div\, w^t &=& 0 & in\quad \Omega\\ u^t = w^t &=& 0 & on\quad \Gamma \end{array}\right. \quad (3.13)$$

where

$$\mathcal{A}(t) := g(t)\mathbb{M}(t)^{-T}\mathbb{M}(t)^{-1},\;\; \mathcal{N}(t)^{-1} = g(t)^{-1}\mathbb{M}(t).$$

Proof. Let $a\in C^1(\Omega)$ be a vector field and $\tilde{a}(x) = a(y(x)) := a(T_t(x))$. Since for all $\phi\in C^1(\Omega)$,

$$(\nabla_y\phi)(y(x)) = (\mathbb{M}^{-T}(t)\nabla_x\tilde{\phi})(x),$$

where $\tilde{\phi}(x) := \phi(y(x))$, the following identities hold

$$\int_{\Omega}(div_y a)(y(x))\tilde{\phi}(x)g(t)dx = \int_{\Omega_t} div_y a\phi(y)dy = -\int_{\Omega_t} a.\nabla_y\phi\, dy$$

$$= -\int_{\Omega} \tilde{a}.\mathbb{M}(t)^{-T}\nabla_x\tilde{\phi}(x)g(t)dx = \int_{\Omega} div_x(g(t)\mathbb{M}(t)^{-1}\tilde{a})\tilde{\phi}(x)dx$$

$$= \int_{\Omega} div_x(\mathcal{N}(t)\tilde{a})\tilde{\phi}(x)dx.$$

Thus,

$$(div_y a)(y(x)) = g(t)^{-1}div_x(\mathcal{N}(t)a(y(x))),\ \ \forall\phi\in C_0^\infty(\Omega).$$

Setting $a(y) = u_t(y)$, we get for free divergence condition,

$$(div_y(u_t))(y(x)) = g(t)^{-1}div_x(\mathcal{N}(t)u_t(y(x))) = g(t)^{-1}div_x u^t(x).$$

As $\mathbb{M}(t)^{-T} = g(t)^{-1}\mathcal{N}(t)^T$,

$$\nabla_y(p_t(y)) = g(t)^{-1}\mathcal{N}(t)\nabla_x p^t(x).$$

Next, set $a := \nabla_y u_t$, and so

$$\tilde{a} = (\nabla_y u_t)(y(x)) = \mathbb{M}(t)^{-T}\nabla_x(u_t(y(x))) = \mathbb{M}(t)^{-T}\nabla_x(\mathcal{N}(t)^{-1}u^t(x))$$

$$= g(t)^{-1}\mathcal{N}(t)^T\nabla_x(\mathcal{N}(t)^{-1}u^t(x)),$$

and we deduce that

$$div_y(\nabla_y u_t(y(x))) = g(t)^{-1}div_x(g(t)^{-1}\mathcal{N}(t)\mathcal{N}(t)^T\nabla_x(\mathcal{N}(t)^{-1}u^t(x)))(x).$$

Using the identity $\Delta = div\nabla$ and $g(t)^{-1}\mathcal{N}(t)\mathcal{N}^T(t) = \mathcal{A}(t)$, we find

$$(\Delta u_t)(y(x)) = g(t)^{-1}div_x(\mathcal{A}(t)\nabla_x(\mathcal{N}(t)^{-1}u^t))(x).$$

Following the same steps for (w_t, q_t) and replacing all the terms in the system (3.8), we obtain the system (3.13) ■

LEMMA 4 *For every desired state $u_d \in L^2(\Omega)$, the coupled system (3.13) has a unique weak solution (u^t, p^t, w^t, q^t) in the space $[\mathcal{H}_0^1(\Omega)\times L_0^2(\Omega)]^2$.*

Proof. The existence of solutions follows from the optimal control problem. Indeed, the unique optimal control implies the existence of the unique optimal state. Therefore the adjoint state is unique, which leads to the existence of the couple (u, w) for the coupled system.

The uniqueness of solutions for the coupled system can be obtained in general by the Fredholm alternative, in our case it is an application of the Lax-Milgram lemma. We provide the outline of the proof, the reader might add all the details. Indeed, upon writing the weak state equation for u in the optimality system in the form $Lu = -\beta\chi_\omega w$ and the weak adjoint state equation for w as $Lw = u - u_d$, we can obtain the weak equation for w in the form $(Lw, Lv)_\Omega + \beta(w, v)_\omega = -(Lu_d, v)_\Omega)$ for all test functions v. The latter equation admits the unique solution for w, and the state equations furnishes the unique u. Therefore, the solution (u, w) of the coupled system is unique. ∎

4. Piola material derivative

This section is devoted to the existence of the *Piola* material derivative. Moreover, we claim that the material derivative is a weak solution of a linear coupled system. Recall that the classical material derivative is defined in the fixed domain by:

$$\dot{u} := \lim_{t\to 0} \frac{u_t \circ T_t - u}{t}.$$

We can adapt this notion to our framework and use the following definition of *Piola* material derivative:

$$\dot{u}_P := \lim_{t\to 0} \frac{\xi_t(u_t) - u}{t} = \lim_{t\to 0} \frac{u^t - u}{t} \tag{4.14}$$

and

$$\dot{w}_P := \lim_{t\to 0} \frac{\xi_t(w_t) - w}{t} = \lim_{t\to 0} \frac{w^t - w}{t}. \tag{4.15}$$

We will derive the system associated to $\dot{u}_P$ and $\dot{w}_P$. In order to give a rigorous formulation of this fact we need some preliminary results concerning the differentiability of the coefficients in the system (3.13).

Lemma 5 *Under the assumptions on the domain transformation, the functions* $t \mapsto g(t)$, $\mathbb{M}(t)$, $\mathcal{N}(t)^{-1}$, $\mathcal{A}(t)$ *are differentiable for all* $t \in [0, \varepsilon]$. *Furthermore, we have*

$$g'(t) = div\zeta.\, g(t) \tag{4.16}$$

$$\mathbb{M}'(t) = D\zeta.\, \mathbb{M}(t) \tag{4.17}$$

$$(\mathcal{N}(t)^{-1})' = [D\zeta - div\zeta\mathbb{I}]\mathcal{N}(t)^{-1} \tag{4.18}$$

$$\mathcal{A}'(t) = [div\zeta\mathbb{I} - M(t)^{-1}D\zeta]\mathcal{A}(t) - [M(t)^{-1}D\zeta\mathcal{A}(t)]^T. \tag{4.19}$$

Proof. See Plotnikov and Sokolowski (2012) and Sokołowski and Zolesio (1992). ■

It is easy to derive formally the equations for the material derivative by differentiating (3.13) with respect to t at zero, this formal procedure giving the following system

$$\begin{aligned}\langle \nabla \dot{u}_P : \nabla v\rangle_\Omega + \beta\langle \dot{w}_P, v\rangle_\omega + \langle (div\zeta\,\mathbb{I} - 2\epsilon(\zeta))\nabla u + \nabla(D\zeta - div\zeta\,\mathbb{I})u : \nabla v\rangle_\Omega \\ + \beta\langle D\zeta\, w, v\rangle_\omega = 0,\ \ \forall v \in \mathcal{H}^1_0(\Omega)\end{aligned} \tag{4.20}$$

$$\begin{aligned}\langle \nabla \dot{w}_P : \nabla v\rangle_\Omega - \langle \dot{u}_P, v\rangle_\Omega + \langle (div\zeta\,\mathbb{I} - 2\epsilon(\zeta))\nabla w + \nabla(D\zeta - div\zeta\,\mathbb{I})w : \nabla v\rangle_\Omega \\ - \langle D\zeta u, v\rangle_\Omega + \langle div\zeta u_d, v\rangle_\Omega + \langle \nabla u_d\zeta, v\rangle_\Omega = 0,\ \ \forall v \in \mathcal{H}^1_0(\Omega)\end{aligned} \tag{4.21}$$

where $\epsilon(\zeta) := \frac{1}{2}(D\zeta^T + D\zeta)$ and $\mathbb{I}$ is the identity matrix.

Of course, the next step is to verify rigorously that the last system characterizes effectively the *Piola* material derivative. In general, there are two approaches for dealing with this question. The first is to introduce some priori estimates for the quotients

$$\frac{u^t - u}{t} \text{ and } \frac{w^t - w}{t}$$

to prove the boundedness of these terms in some Sobolev space. We then show that the weak limits are exactly $\dot{u}_P$ and $\dot{w}_P$ given by the system (3.13); this method is used in most of cases, see for example Plotnikov and Sokolowski (2012) for stationary compressible Navier-Stokes flow and Consiglieri, Nečasova and Sokolowski (2010) for the Maxwell-Boussinesq system. The second approach, the one that we adopt in our case, is to apply a compatible version of the implicit function theorem, see Moubachir and Zolesio (2006). The abstract theorem used here is given by:

Theorem 1 *Let H, Y be two Banach spaces and I an open bounded set in $\mathbb{R}$, consider the map*

$$\begin{aligned}\Psi : I \times H &\longrightarrow Y' \\ (t,h) &\longmapsto \Psi(t,h)\end{aligned}$$

where Y' denotes the dual space of Y.
If

- ***(i)*** *$t \longmapsto \langle \Psi(t,h), y\rangle$ is continuously differentiable for any $y \in Y$ and $(t,h) \longmapsto \langle \partial_t \Psi(t,h), y\rangle$ is continuous;*
- ***(ii)*** *there exists $s : I \to H$ Lipschitz and $\Psi(t, s(t)) = 0$, $\forall t \in I$;*
- ***(iii)*** *$h \longmapsto \langle \Psi(t,h), y\rangle$ is differentiable and $(t,h) \longmapsto \partial_h \Psi(t,h)$ is continuous;*

(iv) *there exists $t_0 \in I$ such that $\partial_h \Psi(t,h)_{|(t_0,s(t_0))}$ is an isomorphism from H to Y',*

then the map

$$\begin{aligned} s : I &\longrightarrow H \\ t &\longmapsto s(t) \end{aligned}$$

is differentiable at $t = t_0$ for the weak topology in H and its weak derivative $\dot{s}(t_0)$ is the solution of

$$\langle \partial_h \Psi(t_0, s(t_0)).\dot{s}(t_0), y\rangle + \langle \partial_t \Psi(t_0, s(t_0)), y\rangle = 0, \ \ \forall y \in Y.$$

Our main result of this section is the following theorem:

Theorem 2 *Suppose that the assumption **H-1** is satisfied, then the Piola material derivatives $\dot{u}_P$ and $\dot{w}_P$ exist in the weak sense in the Sobolev space $\mathcal{H}_0^1(\Omega)$, i.e.,*

$$\frac{u^t - u}{t} \rightharpoonup \dot{u}_P, \ \ \frac{w^t - w}{t} \rightharpoonup \dot{w}_P.$$

Moreover, they are characterized by the two coupled equations (4.21) and (4.20).

Proof. We use the implicit function theorem for a mapping Ψ defined from $[0,\varepsilon) \times \mathcal{H}_0^1(\Omega)^2$ to $[\mathcal{H}_0^1(\Omega)^2]'$, which can be identified with the space $\mathcal{H}_0^1(\Omega)' \times \mathcal{H}_0^1(\Omega)'$. The expression of Ψ is given by

$$\langle \Psi(t; u, w), v\rangle := (\langle \Psi_1(t; u, w), v\rangle, \langle \Psi_2(t; u, w), v\rangle)$$

where the components Ψ_1 and Ψ_2 are defined by

$$\langle \Psi_1(t; u, w), v\rangle := \langle \mathcal{A}(t).\nabla(\mathcal{N}(t)^{-1}u) : \nabla v\rangle_\Omega + \beta g(t)\langle \mathcal{N}(t)^{-1}w, v\rangle_\omega$$

$$\langle \Psi_2(t; u, w), v\rangle := \langle \mathcal{A}(t).\nabla(\mathcal{N}(t)^{-1}w) : \nabla v\rangle_\Omega - g(t)\langle \mathcal{N}(t)^{-1}u - u_d, v\rangle_\Omega.$$

Thus, we need to verify all the assumptions for Ψ:

Step 1: Since the coefficients $\mathcal{A}$, $\mathcal{N}^{-1}$ and g are differentiable with respect to t, the functions,

$$t \mapsto \langle \Psi_1(t; u, w), v\rangle \text{ and } t \mapsto \langle \Psi_2(t; u, w), v\rangle$$

are C^1 from $[0,\varepsilon]$ to $\mathbb{R}$, and the derivatives with respect to t are given by

$$\begin{aligned} \langle \partial_t \Psi_1(t; (u, w), v)\rangle &= \langle \mathcal{A}'(t).\nabla(\mathcal{N}(t)^{-1}u) + \mathcal{A}(t)\nabla((\mathcal{N}(t)^{-1})'u) : \nabla v\rangle_\Omega \\ &\quad + \beta\langle g'(t)\mathcal{N}(t)^{-1}w, v\rangle_\omega + \beta\langle g(t)(\mathcal{N}(t)^{-1})'w, v\rangle_\omega \end{aligned}$$

$$\begin{aligned}\langle \partial_t\Psi_2(t;(u,w),v)\rangle &= \langle \mathcal{A}'(t).\nabla(\mathcal{N}(t)^{-1}w) + \mathcal{A}(t)\nabla((\mathcal{N}(t)^{-1})'w) : \nabla v\rangle_\Omega \\ &- \langle g'(t)\mathcal{N}(t)^{-1}u, v\rangle_\Omega - \langle g(t)(\mathcal{N}(t)^{-1})'u, v\rangle_\Omega \\ &+ \langle g'(t)u_d, v\rangle_\Omega + \langle \nabla u_d\zeta, v\rangle_\Omega .\end{aligned}$$

Since the functions $(t;(u,w)) \mapsto \langle \partial_t\Psi_1(t;(u,w),v\rangle$ and $(t;(u,w)) \mapsto \langle \partial_t\Psi_2(t;(u,w),v\rangle$ are sum and compounds of continuous functions, they are continuous. Moreover, as

$$\mathcal{A}(0) = \mathcal{N}(0) = \mathbb{I}$$

so from the formulas (4.17), (4.16), (4.18) and (4.19),

$$\begin{aligned}\mathcal{A}'(0) &= div\zeta\,\mathbb{I} - 2\epsilon(\zeta) \\ (\mathcal{N}(0)^{-1})' &= D\zeta - div\zeta\,\mathbb{I} \\ g'(0) &= div\,\zeta\end{aligned}$$

thus at $t = 0$, we get

$$\begin{aligned}\langle \partial_t\Psi_1(0;u,w),v\rangle &= \langle (div\zeta\,\mathbb{I} - 2\epsilon(\zeta))\nabla u + \nabla(D\zeta - div\zeta\,\mathbb{I})u : \nabla v\rangle_\Omega \\ &+ \beta\langle D\zeta w, v\rangle_\omega \\ \langle \partial_t\Psi_2(0;u,w),v\rangle &= \langle (div\zeta\,\mathbb{I} - 2\epsilon(\zeta))\nabla w + \nabla(D\zeta - div\zeta\,\mathbb{I})w : \nabla v\rangle_\Omega \\ &- \langle D\zeta u, v\rangle_\Omega + \langle div\zeta u_d, v\rangle_\Omega + \langle \nabla u_d\zeta, v\rangle_\Omega .\end{aligned}$$

Step 2: For (ii), if u^t and w^t are weak solutions of (4.20) and (4.21), then, by construction,

$$\langle \Psi_1(t;u^t,w^t),v\rangle = 0,\ \forall v \in \mathcal{H}_0^1(\Omega)$$

and

$$\langle \Psi_2(t;u^t,w^t),v\rangle = 0,\ \forall v \in \mathcal{H}_0^1(\Omega).$$

Let us check now that the map $t \mapsto (u_t, w_t)$ is Lipschitz from $[0,\varepsilon]$ to $\mathcal{H}_0^1(\Omega)^2$.

Given t_1 and t_2 in $[0,\varepsilon]$ such that

$$a^{t_1}(u^{t_1}, w^{t_1}, v, r) = l^{t_1}(v,r) \text{ and } a^{t_2}(u^{t_2}, w^{t_2}, v, r) = l^{t_2}(v,r)$$

where a^t is a bilinear form on the space $\mathcal{H}_0^1(\Omega)^2 \times \mathcal{H}_0^1(\Omega)^2$, defined by

$$\begin{aligned}a^t(u^t, w^t, v, r) :=& \mu\langle \mathcal{A}(t)\nabla(\mathcal{N}(t)^{-1}u^t) : \nabla v\rangle_\Omega + \beta\langle g(t)\mathcal{N}(t)^{-1}w^t, v\rangle_\omega \\ &+ \mu\langle \mathcal{A}(t)\nabla(\mathcal{N}(t)^{-1}w^t) : \nabla r\rangle_\Omega - \langle g(t)\mathcal{N}(t)^{-1}u^t, r\rangle_\Omega\end{aligned}$$

and l^t is a linear form on $\mathcal{H}_0^1(\Omega)^2$, given by

$$l^t(v,r) := -\langle g(t)u_d, r\rangle_\Omega , \tag{4.22}$$

for t^1, t^2 in $[0,\varepsilon]$, we have

$$\begin{aligned}a^{t_1}(u^{t_1},w^{t_1},v,r)-a^{t_2}(u^{t_2},w^{t_2},v,r) &= a^{t_1}(u^{t_1},w^{t_1},v,r)-a^{t_1}(u^{t_2},w^{t_2},v,r)\\ &+a^{t_1}(u^{t_2},w^{t_2},v,r)-a^{t_2}(u^{t_2},w^{t_2},v,r)\\ &= l^{t_1}(v,r)-l^{t_2}(v,r).\end{aligned}$$

On the other hand, from the identities (4.17), (4.16), (4.18) and (4.19), the derivatives of the coefficients $\mathcal{A}(\cdot)$, $\mathcal{N}(\cdot)^{-1}$ and $g(\cdot)$ being continuous with respect to $t\in[0,\varepsilon]$, it results that then, they are bounded on $[0,\varepsilon]$ and, consequently, all these coefficients are Lipschitz on $[0,\varepsilon]$. Hence, there exist constants $L_i>0$ $(i=1,..5)$ such that we get the following estimates,

$$\begin{aligned}a^{t_1}(u^{t_1}-u^{t_2},w^{t_1}-w^{t_2},v,r) &= l^{t_1}(v,r)-l^{t_2}(v,r)\\ &-a^{t_1}(u^{t_2},w^{t_2},v,r)+a^{t_2}(u^{t_2},w^{t_2},v,r)\\ &=\langle (g(t_2)-g(t_1))u_d,r\rangle_\Omega\\ &+\mu\langle \mathcal{A}(t_1)\nabla(\mathcal{N}(t_1)^{-1}u^{t_2})-\mathcal{A}(t_2)\nabla(\mathcal{N}(t_2)^{-1}u^{t_2}):\nabla v\rangle_\Omega\\ &+\beta\langle g(t_1)\mathcal{N}(t_1)^{-1}w^{t_2}-g(t_2)\mathcal{N}(t_2)w^{t_2},v\rangle_\omega\\ &+\mu\langle \mathcal{A}(t_1)\nabla(\mathcal{N}(t_1)^{-1}w^{t_2})-\mathcal{A}(t_2)\nabla(\mathcal{N}(t_2)^{-1}w^{t_2}):\nabla r\rangle_\Omega\\ &+\langle g(t_1)\mathcal{N}(t_1)^{-1}u^{t_2}-g(t_2)\mathcal{N}(t_2)u^{t_2},r\rangle_\Omega\\ &\le |t_1-t_2|(L_1\|r\|+L_2\|u^{t_2}\|\|v\|+L_3\|w^{t_2}\|\|v\|\\ &+L_4\|w^{t_2}\|\|r\|+L_5\|u^{t_2}\|\|r\|).\end{aligned}\tag{4.23}$$

Now, the states u^t and w^t are bounded, indeed, going back to the initial control problem in the non perturbed domain, the coercivity of the cost functional gives

$$\mathcal{J}(\overline{u},z_0)\ge\mathcal{J}(\overline{u},z)\ge\frac{\alpha}{2}\|z\|^2$$

where $z_0\in L^2(\omega)$ and $z:=\overline{z}$ is the minimum of the functional $\mathcal{J}$. Then, z is bounded and so it is for the adjoint state $w=\beta z$. Now, as the control-state operator is a linear continuous operator, the state u is bounded too. Consequently, by introducing the *Piola* transformation, the functions u^t and w^t remain bounded. Hence, there exists a real constant $C>0$ such that for the test functions $v:=u^{t_1}-u^{t_2}$, $r:=w^{t_1}-w^{t_2}$, we obtain, owing to the coercivity of the bilinear form a^t and inequality (4.23),

$$\alpha\|(u^{t_1}-u^{t_2},w^{t_1}-w^{t_2})\|^2\le a^{t_1}(u^{t_1}-u^{t_2},w^{t_1}-w^{t_2},u^{t_1}-u^{t_2},w^{t_1}-w^{t_2})\tag{4.24}$$

$$\le 2C(\|u^{t_1}-u^{t_2}\|+\|w^{t_1}-w^{t_2}\|)|t_1-t_2|\tag{4.25}$$

which leads, after dividing by $\|(u^{t_1}-u^{t_2},w^{t_1}-w^{t_2})\|$, to

$$\|(u^{t_1}-u^{t_2},w^{t_1}-w^{t_2})\|\le C|t_1-t_2|.$$

Step 3: For the differentiability of Ψ, it is clear that the functions Ψ_1 and Ψ_2 are linear continuous with respect of u and w, and so they are differentiable with respect u and w and their partial derivatives are given by

$$\begin{aligned}
\langle \partial_u \Psi_1(t;u,w)\delta u, v\rangle &= \mu \langle \mathcal{A}(t)\nabla(\mathcal{N}(t)\delta u) : \nabla v\rangle_\Omega \\
\langle \partial_w \Psi_1(t;u,w)\delta w, v\rangle &= \beta g(t)\langle \mathcal{N}(t)^{-1}\delta w, v\rangle_\omega \\
\langle \partial_u \Psi_2(t;u,w)\delta u, v\rangle &= -g(t)\langle \mathcal{N}(t)^{-1}\delta u, v\rangle_\Omega \\
\langle \partial_w \Psi_2(t;u,w)\delta w, v\rangle &= \mu \langle \mathcal{A}(t)\nabla(\mathcal{N}(t)\delta w) : \nabla v\rangle_\Omega \ .
\end{aligned}$$

Since the vector $\zeta \in C^2(\Omega)$, the coefficients $\mathcal{A}(t)$, $\mathcal{N}(t)^{-1}$ and $g(t)$ are continuous, then all the partial derivatives are continuous. The derivative of Ψ with respect to u and w is given by

$$(\partial_u \Psi_1(t;u,w)\delta u + \partial_w \Psi_1(t;u,w)\delta w, \partial_u \Psi_2(t;u,w)\delta u + \partial_w \Psi_2(t;u,w)\delta w).$$

Step 4: To show the last condition of the implicit function theorem, we have to prove existence and uniqueness of solution for the coupled system

$$\begin{cases} \partial_u \Psi_1(0;u,w) + \partial_w \Psi_1(0;u,w) = \kappa \\ \partial_w \Psi_2(0;u,w) + \partial_u \Psi_2(0;u,w) = \varsigma \end{cases} \tag{4.26}$$

for all $(\kappa, \varsigma) \in \mathcal{H}_0^1(\Omega)' \times \mathcal{H}_0^1(\Omega)'$. Since at $t = 0$, we have

$$\begin{aligned}
\langle \partial_u \Psi_1(0;u,w)\delta u, v\rangle &= \mu \langle \nabla \delta u : \nabla v\rangle_\Omega \\
\langle \partial_w \Psi_1(0;u,w)\delta w, v\rangle &= \beta \langle \delta w, v\rangle_\omega \\
\langle \partial_u \Psi_2(0;u,w)\delta u, v\rangle &= -\langle \delta u, v\rangle_\Omega \\
\langle \partial_w \Psi_2(0;u,w)\delta w, v\rangle &= \mu \langle \nabla \delta w : \nabla v\rangle_\Omega
\end{aligned}$$

the system (4.26) in the weak sense can be written as:

$$\begin{cases} \mu \langle \nabla \delta u : \nabla v\rangle_\Omega - \beta \langle \delta w, v\rangle_\omega = \langle \kappa, v\rangle_\Omega \\ \mu \langle \nabla \delta w : \nabla r\rangle_\Omega + \langle \delta u, r\rangle_\Omega = \langle \varsigma, r\rangle_\Omega \end{cases} .$$

This system can be reduced to the following form: Find $(\delta u, \delta w) \in [\mathcal{H}_0^1(\Omega)]^2$ such that

$$\mathcal{B}(\delta u, \delta w, v, r) = \mathcal{F}(v, r), \ \ \forall (v.r) \in [\mathcal{H}_0^1(\Omega)]^2$$

with

$$\begin{aligned}
\mathcal{B}(\delta u, \delta w, v, r) := &\ \mu \langle \nabla \delta u : \nabla v\rangle_\Omega - \beta \langle \delta w, v\rangle_\omega \\
&+ \mu \langle \nabla \delta w : \nabla r\rangle_\Omega + \langle \delta u, r\rangle_\Omega
\end{aligned}$$

and

$$\mathcal{F}(v, r) := \langle \kappa, v\rangle_\Omega + \langle \varsigma, r\rangle_\Omega ,$$

so that, by the Lax-Milgram theorem, we get existence and uniqueness for this system.

Finally, all conditions of Theorem 1 being satisfied, we conclude that the *Piola* material derivatives $\dot{u}_P$, $\dot{w}_P$ exist and they are solutions of the linearized system:

$$\begin{aligned}&\langle \partial_u \Psi_1(0; u, w)\dot{u}_P, v\rangle + \langle \partial_w \Psi_1(0; u, w)\dot{w}_P, v\rangle + \langle \partial_t \Psi_1(0; u, w), v\rangle = 0\\&\langle \partial_u \Psi_2(0; u, w)\dot{u}_P, v\rangle + \langle \partial_w \Psi_2(0; u, w)\dot{w}_P, v\rangle + \langle \partial_t \Psi_2(0; u, w), v\rangle = 0\end{aligned}$$

so that the equations (4.20) and (4.21) hold. ■

5. Shape derivative

In this section, we are interested in the shape derivatives of u_t and w_t, which are defined by the derivatives of the mappings

$$\begin{aligned}&t \mapsto u_t(x + t\zeta(x)), \ \ x \in \Omega\\&t \mapsto w_t(x + t\zeta(x)), \ \ x \in \Omega.\end{aligned}$$

The shape derivatives associated to the optimality system (3.8), are denoted by u' and w'. Formally, the equations for the shape derivatives are derived by an application of the Reynold's transport theorem to the variational formulation of the model in variable domain. It is well known in shape sensitivity analysis that they are related to the classical material derivatives $\dot{u}$ and $\dot{w}$ in the following manner

$$u' = \dot{u} - \nabla u\zeta \tag{5.27}$$

$$w' = \dot{w} - \nabla w\zeta. \tag{5.28}$$

Since we are working in our case with the transformation of *Piola*, which conserves the free divergence condition, we have to establish a relation that expresses the shape derivatives u', w' in terms of the *Piola* material derivatives $\dot{u}_P$, $\dot{w}_P$, instead of $\dot{u}$, $\dot{w}$. The existence of the *Piola* material derivative implies the existence of the shape derivative, provided that the solution (u, w) is sufficiently regular.

From the relation

$$u_t(x + t\zeta) = \mathcal{N}(t)^{-1}u^t(x)$$

we get

$$\dot{u} = \dot{u}_P + (\mathcal{N}(0)^{-1})'u = \dot{u}_P + D\zeta u - div\zeta u \tag{5.29}$$

$$\dot{w} = \dot{w}_P + D\zeta u - div\zeta w, \tag{5.30}$$

and from (5.27) and (5.28), we deduce the following relations between u' and $\dot{u}_P$ (respectively w' and $\dot{w}_P$):

$$u' = \dot{u}_P - u\, div\zeta - \nabla u\zeta + D\zeta u \tag{5.31}$$

$$w' = \dot{w}_P - w\, div\zeta - \nabla w\zeta + D\zeta w. \tag{5.32}$$

We deduce from (5.31-5.32) that the shape derivatives lose the regularity compared to the material derivatives.

Remark 4 *From relations (5.31-5.32) it follows that in general, the shape derivatives are not divergence free, therefore the standard use of such derivatives is rather impossible for numerical methods.*

Lemma 6 *Let ϕ, v and T be smooth fields with ϕ scalar valued, v vector valued and T tensor valued, then we have*

$$div(\phi v) = \phi div v + v\nabla\phi \tag{5.33}$$

$$div(Tv) = v div(T^T) + T : \nabla v \tag{5.34}$$

$$div[(\nabla v)^T] = \nabla(div v). \tag{5.35}$$

Proof. See Gurtin (1981). ■

Using the identities (5.33), (5.34), (5.35) and the fact that we have $div\, \dot{u}_P = div\, \dot{w}_P = 0$, we get

$$\begin{aligned} div u' &= div\, \dot{u}_P - div\,(u\, div\zeta) - div(\nabla u\zeta) + div(D\zeta u) \\ &= 0 - div\zeta div\, u - u\nabla(div\zeta) - \zeta div(Du) - \nabla u : \nabla\zeta \\ &+ u\, div(\nabla\zeta) + D\zeta : \nabla u \\ &= -u\nabla(div\zeta) - \zeta div(Du) - \nabla u : \nabla\zeta + u\, div(\nabla\zeta) + D\zeta : \nabla u, \end{aligned} \tag{5.36}$$

and the same for $div w'$, namely we have

$$div\, w' = -w\nabla(div\zeta) - \zeta div(Dw) - \nabla w : \nabla\zeta + w\, div(\nabla\zeta) + D\zeta : \nabla w. \tag{5.37}$$

We denote $div\, u' := U$ and $div\, w' := W$, and then it follows from the above relations that the linearized system for shape derivatives is given by

$$\left\{ \begin{array}{rcll} -\mu\Delta u' + \nabla p' &=& -\beta w'\chi_\omega & \text{in } \Omega \\ div\, u' &=& U & \text{in } \Omega \\ -\mu\Delta w' + \nabla q' &=& u' & \text{in } \Omega \\ div\, w' &=& W & \text{in } \Omega \\ u' &=& -(Du.n)\zeta_n & \text{on } \Gamma \\ w' &=& -(Dw.n)\zeta_n & \text{on } \Gamma. \end{array} \right. \tag{5.38}$$

6. Shape gradient of the shape functional $\mathcal{J}$

This section is devoted to the identification of the shape gradients for the cost functional $\mathcal{J}(\Omega)$. It is well know that the expression of the shape derivative denoted by $d\mathcal{J}(\Omega;\zeta) = d\mathcal{J}(\Omega)[\zeta]$ can be obtained in terms of the material derivatives or the shape derivatives, note also that by the adjoint state formalism we can express the shape gradient as a density in the moving boundary, see for example, Plotnikov and Sokolowski (2012), Sokolowski and Zolesio (1992), and Zhu and Gao (2019). By a change of variable, the cost functional in the fixed domain Ω is given by

$$\mathcal{J}(\Omega_t) = \frac{1}{2}\int_\Omega \|u_t \circ T_t - u_d \circ T_t\|^2 g(t) + \frac{\beta}{2}\int_\omega \|w_t \circ T_t\|^2 g(t).$$

We will introduce a second level adjoint system in the weak form, which leads, with the material derivatives $\dot{u}$, $\dot{w}$, to the shape gradient of $\mathcal{J}$ in volume or interface expression. Let us start by defining the functional Lagrangian $\mathcal{L}$ by:

$$\mathcal{L}(u,w,\pi,\varrho) := \mathcal{J}(u,w)+\mu\langle\nabla u : \nabla\pi\rangle_\Omega+\beta\langle\, w,\pi\rangle_\omega+\mu\langle\nabla w : \nabla\varrho\rangle_\Omega-\langle u-u_d,\varrho\rangle_\Omega.$$

We can easily calculate the differential with respect to the state variables u, w, more precisely, we have

$$\begin{aligned}\nabla_u\mathcal{L}(u,w,\pi,\varrho)(\vartheta) &= \langle u-u_d,\vartheta\rangle_\Omega + \mu\langle\nabla\pi : \nabla\vartheta\rangle_\Omega - \langle\varrho,\vartheta\rangle_\Omega\\ \nabla_w\mathcal{L}(u,w,\pi,\varrho)(\upsilon) &= \beta\langle w,\upsilon\rangle_\omega + \beta\langle\pi,\upsilon\rangle_\omega + \mu\langle\nabla\varrho : \nabla\upsilon\rangle_\Omega.\end{aligned}$$

Then, the coupled adjoint system in the weak form can be defined as follows:

Find a couple $(\pi,\varrho) \in [\mathcal{H}_0^1(\Omega)]^2$ such that

$$\begin{cases} \mu\langle\nabla\pi : \nabla\vartheta\rangle_\Omega - \langle\varrho,\vartheta\rangle_\Omega = -\langle u-u_d,\vartheta\rangle_\Omega \\ \mu\langle\nabla\varrho : \nabla\upsilon\rangle_\Omega + \beta\langle\pi,\upsilon\rangle_\omega = -\beta\langle w,\upsilon\rangle_\omega. \end{cases} \tag{6.39}$$

The second main result of this paper is constituted by the volume and interface expressions of the shape gradient .

Theorem 3 *We assume that the assumption **H-1** is satisfied and (π,ϱ) is a solution to the adjoint state system (6.39), then the shape gradient of the functional $\mathcal{J}$ in the direction ζ is given by*

$$\begin{aligned} d\mathcal{J}(\Omega)[\zeta] = \frac{1}{2}&\left(\int_\Omega div\zeta\|u-u_d\|^2 + \beta\int_\omega div\zeta\|w\|^2\right) - \langle u-u_d,\nabla u_d\zeta\,\rangle_\Omega \\ &+ \mu\langle\mathcal{A}'(0)\nabla u : \nabla\pi\rangle_\Omega + \beta\langle div\zeta w,\pi\rangle_\omega \\ &+ \mu\langle\mathcal{A}'(0)\nabla w : \nabla\varrho\rangle_\Omega - \langle div\zeta(u-u_d) - \nabla u_d\zeta,\varrho\rangle_\Omega. \end{aligned} \tag{6.40}$$

Proof. Using the formula (4.16) and the material derivatives of u, w, we deduce that the shape derivative of the quadratic functional $\mathcal{J}$ in terms of $\dot{u}$, $\dot{w}$ is given by

$$
\begin{aligned}
d\mathcal{J}(\Omega)[\zeta] = \frac{1}{2}\left(\int_\Omega div\zeta\|u-u_d\|^2 + \beta\int_\omega div\zeta\|w\|^2\right) \\
+ \langle u-u_d, \dot{u} - \nabla u_d\zeta\rangle_\Omega + \beta\langle w, \dot{w}\rangle_\omega .
\end{aligned}
$$

If we test the adjoint problem with the material derivatives $\dot{u}$, $\dot{w}$ and use the variational formulations (6.39), we find

$$
\begin{aligned}
d\mathcal{J}(\Omega)[\zeta] = \frac{1}{2}\left(\int_\Omega div\zeta\|u-u_d\|^2 + \beta\int_\omega div\zeta\|w\|^2\right) - \langle u-u_d, \nabla u_d\zeta\rangle_\Omega \\
- \mu\langle\nabla\pi : \nabla\dot{u}\rangle_\Omega + \langle\varrho, \dot{u}\rangle_\Omega - \mu\langle\nabla\varrho : \nabla\dot{w}\rangle_\Omega - \beta\langle\pi, \dot{w}\rangle_\omega .
\end{aligned}
$$

The classical material derivatives can be expressed in terms of *Piola* material derivatives $\dot{u}_P$, $\dot{w}_P$, using the formulas (5.29), (5.30), and then we can use the system (4.20), (4.21), which characterizes $\dot{u}_P$ and $\dot{w}_P$, to get the last formula (6.40). More precisely, we have

$$
\begin{aligned}
d\mathcal{J}(\Omega)[\zeta] &= \frac{1}{2}\left(\int_\Omega div\zeta\|u-u_d\|^2 + \beta\int_\omega div\zeta\|w\|^2\right) - \langle u-u_d, \nabla u_d\zeta\rangle_\Omega \\
&- \mu\langle\nabla\pi : \nabla\dot{u}_P + \nabla((D\zeta - div\zeta\mathbb{I})u)\rangle_\Omega + \langle\varrho, \dot{u}_P + (D\zeta - div\zeta\mathbb{I})u\rangle_\Omega \\
&- \mu\langle\nabla\varrho : \nabla\dot{w}_P + \nabla((D\zeta - div\zeta\mathbb{I})w)\rangle_\Omega - \beta\langle\pi, \dot{w}_P + (D\zeta - div\zeta\mathbb{I})w)\rangle_\omega \\
&= \frac{1}{2}\left(\int_\Omega div\zeta\|u-u_d\|^2 + \beta\int_\omega div\zeta\|w\|^2\right) - \langle u-u_d, \nabla u_d\zeta\rangle_\Omega \\
&+ \mu\langle\mathcal{A}'(0)\nabla u : \nabla\pi\rangle_\Omega + \beta\langle div\zeta w, \pi\rangle_\omega \\
&+ \mu\langle\mathcal{A}'(0)\nabla w : \nabla\varrho\rangle_\Omega - \langle div\zeta(u-u_d), \varrho\rangle_\Omega + \langle\nabla u_d\zeta, \varrho\rangle_\Omega .
\end{aligned}
$$

■

Note that for the volume expression of the shape gradient, we have not used the fact that either the states functions u, w or adjoint states functions π, ϱ are more regular than $\mathcal{H}_0^1(\Omega)$. The next result shows that under additional regularity assumptions, the expression of $d\mathcal{J}$ can be formulated according to the structure theorem, see Sokolowski and Zolesio (1992).

THEOREM 4 *Let u, w, π, $\varrho \in \mathcal{H}_0^1(\Omega) \cap [H^2(\Omega)]^2$, suppose that ζ satisfies the assumption **H-1**, then the shape gradient of $\mathcal{J}$ can be equivalently represented by*

$$
d\mathcal{J}(\Omega)[\zeta] = \int_{\Gamma_\omega}\left[\frac{1}{2}(\|u-u_d\|^2 + \beta\|w\|^2) - \mu(D\pi.n, Du.n) - \mu(D\varrho.n, Dw.n)\right]\zeta_n . \tag{6.41}
$$

Proof. At first we observe that we have

$$\frac{1}{2}div(\zeta\|u-u_d\|^2) = \frac{1}{2}div\zeta\|u-u_d\|^2 + (u-u_d)\nabla u\zeta - (u-u_d)\nabla u_d\zeta. \quad (6.42)$$

We need also to employ the following identity to simplify the terms that depend on the coefficient $\mathcal{A}'(0)$ (see Berggren, 2010)

$$\zeta\left[\nabla(\nabla r\nabla s)\right] + \left[2\epsilon(\zeta)\nabla r\right]\nabla s = \left[\nabla(\zeta\nabla r)\right]\nabla s + \left[\nabla(\zeta\nabla s)\right]\nabla r \quad (6.43)$$

for any vector ζ and scalar quantities r and s. Since the functions u, w, π and ϱ are in the space $[H^2(\Omega)]^2$, we can use (6.43) for each component u_i, π_i of u and π (for w and ϱ the steps are exactly the same),

$$\begin{aligned}\langle\nabla u_i, 2\epsilon(\zeta)\nabla\pi_i\rangle_\Omega &= -\langle\nabla(\nabla u_i\nabla\pi_i),\zeta\rangle_\Omega + \langle\nabla(\zeta\nabla u_i),\nabla\pi_i\rangle_\Omega \\ &+ \langle\nabla u_i,\nabla(\zeta\nabla\pi_i)\rangle_\Omega.\end{aligned}$$

Using this identity and the Green's formula, we get

$$\begin{aligned}\mu\langle\mathcal{A}'(0)\nabla u:\nabla\pi\rangle_\Omega &= \mu\langle(div\zeta\mathbb{I} - 2\epsilon(\zeta))\nabla u:\nabla\pi\rangle_\Omega \\ &= \mu\sum_{i=1}^{3}\Big[\langle div\zeta\nabla u_i,\nabla\pi_i\rangle_\Omega + \langle\nabla(\nabla u_i\nabla\pi_i),\zeta\rangle_\Omega - \langle\nabla(\zeta\nabla u_i),\nabla\pi_i\rangle_\Omega \\ &- \langle\nabla u_i,\nabla(\zeta\nabla\pi_i)\rangle_\Omega\Big] \\ &= \mu\sum_{i=1}^{3}\Big[\langle\nabla u_i,\nabla\pi_i\zeta_n\rangle_\Gamma + \langle\Delta\pi_i,\nabla u_i\zeta\rangle_\Omega + \langle\Delta u_i,\nabla\pi_i\zeta\rangle_\Omega \\ &- \langle\frac{\partial\pi_i}{\partial n},\nabla u_i\zeta\rangle_\Gamma - \langle\frac{\partial u_i}{\partial n},\nabla\pi_i\zeta\rangle_\Gamma\Big].\end{aligned}$$

Note that the tangential gradient ∇_Γ on the boundary is defined by

$$\nabla_\Gamma u_i := \nabla u_i - \frac{\partial u_i}{\partial n}n, \quad \text{for } i = 1,...,3$$

since we have the homogeneous Dirichlet conditions on the boundary, the tangential gradients of u and π vanish on Γ (see Sokolowski and Zolesio, 1992), and so we have

$$\begin{aligned}\mu\langle\mathcal{A}'(0)\nabla u:\nabla\pi\rangle_\Omega &= \mu\sum_{i=1}^{3}\Big[\langle\frac{\partial u_i}{\partial n},\frac{\partial\pi_i}{\partial n}\zeta_n\rangle_\Gamma + \langle\Delta\pi_i,\nabla u_i\zeta\rangle_\Omega + \langle\Delta u_i,\nabla\pi_i\zeta\rangle_\Omega \\ &- \langle\frac{\partial\pi_i}{\partial n},\frac{\partial u_i}{\partial n}\zeta_n\rangle_\Gamma - \langle\frac{\partial u_i}{\partial n},\frac{\partial\pi_i}{\partial n}\zeta_n\rangle_\Gamma\Big] \\ &= \mu\langle\Delta\pi,\nabla u\zeta\rangle_\Omega + \mu\langle\Delta u,\nabla\pi\zeta\rangle_\Omega - \mu\langle(Du.n),(D\pi.n)\zeta_n\rangle_\Gamma\ .\end{aligned}$$

Now, returning to the volume expression of $d\mathcal{J}(\Omega)$, using (6.42) and replacing the terms $\langle\mathcal{A}'(0)\nabla u:\nabla\pi\rangle_{\Omega}$, $\langle\mathcal{A}'(0)\nabla w:\nabla\varrho\rangle_{\Omega}$, we find

$$
\begin{aligned}
d\mathcal{J}(\Omega)[\zeta] = &\frac{1}{2}\left(\int_{\Omega} div(\zeta\|u-u_d\|^2)+\beta\int_{\omega} div(\zeta\|w\|^2)\right)-\langle(u-u_d),\nabla u\zeta\rangle_{\Omega}\\
&-\beta\langle w,\nabla w\zeta\rangle_{\omega}-\mu\langle(Du.n),(D\pi.n)\zeta_n\rangle_{\Gamma}-\mu\langle(Dw.n),(D\varrho.n)\zeta_n\rangle_{\Gamma}\\
&+\langle\nabla p+\beta w\chi_{\omega},\nabla\pi\zeta\rangle_{\Omega}+\langle\nabla\overline{p}-\varrho+u-u_d,\nabla u\zeta\rangle_{\Omega}\\
&+\langle\nabla q-(u-u_d),\nabla\varrho\zeta\rangle_{\Omega}+\langle\nabla\overline{q}+(\pi+w)\beta\chi_{\omega},\nabla w\zeta\rangle_{\Omega}\\
&+\beta\langle div\zeta w,\pi\rangle_{\omega}-\langle div\zeta(u-u_d),\varrho\rangle_{\Omega}+\langle\nabla u_d\zeta,\varrho\rangle_{\Omega} \qquad (6.44)
\end{aligned}
$$

where $\overline{p}$ and $\overline{q}$ are the adjoint pressures in the system (6.39) in the strong form. Note that we have

$$
\begin{aligned}
&-\langle div\zeta u,\varrho\rangle_{\Omega}-\langle u,\nabla\varrho\zeta\rangle_{\Omega}-\langle\varrho,\nabla u\zeta\rangle_{\Omega}=-\int_{\Omega} div((\varrho u)\zeta)=-\int_{\Gamma}(\varrho u)\zeta_n=0\\
&+\langle div\zeta w,\pi\rangle_{\omega}+\langle w,\nabla\pi\zeta\rangle_{\omega}+\langle\pi,\nabla w\zeta\rangle_{\omega}=\int_{\Gamma}(\pi w)\zeta_n=0\\
&+\langle div\zeta u_d,\varrho\rangle_{\Omega}+\langle u_d,\nabla\varrho\zeta\rangle_{\Omega}+\langle\varrho,\nabla u_d\zeta\rangle_{\Omega}=\int_{\Gamma}(\varrho u_d)\zeta_n=0
\end{aligned}
$$

which is due to the homogeneous Dirichlet conditions on the boundary for the state and adjoint state functions. For terms that depend on pressures, we have free divergence condition for the state (u,w) and the adjoint state (π,ϱ), and then, by the Green's formula, we deduce that all pressure terms in the expansion (6.44) are zero. Finally, the boundary terms that remain in (6.44) represent the shape gradient of the functional $\mathcal{J}$, the assumption **H-1** implies that the vector field ζ is tangential to $\Gamma\setminus\Gamma_{\omega}$, then $d\mathcal{J}(\Omega)$ is supported on Γ_{ω}. ■

7. Numerical results

For the numerical computations we suppose that Ω is a square domain $(-3,3)\times(-3,3)$ located at the origin in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. We suppose that the source term z is the function from $L^2(\Omega)$, which controls the system (2.1) and is defined in a subdomain $\omega\subset\Omega$ with its boundary $\partial\omega$. We take ω as a circular subdomain $\omega\subset\Omega$, as is presented in Fig.3, with the center (a,b) located at some point of the boundary Γ and with radius $R=0.5$, 1 or 2, respectively, in different examples. In such subdomain ω the control z is defined as a vector to be optimized and it acts on Ω.

The numerical method has two parts: • finding of an optimal control z of the system, • finding an optimal position of the control region.

7.1. Optimal control

In this section we compute the optimal control z, acting from a fixed subdomain $\omega\subset\Omega$. We start the computations with an initial value of control $z=[z_1,z_2]$

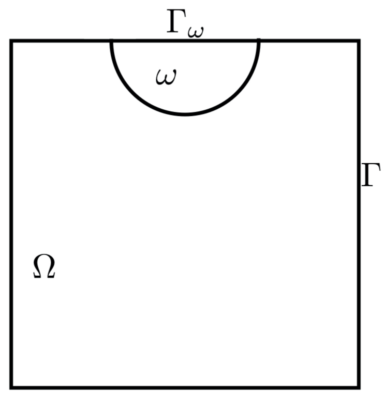

Figure 3. The domain in numerical experiments

since control must be a vector. We set also an initial value for $\alpha \in (0,1)$, which stands for the importance of the second term in the objective function $\mathcal{J}$. We choose a reference function u_d. First, we solve the Stokes problem (2.1) and the corresponding adjoint problem (2.4). In this way, we get the solution u, directly inserted into the shape functional $\mathcal{J}$, and the solution w of the adjoint problem. Due to the solution w we adapt the control $z = -\frac{1}{\alpha}w$, which we insert into the functional. Now we are able to compute the value of $\mathcal{J}$. The above procedure is repeated until the changes of values of $\mathcal{J}$ remain small enough.

To check this, we compare the last two values of the functional and if the difference is smaller than some ε, we stop the computations. In this way we obtain the control z, which minimizes the shape functional $\mathcal{J}$, given by (2.2). This control z remains optimal in the next part of the algorithm.

7.2. Optimal shape of subregion ω obtained by changing its location

In this part we search for an optimal position of the control region ω by moving its position along the boundary $\partial\Omega$. In such an approach there is no need to use the shape derivative since the boundary of the subregion ω does not change the shape. The optimal shape is determined through the minimization of the functional $\mathcal{J}(\Omega_t)$, defined in perturbed domain and depending on the solutions to the perturbed system (3.8). The procedure, constructed for finding an optimal shape, is the following: In each iteration we fix the position of the subregion ω and for this given position we solve the appropriate problems – the Stokes equation (2.1) and the adjoint equation (2.4) – and compute the current value of the shape functional $\mathcal{J}$. Then we move the position of ω along the boundary Γ and we pass to the next iteration. In this way we collect in the matrix of data the values of the functional $\mathcal{J}$ and the corresponding positions of subdomain ω. Finally, it remains to choose the minimal values of the functional and thus we find the optimal position of the subregion ω. All computations were made in the FreeFem++ environment, combined with Matlab programming tools.

7.2.1. Example 1

We suppose that $\alpha = 0.4$, we set the initial control $z = [-5, 0]$ and the reference function $u_d = [1, 0]$. We place the control subdomain at the point $(0, -3)$ and set its radius to $R = 1$. The optimal control is obtained in about third iteration of minimization of the functional $\mathcal{J}$, see Fig. 4. Next, we move the control

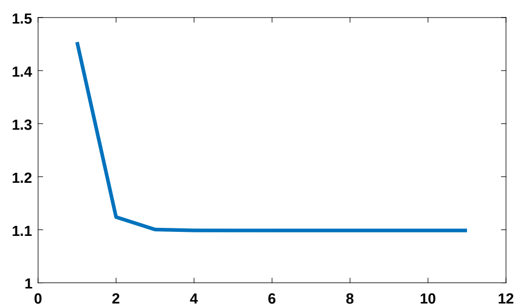

Figure 4. Values of the functional $\mathcal{J}$ during the minimization process of finding the optimal control z

region along the boundary of the domain Ω in order to find the optimal shape of this domain. In this case the optimal position of circular control subdomain is at point $(3, 0)$ with the minimal value of the functional $\mathcal{J}$. Figure 5 presents the solution u to the Stokes equation in the optimal domain and the pressure p in this domain.

7.2.2. Example 2

In this example, we choose the reference function $u_d = [\sin(x + y), -\sin(x + y)]$ while we set the same initial control region and we keep the same initial control as previously. The optimal control is obtained in few iterations of minimization of the functional $\mathcal{J}$, see Fig. 6.

After moving the control region along the boundary of the domain Ω in order to find the optimal shape of this domain, we find the optimal position at point $(2, 3)$ with the minimal value of the functional $\mathcal{J}$. Figure 7 presents the solution (u, p) to the Stokes equation in the optimal domain.

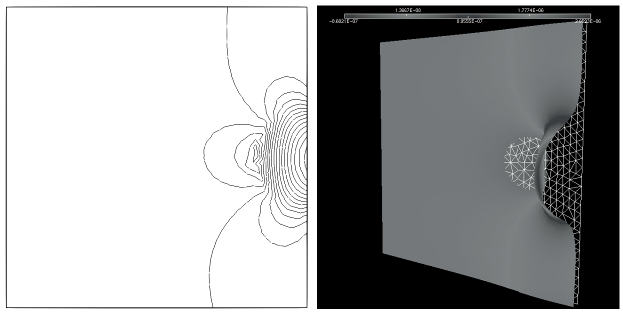

Figure 5. Optimal solution to the shape optimization problem for the reference function $u_d = [1, 0]$, a) stream lines for the solution to the Stokes equation in optimal domain, b) pressure p for the corresponding optimal domain

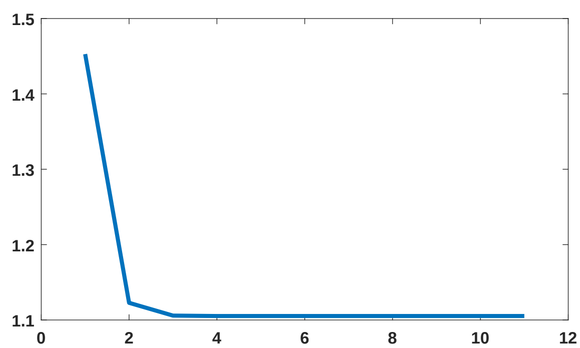

Figure 6. Values of the functional $\mathcal{J}$ during the minimization process of finding the optimal control z

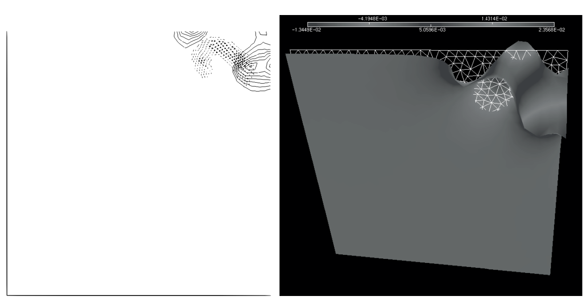

Figure 7. Optimal solution to the shape optimization problem for the reference function $u_d = [\sin(x + y), -\sin(x + y)]$, a) stream lines for the solution to the Stokes equation in optimal domain, b) pressure p for the corresponding optimal domain

7.2.3. Example 3

In this case, we choose the constant reference function $u_d = (1, 0)$ and we set the constant initial control $z = (1, 1)$, the control region ω is now placed at point $(0, 3)$ and its radius is $R = 2$. The minimization of the functional $\mathcal{J}$ that gives the optimal control is presented in Fig. 8. After moving the control region

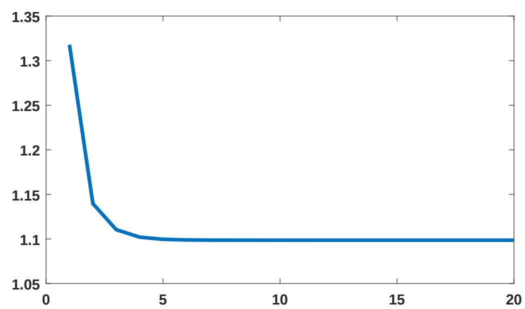

Figure 8. Values of the functional $\mathcal{J}$ during the minimization process of finding the optimal control z

along the boundary of the domain Ω in order to find the optimal shape of this domain, we find the optimal position at point $(2, -3)$ with the minimal value of the functional $\mathcal{J}$. Figure 9 presents the solution (u, p) to the Stokes equation in the optimal domain.

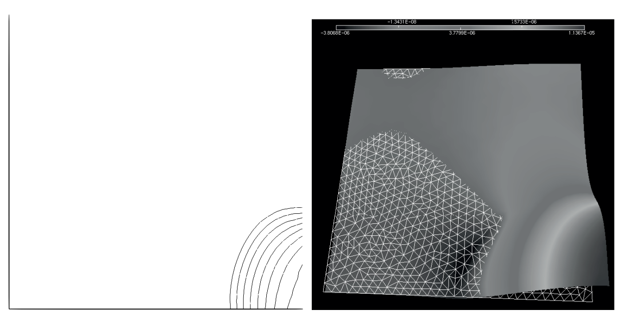

Figure 9. Optimal solution to shape optimization problem for the reference function $u_d = [1, 0]$, a) stream lines for the solution to the Stokes equation in the optimal domain, b) pressure p for the corresponding optimal domain

7.3. Optimal shape of subregion ω obtained by changing its boundary

In this section we search for an optimal shape of the region ω by moving its boundary Σ. The perturbation of ω is such that the functional $\mathcal{J}$ is minimized. Changes of the boundary Σ are obtained by the smoothest possible transformation described below.

Let us suppose that the displacement in Ω is denoted by $u = u(x)$. For the numerical experiments we limit ourselves to the 2d case. We introduce the transformation $\Phi : \omega \to \omega(u)$ such that

$$\Phi(u) = x + \phi(x) = x + \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} \tag{7.45}$$

where for $i = 1, 2$ the function $\phi_i(u_i)(x)$ is a harmonic extension of the boundary data u_i, see Lasiecka, Szulc and Zochowski (2018).

$$\begin{aligned} \Delta\phi_i = &\quad 0 \text{ in } \quad \Omega \\ \phi_i = &\quad u_i \text{ on } \quad \Sigma \\ \phi_i = &\quad 0 \text{ on } \quad \partial\Omega \setminus \Sigma \end{aligned} \tag{7.46}$$

The so defined transformation "lifts" the boundary trace of u from the boundary Σ into $\Omega(u) = \Phi(\Omega)$.

Examples

We set $\alpha = 0.4$, the initial control $z = (-5, 0)$ and the reference function $u_d = (1, 0)$. We suppose that the subdomain ω has a circular shape and its center is located at points $(0, 3), (0, -3), (3, 0)$, i.e. at the upper, lower or right boundary of Ω, respectively; the radius of the circle is equal to 2. Similarly to the previous examples, first we find the optimal control which minimises the shape functional $\mathcal{J}$, the procedure is the same as before and as was described at the beginning of this chapter. Next we minimise the functional $\mathcal{J}$ according to the perturbation of the subdomain ω. The results are presented in Figs. 10, 11 and 12.

8. Conclusion

In this paper we have derived shape sensitivities for the control problem of the Stokes systems. The results can be easily extended to other situations, for example, the sensitivity of an obstacle in the interior of the domain, the case of the linearized Navier Stokes system. However, the case of the constraints on the control, which brings us to a system of nonlinear optimality because of the presence of the projection operator, remains always in question. The topological sensitivity is beyond the scope of this paper and will be the subject for a forthcoming publication.

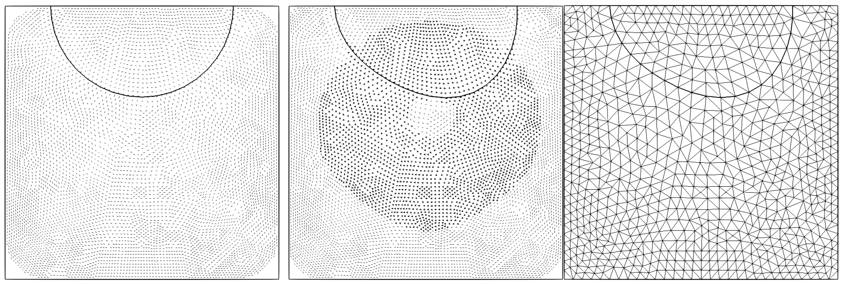

Figure 10. Optimal solution to the shape optimisation problem for the reference function $u_d = (1,0)$, a) initial domain Ω with circular subregion ω located at point $(0,3)$ of the upper boundary, b) optimal perturbation of ω, c) visualisation of triangulation

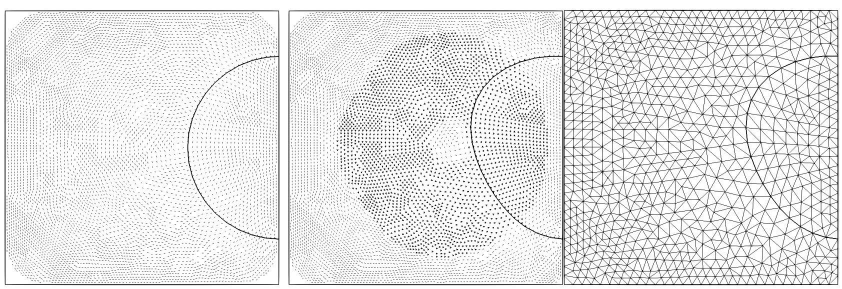

Figure 11. Optimal solution to the shape optimisation problem for the reference function $u_d = (1,0)$, a) initial domain Ω with circular subregion ω located at point $(3,0)$ of the right boundary, b) optimal perturbation of ω, c) visualisation of triangulation

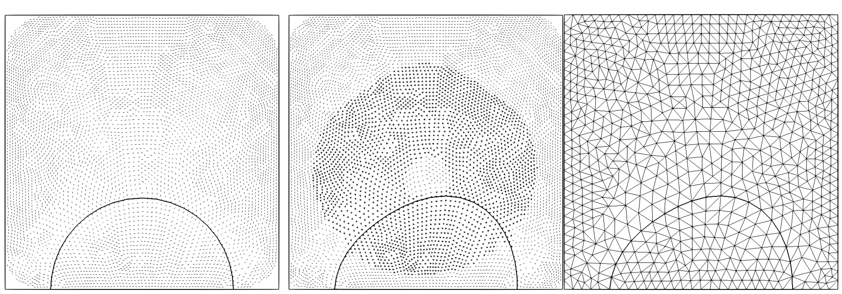

Figure 12. Optimal solution to shape optimisation problem for the reference function $u_d = (1,0)$, a) initial domain Ω with circular subregion ω located at point $(0,-3)$ of the lower boundary, b) optimal perturbation of ω, c) visualisation of triangulation

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References

Abdelwahab, M. and Hassine, M. (2009) Topological optimization method for a geometric control problem in Stokes flow. *Applied Numerical Mathematics* **59**. Elsevier, 1823-1838.

Allaire, G. (2007) *Conception optimale de structures. Mathématiques & Applications (Berlin) [Mathematics & Applications]*, **58**, Springer-Verlag, Berlin.

Ammari, H. (2002) An inverse initial boundary value problem for the wave equation in the presence of imperfections of small volume. *SIAM J. Control Optim.* 41, 1194-1211.

Amstutz, S. (2005) The topological asymptotic for the Navier-Stokes equations. *ESAIM: COCV*, **11**, 401-425, .

Berggren, M. (2010) A unified discrete-continuous sensitivity analysis method for shape optimization. In: W. Fitzgibbon et al. (eds.), *Applied and Numerical Partial Differential Equations. Computational Methods in Applied Sciences*, **15**, Springer, 2539.

Boisgérault, S. (2000) Shape Optimisation: Nonlinear Systems and Fluid Mechanics. PhD Thesis, Ecole des Mines de Paris https://eul.ink/shape-optimization/

Boyer, F. and Fabrie, P. (2006) *Eléments d'analyse pour l'étude de quelques modèles d'écoulements de fluides visqueux incompressibles, Mathématiques & Applications (Berlin) [Mathematics & Applications]*, **52**, Springer-Verlag, Berlin.

Caubet, F. and Dambrine, M. (2012) Localization of small obstacles in Stokes flow. *Inverse Problems*, **28**(10), 105007, 31.

Consiglieri, L., Nečasovà, Š. and Sokolowski, J. (2010) New approach to the incompressible Maxwell-Boussinesq approximation: Existence, uniqueness and shape sensitivity. *J. Differential Equations* 249, 3052-3080.

Delfour, M. C. and Zolesio, J. P. (2012) *Shapes and Geometries*, second ed., *Advances in Design and Control*, **22**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011, *Metrics, Analysis, Differential Calculus, and Optimization.* MR 2731611.

Fremiot, G. (2000) Structure de la semi-dérivée eulérienne dans le cas de domaines fissurés et quelques applications. PhD Thesis of University Henri Poincaré-Nancy 1.

Fursikov, A. V. (1999) *Optimal Control of Distributed Systems. Theory and application.* AMS, Providence.

GALDI, G.P. (1994) *An Introduction to the Mathematical Theory of the Navier-Stokes Equations.* Vol. I and II, *Springer Tracts in Natural Philosophy*, **38** and **39**. Springer-Verlag, New York.

GUILLAUME, PH. AND HASSINE, M. (2008) Removing holes in topological shape optimization. *ESAIM: COCV*, **14**, 1, 160-191.

GURTIN, M. E. (1981) *An Introduction to Continuum Mechanics. Mathematics in Science and Engineering*, **158**. Academic Press, New York.

HADAMARD, J. (1907) Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Bull. Soc. Math.* France.

HENROT, A. AND PIERRE, M. (2005) *Variation et optimisation de formes. Mathématiques et Applications*, **48**, Springer-Verlag, Berlin.

LASIECKA, I. (2002)*Mathematical Control Theory of Coupled PDEs.* SIAM, Philadelphia.

LASIECKA, I., SZULC, K. AND ZOCHOWSKI, A. (2018) Boundary control of small solutions to fluid structure interactions arising in coupling of elasticity with Navier stokes equation under mixed boundary conditions. *Nonlinear Analysis: Real World Applications*, **44**, 54-85.

LAURAIN, A. (2006) Domaines singulièrement perturbés en optimisation de formes. PhD Thesis, Nancy; https://tel.archives-ouvertes.fr/tel-01748151 v2/document

LEUGERING, G., NOVOTNY, A. A., PERLA MENZALA, G. AND SOKOŁOWSKI, J. (2011) On shape optimization for an evolution coupled system. *Appl. Math. Optim.* 64:441-466.

LIONS, J.L. (1971) *Optimal Control of Systems Governed by Partial Differential Equations.* Springer, Berlin.

MOHAMMADI B. AND PIRONNEAU, O. (2010) *Applied Shape Optimization for Fluids*, second ed. *Numerical Mathematics and Scientific Computation*, Oxford University Press, Oxford. MR 2567067.

MOUBACHIR, M. AND ZOLESIO, J.P. (2006) *Moving Shape Analysis and Control Applications to Fluid Structure Interactions.* Chapman & Hall/CR.

MURAT, F. AND SIMON, J. (1976) Sur le contrôle par un domaine géométrique. *Publication du Laboratoire d'Analyse Numérique de l'Université Paris*, **6** 189.

NOVOTNY, A. A. AND SOKOŁOWSKI, J. (2013) *Topological Derivatives in the Shape Optimization.* Series: *Interaction of Mechanics and Mathematics.* Springer-Verlag.

PIRONNEAU, O. (1984) *Optimal Shape Design for Elliptic Systems.* Springer, Berlin.

PLOTNIKOV, P. AND SOKOŁOWSKI, J. (2012) *Compressible Navier Stokes Equations. Theory and shape optimization.* Birkhäuser.

PLOTNIKOV, P. AND SOKOŁOWSKI, J. (2010) Shape derivative of drag functional. *SIAM J. Control Optim.* 48, 4680-4706.

RÖSCH, A. AND VEXLER, B. (2006) Optimal control of the Stokes equations, a priori error analysis for finite element. *SIAM J. Numer. Anal.* 44, 1903-1920.

Sokołowski, J. and Zolesio, J.P. (1992) *Introduction to Shape Optimization, Shape Sensitivity Analysis. Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, New York.

Sokołowski, J. and Zochowski, A. (2005) Topological derivatives for contact problems. *Numer. Math.* 102, 1, 145-179.

Tröltzsch, F. (2000) *Optimal Control of Partial Differential Equations. Theory, Methods and Applications.* AMS.

Zhu, S. and Gao, Z. (2019) Convergence analysis of mixed finite element approximations to shape gradients in Stokes equation. *Comput. Methods Appl. Mech. Engrg.*, 343 (127-150).