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CHAOTIC EXPANSION IN THE G-EXPECTATION SPACE

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Abstract. In this paper, we are motivated by uncertainty problems in volatility. We prove the equivalent theorem of Wiener chaos with respect to G-Brownian motion in the framework of a sublinear expectation space. Moreover, we establish some relationship between Hermite polynomials and G-stochastic multiple integrals. An equivalent of the orthogonality of Wiener chaos was found.

Keywords: G-expectation, G-Brownian motion, G-multiple integrals, Hermite polynomials, G-Wiener chaos.

Mathematics Subject Classification: 60H10, 60H05, 60H30.

1. INTRODUCTION

The notion of multiple stochastic integrals was first introduced by Wiener in connection with his studies of classical Brownian motion. Itô was the first to establish the existence of these multiple integrals and he was able to establish an orthogonal decomposition of the space L^2. This decomposition agrees exactly with the one obtained by Cameron and Martin using Hermite polynomials and clarifies much of the work started by Wiener in "homogenous chaos" [12]. However, the classical Brownian motion was constructed in a linear space (a probability space where the classical linear expectation plays the same role as the nonlinear expectation considered in our context), such linearity assumption is not feasible in many applications because some uncertain phenomena cannot be modelled well using linear expectations (see [4, 9]). More specially, motivated by uncertainty problems regarding stochastic volatility and risk measures in finance, Peng in [10] has introduced a new notion of a nonlinear expectation space; the so-called G-expectation which can take the uncertainty into consideration, in a systematic, beautiful and robust way.

The G-expectation has been developed very recently and opened the way to the introduction of a G-normal random variable under the framework of a G-expectation space, under which the canonical process $(B_t)_{t\geq 0}$ is a G-Brownian motion; and the related stochastic calculus was established especially, stochastic integrals of Itô's type with respect to G-Brownian motion and its quadratic variation [10].

The main difficulty lies in the fact that the G-expectation is intrinsic in the sense that it is not based on a given linear probability space, this give us hope to further develop the stochastic analysis including the theorem of the G-Wiener chaos which is proved after applying some properties of the generating function on the new formula for G-multiple integrals that we have established and led us to find the equivalent of "the orthogonality" of Wiener chaos. We also obtain the relationship between Hermite polynomials and G-multiple integrals, all of them are an extension of the classical case.

This paper is organized as follows. In Section 2 we introduce the necessary notations and the basic notions on the nonlinear space. In Section 3 we give a short overview of some elements of G-stochastic analysis which will be used in what follows. In Section 4 we state the main results of this paper: the theorem which gives the equivalence of the orthogonality of Wiener chaos and the theorem of G-Wiener chaos. In Section 5 we get the relationship between Hermite polynomials and G-stochastic multiple integrals and some immediate consequences of the properties of the generating function G. The last two Sections 6 and 7 are devoted to proofs of the two theorems mentioned above.

2. PRELIMINARIES

We present some notations and preliminaries of the theory of sublinear expectations and the related G-Brownian motion. More details can be found in Peng [10]. We denote by $\Omega = C_0^n(\mathbb{R}_+)$ the space of all $\mathbb{R}^n$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$, equipped with the distance

$$\rho\left(\omega^1,\omega^2\right) = \sum_{i=0}^{\infty} 2^{-i}\left[\left(\max_{t\in[0,\infty]}\left|\omega_t^1-\omega_t^2\right|\right)\wedge 1\right].$$

We will consider the following type of spaces of sublinear expectations: let $\mathcal{H}$ be a linear space of real functions defined on Ω such that if $X_1,\ldots,X_n \in \mathcal{H}$ then $\varphi(X_1,\ldots,X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of functions φ such that

$$|\varphi(x)-\varphi(y)| \leq C(1+|x|^m+|y|^m)|x-y|, \quad \forall x,y\in\mathbb{R}^n,$$

for some $C > 0$, $m \in \mathbb{N}$ depending on φ.

$\mathcal{H}$ is then considered as the space of random variables on Ω.

Here we use $C_{l.Lip}(\mathbb{R}^n)$ in our framework only for convenience. In fact our essential requirement is that $\mathcal{H}$ contains all constants and, moreover, $X \in \mathcal{H}$ implies that $|X| \in \mathcal{H}$. In general $C_{l.Lip}(\mathbb{R}^n)$ can be replaced by the following spaces of functions defined on $\mathbb{R}^n$:

$L^{\infty}(\mathbb{R}^n)$ – the space of all bounded Borel-measurable functions,
$C_{unif}(\mathbb{R}^n)$ – the space of all bounded and uniformly continuous functions,
$C_{b.Lip}(\mathbb{R}^n)$ – the space of all bounded and Lipschitz continuous functions,
$Lip(\mathbb{R}^n)$ – the space of all Lipschitzian functions on $\mathbb{R}^n$.

Definition 2.1. A sublinear expectation $\widehat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\widehat{\mathbb{E}} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties for all $X, Y \in \mathcal{H}$:

(a) Monotonicity: If $X \geq Y$, then $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$.
(b) Preserving of constants: $\widehat{\mathbb{E}}[c] = c$.
(c) Sub-additivity: $\widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y] \leq \widehat{\mathbb{E}}[X - Y]$.
(d) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$ for all $\lambda \geq 0$.

The triplet $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a sublinear expectation space.

Definition 2.2. In a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$, under $\widehat{\mathbb{E}}$ if for each test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have

$$\widehat{\mathbb{E}}\left[\varphi(X, Y)\right] = \widehat{\mathbb{E}}\left[\widehat{\mathbb{E}}\left[\varphi(x, Y)\right]_{x=X}\right],$$

where $\widehat{\mathbb{E}}\left[\varphi(x, Y)\right]_{x=X}$ means $\psi(X)$, where $\psi(x) = \widehat{\mathbb{E}}\left[\varphi(x, Y)\right]$.

Note that this definition is correct. To see this it suffices to show that for each $x \in \mathbb{R}^m, \varphi(x, Y) \in \mathcal{H}$ and $\widehat{\mathbb{E}}\left[\varphi(\cdot, Y)\right]$ belongs to $C_{l.Lip}(\mathbb{R}^m)$. We have to show that $\varphi(x, \cdot) \in C_{l.Lip}(\mathbb{R}^n)$. Indeed, for all $(x, y), (u, z) \in \mathbb{R}^m \times \mathbb{R}^n$, we have

$$|\varphi(x, y) - \varphi(u, z)| \leq C\left|(x, y) - (u, z)\right| \left(1 + |(x, y)|^k + |(u, z)|^k\right), \quad C > 0, k \in \mathbb{N},$$

and so

$$\begin{aligned}|\varphi(x, y) - \varphi(x, z)| &\leq C|y - z|\left(1 + |(x, y)|^k + |(x, z)|^k\right) \leq \\ &\leq C|y - z|\left(1 + \left(|x|^2 + |y|^2\right)^{\frac{k}{2}} + \left(|x|^2 + |z|^2\right)^{\frac{k}{2}}\right) \leq \\ &\leq C|y - z|\left(1 + 2^{\frac{k}{2}}\left(|x|^k + |y|^k\right) + 2^{\frac{k}{2}}\left(|x|^k + |z|^k\right)\right) \leq \\ &\leq C'|y - z|\left(1 + \left(|x|^k + |z|^k\right)\right),\end{aligned}$$

where we used the inequality $(a + b)^p \leq 2^p\left(a^p + b^p\right)$ for $a, b \geq 0$ and put $C' = C\max\left(2^{\frac{k}{2}}, 1 + 2^{\frac{k}{2}+1}|x|^k\right)$. Therefore, $\varphi(x, \cdot) \in C_{l.Lip}(\mathbb{R}^n)$.

Similarly, we have

$$|\varphi(x, Y) - \varphi(u, Y)| \leq C|x - u|\left(1 + 2^{\frac{k}{2}}\left(|x|^k + |Y|^k\right) + 2^{\frac{k}{2}}\left(|u|^k + |Y|^k\right)\right),$$

which implies that

$$\begin{aligned}
&\left|\widehat{\mathbb{E}}\left[\varphi(x,Y)\right]-\widehat{\mathbb{E}}\left[\varphi(u,Y)\right]\right| \leq \\
&\leq \widehat{\mathbb{E}}\left|\varphi(x,Y)-\varphi(u,Y)\right| \leq \\
&\leq C\left|x-u\right|\left(1+2^{\frac{k}{2}}\left(\left|x\right|^k+\widehat{\mathbb{E}}\left(\left|Y\right|^k\right)\right)+2^{\frac{k}{2}}\left(\left|u\right|^k+\widehat{\mathbb{E}}\left(\left|Y\right|^k\right)\right)\right) \leq \\
&\leq C''\left|x-u\right|\left(1+\left|x\right|^k+\left|u\right|^k\right),
\end{aligned}$$

where $C''=C\max\left(2^{\frac{k}{2}},1+2^{\frac{k}{2}+1}\widehat{\mathbb{E}}\left(\left|Y\right|^k\right)\right)$, which means that $\widehat{\mathbb{E}}\left[\varphi\left(\cdot,Y\right)\right]$ belongs to $C_{l.Lip}\left(\mathbb{R}^m\right)$.

Definition 2.3 (G-normal distribution)**.** A random variable X on a sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$ is called *G-normal distributed*, denoted by $X\sim\mathcal{N}(0,[\underline{\sigma}^2,\overline{\sigma}^2])$, if for all $a,b\geq 0$ the random variables $aX+b\widetilde{X}$ and $\sqrt{a^2+b^2}X$ have the same distribution, where $\tilde{X}$ is an independent copy of X.

Here the subset $[\underline{\sigma}^2,\overline{\sigma}^2]$ characterizes the variance-uncertainty of X, where

$$\bar{\sigma}^2=\widehat{\mathbb{E}}[X^2],\qquad \underline{\sigma}^2=-\widehat{\mathbb{E}}[-X^2],$$

and the letter G denotes the generating function defined by $G\left(\alpha\right)=\frac{1}{2}\widehat{\mathbb{E}}\left(\alpha X^2\right)$ by Peng. However, the positive homogeneity of G is equivalent to the following condition

$$\widehat{\mathbb{E}}\left(\lambda X\right)=\lambda^+\widehat{\mathbb{E}}\left(X\right)+\lambda^-\widehat{\mathbb{E}}\left(-X\right),\quad \forall\lambda\in\mathbb{R},$$

which in turn implies that

$$\widehat{\mathbb{E}}\left(\alpha X^2\right)=\alpha^+\widehat{\mathbb{E}}\left(X^2\right)+\alpha^-\widehat{\mathbb{E}}\left(-X^2\right)$$

and

$$G\left(\alpha\right)=\frac{1}{2}\left(\alpha^+\overline{\sigma}^2-\alpha^-\underline{\sigma}^2\right).$$

Recall that $\tilde{X}$ is a copy of X, if they have the same distributions:

$$\widehat{\mathbb{E}}[\varphi(\tilde{X})]=\widehat{\mathbb{E}}\left[\varphi\left(X\right)\right],\quad \forall\varphi\in C_{l.Lip}(\mathbb{R}^n)$$

(For more details see [8]).

Definition 2.4. A process $\{B_t(\omega)\}_{t\geq 0}$ in a sublinear expectation space $(\Omega,\mathcal{H},\widehat{\mathbb{E}})$ is called a *G-Brownian motion* if for each $n\in\mathbb{N}$ and $0<t_1<\ldots<t_n<\infty$, $B_{t_1},\ldots,B_{t_n}\in\mathcal{H}$ and the following properties are satisfied:

(i) $B_0\left(\omega\right)=0$,
(ii) for all $t,s\geq 0$, the increment $B_{t+s}-B_t$ is $\mathcal{N}(0,[\underline{\sigma}^2 s,\overline{\sigma}^2 s])$-distributed,
(iii) for all $t,s\geq 0$, $B_{t+s}-B_t$ is independent to $(B_{t_1},\ldots,B_{t_n})$ for all $n\in\mathbb{N}$ and $0\leq t_1\leq\ldots\leq t_n\leq t$.

3. G-STOCHASTIC INTEGRAL

We recall some notions on G-stochastic calculus.

For each fixed $T \in [0, \infty)$, we set $\Omega_T = \{\omega_{.\wedge T} : \omega \in \Omega\}$. The spaces of Lipschitzian functions on Ω are denoted by:

$$\begin{aligned} Lip\,(\Omega_T) &= \{\varphi\,(B_{t_1\wedge T}, \ldots, B_{t_n\wedge T}) : t_1, \ldots, t_n \in [0, \infty), \varphi \in C_{l.Lip}(\mathbb{R}^n)\}\,, \\ Lip\,(\Omega) &= \bigcup_{n=1}^{\infty} Lip\,(\Omega_n)\,, \end{aligned}$$

where B_t denotes the canonical process, that is, $B_t(\omega) = \omega_t$.

Let us consider the G-expectation $\widehat{\mathbb{E}} : Lip(\Omega) \to \mathbb{R}$ defined by Peng in [6] when he constructed a sublinear expectation on $(\Omega, Lip\,(\Omega))$ such that the canonical process $(B_t)_{t\geq 0}$ is a G-Brownian motion. For this purpose, let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of random variables on a sublinear expectation space $\left(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}}\right)$ such that ξ_i is G-normal distributed, and ξ_{i+1} is independent of $(\xi_1, \ldots, \xi_i)$ for each $i \geq 1$. Then a sublinear expectation $\widehat{\mathbb{E}}$ on $Lip\,(\Omega)$ is constructed by the following procedure: for each $X \in Lip\,(\Omega)$ with

$$X = \varphi\left(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}\right)$$

for some $\varphi \in C_{l.Lip}(\mathbb{R}^m)$ and $0 < t_1 < \ldots < t_m < \infty$, we set

$$\widehat{\mathbb{E}}\left[\varphi\left(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}\right)\right] = \widetilde{\mathbb{E}}\left[\varphi\left(\sqrt{t_1}\xi_1, \ldots, \sqrt{t_m - t_{m-1}}\xi_m\right)\right].$$

Peng proved that the corresponding canonical process $(B_t)_{t\geq 0}$ on the sublinear expectation space $(\Omega, Lip(\Omega), \widehat{\mathbb{E}})$ called the G-expectation space is a G-Brownian motion. In what follows, we consider this canonical G-Brownian motion on $\left(\Omega, Lip\,(\Omega)\,, \widehat{\mathbb{E}}\right)$ in the case $\Omega = C_0^1\,(\mathbb{R}_+)$.

Denis and Martini [2] introduced a new approach of quasi-sure stochastic analysis. However, they used a quite different approach which utilizes the set of all probability measures on Ω. Although the constructions of the quasi-sure analysis and the G-expectations are substantially different, these theories are very closely related as proved recently by Denis, Hu and Peng [1]. They also provided a dual representation of the G-expectation and they established that the sublinear expectation $\widehat{\mathbb{E}}$ can be represented as an upper expectation of classical expectations, i.e. there exists a set of probability measures $\mathcal{P}$ such that for each random variable X we have

$$\widehat{\mathbb{E}}\,[X] = \sup_{P\in\mathcal{P}} \mathbb{E}^P\,[X]\,,$$

where $\mathbb{E}^P$ is the classical expectation under the probability measure P (see also [5] for a simple proof).

As in [11], we utilize the terminology of Denis and Martini and we say that a property holds $\mathcal{P}$-quasi-surely, abbreviated as $\mathcal{P}$-*q.s.*, if it holds P-almost surely for all $P \in \mathcal{P}$. For each partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0,T]$ such that $0 = t_0 < t_1 < \ldots < t_N = T$, we set

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, \ldots, N-1\}.$$

Let $\left(\pi_T^N\right)$ be a sequence of partitions of $[0,T]$ such that $\lim_{N\to\infty} \mu(\pi_T^N) = 0$. We consider the following type of simple processes: for a given partition $\pi_T = \{t_0, \ldots, t_N\}$ of $[0,T]$, we set

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t),$$

where ξ_k are given such that $\widehat{\mathbb{E}}\left[|\xi_k|^p\right] < \infty,\ \ k = 0, 1, \ldots, N-1$. The collection of these processes is denoted by $M_G^{p,0}(0,T)$.

We can introduce a natural norm

$$\|\eta\|_{M_G^p} = \left\{\widehat{\mathbb{E}}\left(\int_0^T |\eta_s|^p\, ds\right)\right\}^{\frac{1}{p}},$$

under which $M_G^{p,0}(0,T)$ can be extended to the Banach space $M_G^p(0,T)$. We denote the completion of $Lip(\Omega_T)$ under the norm $\|X\|_p = \left(\widehat{\mathbb{E}}\left[|X|^p\right]\right)^{\frac{1}{p}}$ by $L_G^p(\Omega_T)$.

Definition 3.1. For each $\alpha \in M_G^{2,0}(0,T)$ of the form

$$\alpha_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j, t_{j+1}[}(t),$$

we define the *stochastic integral* by

$$I(\alpha) = \int_0^T \alpha_t dB_t = \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

Note that $I(\alpha)$ is completely independent of G and that $\widehat{\mathbb{E}}\left(\int_0^T \alpha_s dB_s\right) = 0$ for each $0 \le t \le T$ (see [3, 13]).

Proposition 3.2. *$I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is a continuous linear mapping and thus can be continuously extended to $M_G^2(0,T)$. For any $\alpha \in M_G^2(0,T)$, we have*

$$\underline{\sigma}^2 \widehat{\mathbb{E}}\left[\int_0^T \alpha_t^2\, dt\right] \le \widehat{\mathbb{E}}\left[\left(\int_0^T \alpha_t dB_t\right)^2\right] \le \overline{\sigma}^2 \widehat{\mathbb{E}}\left[\int_0^t \alpha_t^2\, dt\right] \tag{3.1}$$

(See [7] and [10]).

In what follows, we will need some properties of G.

Properties of G:

1)

$$G(\alpha) = \frac{1}{2} \sup_{\underline{\sigma} \leq \sigma \leq \overline{\sigma}} \sigma^2 \alpha \quad \text{for each } \alpha \in \mathbb{R}.$$

2)

$$G(\alpha x) = \alpha G(x) \quad \text{for each } \alpha \geq 0.$$

3) G is an increasing function.

4)

$$G^n(\alpha) = \begin{cases} \frac{\overline{\sigma}^{2n}}{2^n}\alpha & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ \frac{\underline{\sigma}^{2n}}{2^n}\alpha & \text{if } \alpha < 0, \end{cases} \text{ where } G^n = \underbrace{G \circ G \circ \ldots \circ G}_{n-fold}, n \geq 1 \text{ and } G^1 = G.$$

5)

$$\widehat{\mathbb{E}}\left(\int_0^t G(\alpha_s)\, ds\right) \leq G\left(\int_0^t \widehat{\mathbb{E}}\ (\alpha_s)\, ds\right) \quad \text{for each } \alpha \text{ in } M_G^2(0,T).$$

Proof. 1) If $\alpha \geq 0$, then $\alpha^+ = \alpha$ and $\alpha^- = 0$, thus

$$G(\alpha) = \frac{1}{2}\left(\alpha^+\overline{\sigma}^2 - \alpha^-\underline{\sigma}^2\right) = \frac{1}{2}\alpha\overline{\sigma}^2 = \frac{1}{2}\sup_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}}\left(\alpha\sigma^2\right).$$

If $\alpha < 0$, then $\alpha^+ = 0$ and $\alpha^- = -\alpha$, thus

$$G(\alpha) = \frac{1}{2}\alpha\underline{\sigma}^2 = \frac{1}{2}\sup_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}}\left(\alpha\sigma^2\right).$$

2) and 3) can be immediately obtained from 1).

4) follows by induction, 2) and the fact that

$$G(\alpha) = \begin{cases} \frac{1}{2}\alpha\overline{\sigma}^2 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ \frac{1}{2}\alpha\underline{\sigma}^2 & \text{if } \alpha < 0. \end{cases}$$

5) According to the Fubini theorem, for each $P \in \mathcal{P}$ we get

$$\mathbb{E}^P\left(\int_0^t G(\alpha_s)\, ds\right) \leq \int_0^t \mathbb{E}^P\left(G(\alpha_s)\right) ds.$$

On the other hand, for a given $\varepsilon > 0$, there exists $\sigma_{\varepsilon,s} \in [\underline{\sigma}, \overline{\sigma}]$ such that

$$G(\alpha_s) \leq \frac{1}{2}\sigma_{\varepsilon,s}^2\alpha_s + \varepsilon,$$

thus

$$\mathbb{E}^P\left(\int\limits_0^t G(\alpha_s)\,ds\right) \le \frac{1}{2}\int\limits_0^t \mathbb{E}^P\left(\sigma_{\varepsilon,s}^2\alpha_s\right)ds + \varepsilon t \le \frac{1}{2}\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\sigma^2\int\limits_0^t \mathbb{E}^P(\alpha_s)\,ds + \varepsilon t \le$$

$$\le \frac{1}{2}\sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\sigma^2\int\limits_0^t \widehat{\mathbb{E}}(\alpha_s)\,ds + \varepsilon t \le G\left(\int\limits_0^t \widehat{\mathbb{E}}(\alpha_s)\,ds\right) + \varepsilon t,$$

hence

$$\mathbb{E}\left(\int\limits_0^t G(\alpha_s)\,ds\right) \le G\left(\int\limits_0^t \widehat{\mathbb{E}}(\alpha_s)\,ds\right). \qquad \square$$

Definition 3.3. The *quadratic variation* of $\left(\langle B\rangle_t\right)_{t\ge 0}$ is defined by

$$\langle B\rangle_t = \lim_{\mu\left(\pi_t^N\right)\to 0}\sum_{j=0}^{N-1}\left(B_{t_{j+1}} - B_{t_j}\right)^2 = B_t^2 - 2\int\limits_0^t B_s dB_s.$$

Definition 3.4. For each $\alpha \in M_G^{1,0}(0,T)$ such that

$$\alpha_t(\omega) = \sum_{k=0}^{N-1}\xi_k(\omega)\,I_{[t_k,t_{k+1})}(t),$$

we define $Q(\cdot) : M_G^{1,0}(0,T) \to L_G^1(\Omega_T)$

$$Q(\eta) = \int\limits_0^T \alpha_t d\langle B\rangle_t = \sum_{i=0}^{N-1}\xi_i\left(\langle B\rangle_{t_{i+1}} - \langle B\rangle_{t_i}\right).$$

$Q(\cdot)$ is a continuous linear mapping, thus $Q(\cdot)$ can be continuously extended to $M_G^1(0,T)$.

The following lemma plays a crucial role in this paper.

Lemma 3.5. *For $\alpha, \beta \in M_G^2(0,T)$, we have*

$$-2G\left(\int\limits_0^T \widehat{\mathbb{E}}(-\alpha_t\beta_t)\,dt\right) \le \widehat{\mathbb{E}}\left(I(\alpha)\,I(\beta)\right) \le 2G\left(\int\limits_0^T \widehat{\mathbb{E}}(\alpha_t\beta_t)\,dt\right). \qquad (3.2)$$

Proof. Let

$$\alpha_t(\omega) = \sum_{k=0}^{N-1}\alpha_k(\omega)\,I_{[t_k,t_{k+1})}(t), \quad \beta_t(\omega) = \sum_{k=0}^{N-1}\beta_k(\omega)\,I_{[t_k,t_{k+1})}(t).$$

From the definition of the Itô integral and by using standard arguments of G-stochastic analysis, we can write

$$\widehat{\mathbb{E}}\left(I\left(\alpha\right)I\left(\beta\right)\right)=\widehat{\mathbb{E}}\left[\int_0^{T_{N-1}}\alpha_s dB_s\int_0^{T_{N-1}}\beta_s dB_s+\alpha_{N-1}\beta_{N-1}\left(B_{t_N}-B_{t_{N-1}}\right)^2\right].$$

Then we can repeat this procedure to obtain

$$\widehat{\mathbb{E}}\left(I\left(\alpha\right)I\left(\beta\right)\right)=\widehat{\mathbb{E}}\left[\sum_{i=0}^{N-1}\alpha_i\beta_i\left(B_{t_{i+1}}-B_{t_i}\right)^2\right]=\widehat{\mathbb{E}}\left[\sum_{i=0}^{N-1}\alpha_i\beta_i\left(\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}\right)\right].$$

Otherwise, we have

$$\underline{\sigma}^2\left(t_{i+1}-t_i\right)\le\left(\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}\right)\le\overline{\sigma}^2\left(t_{i+1}-t_i\right).$$

Multiplying the above inequalities by $\alpha_i\beta_i$ and summing them with respect to i we obtain

$$\underline{\sigma}^2\sum_{\alpha_i\beta_i\ge 0}\alpha_i\beta_i\left(t_{i+1}-t_i\right)+\overline{\sigma}^2\sum_{\alpha_i\beta_i\le 0}\alpha_i\beta_i\left(t_{i+1}-t_i\right)\le\sum_{i=0}^{N-1}\alpha_i\beta_i\left(\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}\right)\tag{3.3}$$

and

$$\sum_{i=0}^{N-1}\alpha_i\beta_i\left(\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}\right)\le\overline{\sigma}^2\sum_{\alpha_i\beta_i\ge 0}\alpha_i\beta_i\left(t_{i+1}-t_i\right)+\underline{\sigma}^2\sum_{\alpha_i\beta_i\le 0}\alpha_i\beta_i\left(t_{i+1}-t_i\right).\tag{3.4}$$

The first member of (3.3) is equal to

$$\underline{\sigma}^2\sum_{i=0}^{N}\left(-\alpha_i\beta_i\right)^-\left(t_{i+1}-t_i\right)-\overline{\sigma}^2\sum_{i=0}^{N}\left(-\alpha_i\beta_i\right)^+\left(t_{i+1}-t_i\right)=$$

$$=-2\sum_{i=0}^{N}G\left(-\alpha_i\beta_i\right)\left(t_{i+1}-t_i\right)=-2\int_0^T G\left(-\alpha_s\beta_s\right)ds.$$

By the same argument, the second member of (3.4) takes the form

$$2\int_0^T G\left(\alpha_s\beta_s\right)ds.$$

As a consequence, we deduce that

$$-2\int_0^T G\left(-\alpha_s\beta_s\right)ds\le\sum_{i=0}^{N-1}\alpha_i\beta_i\left(\langle B\rangle_{t_{i+1}}-\langle B\rangle_{t_i}\right)\le 2\int_0^T G\left(\alpha_s\beta_s\right)ds.$$

Finally, we obtain

$$-2\widehat{\mathbb{E}}\left(\int_0^T G\left(-\alpha_t\beta_t\right)dt\right) \leq \widehat{\mathbb{E}}\left(I\left(\alpha\right)I\left(\beta\right)\right) \leq 2\widehat{\mathbb{E}}\left(\int_0^T G\left(\alpha_t\beta_t\right)dt\right).$$

The formula (3.2) follows from the property 5) of G. □

Definition 3.6. A *G-Itô* process X (1-dimensional case) is defined by

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s dB_s + \int_0^t \eta_s d\langle B\rangle_s, \quad t \in [0,T],$$

where $X_0 \in L_G^2(\Omega_T)$, such that α, β and η are bounded processes in $M_G^2(0,T)$.

For the G-Itô formula, we consider an n-dimensional G-Itô process $X_s = \left(X_s^1, \ldots, X_s^n\right)^t$, where

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_s^\nu ds + \int_0^t \beta_s^\nu dB_s + \int_0^t \eta_s^\nu d\langle B\rangle_s, \quad t \in [0,T], \ \nu = 1, \ldots, n.$$

Theorem 3.7 (The G-Itô formula [8]). *Let Φ be a C^2-function on $\mathbb{R}^n$ such that $\partial^2_{x_\nu x_\mu}\Phi$ satisfy the polynomial growth condition for $\nu, \mu = 1, \ldots, n$. Let α^ν, β^ν and η^ν, $\nu = 1, \ldots, n$, be bounded processes in $M_G^2(0,T)$. Then for all $s, t \geq 0$ we have in $L_G^2\left(\Omega_t\right)$:*

$$\Phi\left(X_t\right) = \Phi\left(X_s\right) + \sum_{\nu=1}^n \left(\int_s^t \partial_{x_\nu}\Phi\left(X_u\right)\alpha_u^\nu du + \int_s^t \partial_{x_\nu}\Phi\left(X_u\right)\beta_u^\nu dB_u\right) +$$
$$+ \int_s^t \left[\sum_{\nu=1}^n \partial_{x_\nu}\Phi\left(X_u\right)\eta_u^\nu + \frac{1}{2}\sum_{\nu,\mu=1}^n \partial^2_{x_\nu x_\mu}\Phi\left(X_u\right)\left(\beta_u^\nu\right)^2\right] d\langle B\rangle_u.$$

4. MAIN RESULTS

Before we state the main results, we give the definition of G-multiple integrals as in [6]. Let us consider $L^2\left(\Delta_n\right) = L^2\left(\Delta_n; dt_1 \ldots dt_n\right)$, where

$$\Delta_{n\wedge t} = \left\{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \ldots \leq t_n \leq t\right\}(n \geq 1, t \geq 0) \text{ and } \Delta_n = \Delta_{n\wedge T}.$$

For any φ in $L^2\left(\Delta_n\right)$, we can form the (n-fold) iterated G-Itô integral by

$$\int_{\Delta_n \wedge t} \varphi dB = \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} \varphi\left(t_1, \ldots, t_n\right) dB_{t_1} \ldots dB_{t_n}.$$

Note that each G-Itô integration with respect to dB_{t_i} is included in $M_G^2(0, t_{i+1})$. More precisely, we have

$$\widehat{\mathbb{E}}\left(\int_0^{t_{i+1}}\int_0^{t_i}\cdots\int_0^{t_2}\varphi(t_1,\ldots,t_n)\,dB_{t_1}\ldots dB_{t_i}\right)^2 \leq$$
$$\leq \overline{\sigma}^{2i}\int_0^{t_{i+1}}\int_0^{t_i}\cdots\int_0^{t_2}\varphi^2(t_1,\ldots,t_n)\,dt_1\ldots dt_i.$$

This follows by induction. To illustrate the argument we prove the above inequality in the case of $i=2$. Using the right inequality of (3.1) we have

$$\widehat{\mathbb{E}}\left(\int_0^{t_3}\int_0^{t_2}\varphi(t_1,\ldots,t_n)\,dB_{t_1}dB_{t_2}\right)^2 \leq \overline{\sigma}^2\int_0^{t_3}\widehat{\mathbb{E}}\left(\int_0^{t_2}\varphi(t_1,\ldots,t_n)\,dB_{t_2}\right)^2 dt_1 \leq$$
$$\leq \overline{\sigma}^4\int_0^{t_3}\int_0^{t_2}\varphi^2(t_1,\ldots,t_n)\,dt_1dt_2,$$

as claimed.

We consider the linear spaces

$$\mathcal{Z}_n = \left\{\int_{\Delta_n}\varphi dB : \varphi\in L^2(\Delta_n)\right\}$$

with $\mathcal{Z}_0 = \{constants\}$, where $\int_{\Delta_n}\varphi dB = \int\limits_{\Delta_n\wedge T}\varphi dB.$

For simplicity we use the following notation: for all $\varphi\in L^2(\Delta_n\wedge t)$ and for all $(t_1,\ldots,t_n)\in\Delta_n\wedge t$, we set

$$\varphi^l(t_{l+1},\ldots,t_n) = \varphi(t_1,\ldots,t_n) \text{ if } l\leq n.$$

The following theorem gives us the equivalent of “the orthogonality“ of the spaces $\mathcal{Z}_n$ in the classical case.

Theorem 4.1. *The map* $\varphi\longmapsto\int\limits_{\Delta_n}\varphi dB$ *defines a linear mapping from* $L^2(\Delta_n)$ *to* $\mathcal{Z}_n$ *such that for all* $\varphi\in L^2(\Delta_n)$ *and* $\psi\in L^2(\Delta_{n+p})$ *and for each* $l\leq n$ *we have:*

1)

$$I_{n,p}(\varphi,\psi)\leq 2^lG^l\left(\int_{\Delta_l}\widehat{\mathbb{E}}\left[\int_{\Delta_{n-l}\wedge t_l}\varphi^l dB\int_{\Delta_{n+p-l}\wedge t_l}\psi^l dB\right]dt_1\ldots dt_l\right),$$

2)

$$-2^lG^l\left(\int_{\Delta_{n-l}}\widehat{\mathbb{E}}\left[-\int_{\Delta_l\wedge t_l}\varphi^l dB\int_{\Delta_{n+p-l}\wedge t_l}\psi^l dB\right]dt_1\ldots dt_l\right)\leq I_{n,p}(\varphi,\psi),$$

where

$$I_{n,p}(\varphi,\psi)=\widehat{\mathbb{E}}\left(\int\limits_{\Delta_n}\varphi dB\int\limits_{\Delta_{n+p}}\psi dB\right).$$

We can formulate a corollary which is an immediate consequence of Theorem 4.1 when $l=n$, $m=n+p$, and the properties of G.

Corollary 4.2. *For all $\varphi,\psi\in L^2(\Delta_n)$ we have:*

1)

$$-2^nG^n\left(-\langle\varphi,\psi\rangle_{L^2(\Delta_n)}\right)\leq\widehat{\mathbb{E}}\left(\int\limits_{\Delta_n}\varphi dB\int\limits_{\Delta_n}\psi dB\right)\leq 2^nG^n\left(\langle\varphi,\psi\rangle_{L^2(\Delta_n)}\right).$$

In particular, if $\varphi=\psi$, then

$$\underline{\sigma}^{2n}\|\varphi\|^2_{L^2(\Delta_n)}\leq\widehat{\mathbb{E}}\left(\int\limits_{\Delta_n}\varphi dB\right)^2\leq\overline{\sigma}^{2n}\|\varphi\|^2_{L^2(\Delta_n)}.$$

2)

$$\widehat{\mathbb{E}}\left(\int\limits_{\Delta_n}\varphi dB\int\limits_{\Delta_m}\psi dB\right)=0.$$

Note that the last formula is equivalent to the orthogonality of $\mathcal{Z}_n$ and $\mathcal{Z}_m$ in the classical case.

The next theorem is the most important result of this paper.

Theorem 4.3 (The G-Wiener chaos)**.** *We have*

$$L^2_G(\Omega_T)=\bigoplus_{n\in\mathbb{N}}Z_n\ \mathcal{P}\text{−}q.s.,$$

that is, each $X\in L^2_G(\Omega_T)$ admits $\mathcal{P}$−q.s. a unique representation $X=X_0+X_1+\ldots+X_n+\ldots$, where $X_n\in Z_n$ for each $n\in\mathbb{N}$.

5. G-STOCHASTIC MULTIPLE INTEGRALS AND HERMITE POLYNOMIALS

Recall that the Hermite polynomials in one variable $H_n(x)$ and in two variables $h_n(t,x)$ are defined as follows. For each $t,x,\lambda\in\mathbb{R}$,

$$e^{\lambda x-\frac{\lambda^2}{2}}=\sum_{n=0}^{\infty}\lambda^nH_n(x)$$

and

$$e^{\lambda x-\frac{\lambda^2}{2}t}=\sum_{n=0}^{\infty}\lambda^nh_n(t,x).$$

Then H_n and h_n satisfy the following properties:

— $h_0(t,x) = H_0(x) = 1$, $h_1(t,x) = H_1(x) = x$ and $h_n(0,0) = 0$ for $n \geq 1$,
— $h_n(1,x) = H_n(x)$,
— $H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} D^n\left(e^{-\frac{x^2}{2}}\right)$, where $D = \frac{d}{dx}$,
— $\partial_x h_n(t,x) = h_{n-1}(t,x)$ for $n \geq 1$,
— $\partial_t h_n(t,x) = -\frac{1}{2}\partial^2_{xx} h_n(t,x) = -\frac{1}{2} h_{n-2}(t,x)$ for $n \geq 2$.

The next proposition describes the relation between h_n and the multiple stochastic integrals.

Proposition 5.1. *Let $\varphi \in L^2([0,T], d\langle B\rangle_t)$ $\mathcal{P}-q.s.$ Then for all $t \geq 0$ and $n \geq 2$ we have:*

1)

$$h_n\left(\int_0^t \varphi^2(s)d\langle B\rangle_s, \int_0^t \varphi(s)dB_s\right) = \int_0^t \varphi(s)h_{n-1}\left(\int_0^s \varphi^2(u)d\langle B\rangle_u, \int_0^s \varphi(u)dB_u\right)dB_s \tag{5.1}$$

in $L^2_G(\Omega_t)$.

2) *Assume that $\int_0^T \varphi^2(t)d\langle B\rangle_t = 1$ $\mathcal{P}-q.s.$. Then we have*

$$H_n\left(\int_0^T \varphi(t)dB_t\right) = \int_{\Delta_n} \overset{n}{\varphi} dB \ \mathcal{P}-q.s., \quad n \geq 2, \tag{5.2}$$

where $\overset{l}{\varphi}(t_1,\dots,t_l) = \varphi(t_1)\dots\varphi(t_l)$ for each $l \leq n$.

Proof. According to G-Itô's formula with $\Phi(t,x) = h_n(t,x)$ and

$$X_t = \left(\int_0^t \varphi^2(u)d\langle B\rangle_u, \int_0^t \varphi(u)dB_u\right)$$

we have

$$h_n(X_t) = h_n(0,0) + \int_0^t \varphi(s)\partial_x h_n(X_s)dB_s +$$
$$+ \int_0^t \left(\varphi^2(s)\partial_t h_n(X_s) + \frac{1}{2}\varphi^2(s)\partial^2_{xx} h_n(X_s)\right)d\langle B\rangle_s.$$

The formula (5.1) follows from the properties of h_n.

By repeating the formula (5.1) we obtain

$$H_n\left(\int_0^T \varphi(s)dB_s\right) = \int_0^T \varphi(s)h_{n-1}\left(X_s\right)dB_s =$$
$$= \int_{\Delta_2} \overset{2}{\varphi}\left(t_1,t_2\right)h_{n-2}\left(X_{t_2}\right)dB_{t_1}dB_{t_2} =$$
$$\vdots$$
$$= \int_{\Delta_n} \overset{n}{\varphi}\left(t_1,\ldots,t_n\right)h_0\left(X_{t_{n-1}}\right)dB_{t_1}dB_{t_2}\ldots dB_{t_n} =$$
$$= \int_{\Delta_n} \overset{n}{\varphi}dB \quad \mathcal{P}{-}q.s. \qquad \square$$

Lemma 5.2. *Let $\lambda_1,\ldots,\lambda_p \in \mathbb{R}$ be such that $\sum_{k=1}^p \lambda_k^2 = 1$, and let $x_1,\ldots,x_p \in \mathbb{R}$. Then*

$$H_n\left(\sum_{k=1}^p \lambda_k x_k\right) = \sum_{\alpha_1,\ldots,\alpha_p=n} C^{\alpha_1,\ldots,\alpha_p} \prod_{k=1}^p \lambda_k^{\alpha_k} H_{\alpha_k}\left(x_k\right),$$

where $C^{\alpha_1,\ldots,\alpha_p}$ is a constant depending only on $\alpha_1,\ldots,\alpha_p$.

Proof. For each $\theta \in \mathbb{R}$, we have

$$e^{\theta\sum\limits_{k=1}^p \lambda_k x_k - \frac{1}{2}\sum\limits_{k=1}^p \lambda_k^2\theta^2} = \prod_{k=1}^p e^{\theta\lambda_k x_k - \frac{1}{2}\lambda_k^2\theta^2}.$$

Note that

$$e^{\theta\sum\limits_{k=1}^p \lambda_k x_k - \frac{1}{2}\sum\limits_{k=1}^p \lambda_k^2\theta^2} = \sum_{n=0}^\infty \theta^n h_n\left(\sum_{k=1}^p \lambda_k^2, \sum_{k=1}^p \lambda_k x_k\right) = \sum_{n=0}^\infty \theta^n H_n\left(\sum_{k=1}^p \lambda_k x_k\right),$$

while on the other hand, we get

$$\prod_{k=1}^p e^{\theta\lambda_k x_k - \frac{1}{2}\lambda_k^2\theta^2} = \prod_{k=1}^p \sum_{\alpha=0}^\infty \theta^\alpha \lambda_k^\alpha H_\alpha\left(x_k\right) = \sum_{n=0}^\infty \theta^n \sum_{\alpha_1+\ldots+\alpha_p=n} C^{\alpha_1,\ldots,\alpha_p} \lambda_k^{\alpha_k} H_{\alpha_k}\left(x_k\right).$$

This implies that

$$H_n\left(\sum_{k=1}^p \lambda_k x_k\right) = \sum_{\alpha_1+\ldots+\alpha_p=n} C^{\alpha_1,\ldots,\alpha_p} \prod_{k=1}^p \lambda_k^{\alpha_k} H_{\alpha_k}\left(x_k\right). \qquad \square$$

Let us denote by $\bigvee_{n\in\mathbb{N}} \mathcal{Z}_n$ the linear space generated by $\bigcup_{n\in\mathbb{N}} \mathcal{Z}_n$, that is, $\bigvee_{n\in\mathbb{N}} \mathcal{Z}_n = \mathcal{Z}_0 + \mathcal{Z}_1 + \ldots + \mathcal{Z}_n + \ldots$. We obtain the following proposition.

Proposition 5.3. *Each* $X \in \bigvee_{n\in\mathbb{N}} \mathcal{Z}_n$ *has a unique representation* $\mathcal{P}-q.s.$

$$X = X_0 + \ldots + X_n + \ldots,$$

where $X_n \in \mathcal{Z}_n$ *for each* $n \in \mathbb{N}$.

We write $\bigvee_{n\in\mathbb{N}} \mathcal{Z}_n = \bigoplus_{n=0}^{\infty} \mathcal{Z}_n$ $\mathcal{P}$*-q.s.*

Proof. It suffices to prove that the random variable 0 admits the unique representation, the trivial one. Let

$$0 = X_0 + \ldots + X_n + \ldots$$

be a representation of 0. Since $X_{n_0}^2 = -\sum_{i\neq n_0} X_{n_0}\ X_i$ for each $n_0 \in \mathbb{N}$, then by "the orthogonality" of $\mathcal{Z}_i$ and $\mathcal{Z}_{n_0}$, we obtain that

$$0 \le \widehat{\mathbb{E}}\left(X_{n_0}^2\right) \le \sum_{i\neq n_0} \widehat{\mathbb{E}}\left(-X_{n_0}\ X_i\right) = 0,$$

which implies that $X_{n_0} = 0$ $\mathcal{P}-q.s.$ □

6. PROOF OF THEOREM 4.1

1) According to the fact that

$$\int_{\Delta_n} \varphi dB = \int_0^T \left(\int_{\Delta_{n-1}\wedge t_1} \varphi^1 dB \right) dB_{t_1}, \quad \int_{\Delta_n} \psi dB = \int_0^T \left(\int_{\Delta_{n-1}\wedge t_1} \psi^1 dB \right) dB_{t_1},$$

the definition of the multiple stochastic integrals and the second inequality from the formula (3.2), we have

$$I_{n,p}(\varphi,\psi) \le 2G\left(\int_0^T \widehat{\mathbb{E}}\left[\int_{\Delta_{n-1}\wedge t_1} \varphi^1 dB \int_{\Delta_{n+p-1}\wedge t_1} \psi^1 dB \right] dt_1 \right).$$

Similarly, we deduce that

$$\widehat{\mathbb{E}}\left[\int_{\Delta_{n-1}\wedge t_1} \varphi^1 dB \int_{\Delta_{n+p-1}\wedge t_1} \psi^1 dB \right] \le 2G\left(\int_0^{t_1} \widehat{\mathbb{E}}\left[\int_{\Delta_{n-2}\wedge t_2} \varphi^2 dB \int_{\Delta_{n+p-2}\wedge t_2} \psi^2 dB \right] dt_2 \right),$$

and so

$$I_{n,p}(\varphi,\psi) \le 2^2G^2\left(\int_0^T \int_0^{t_1} \widehat{\mathbb{E}}\left[\int_{\Delta_{n-2}\wedge t_2} \varphi^2 dB \int_{\Delta_{n+p-2}\wedge t_2} \psi^2 dB \right] dt_1 dt_2 \right).$$

By repeating this procedure, we conclude that

$$I_{n,p}(\varphi,\psi) \leq 2^l G^l \left(\int\limits_{\Delta_l} \widehat{\mathbb{E}} \left[\int\limits_{\Delta_{n-l}\wedge t_l} \varphi^l dB \int\limits_{\Delta_{n+p-l}\wedge t_l} \psi^l dB \right] dt_1 \dots dt_l \right). \tag{6.1}$$

2) By the first inequality of the formula (3.2), we obtain

$$I_{n,p}(\varphi,\psi) = \widehat{\mathbb{E}} \left[\int\limits_0^T \left(\int\limits_{\Delta_{n-1}\wedge t_1} \varphi^1 dB \right) dB_{t_1} \int\limits_0^T \left(\int\limits_{\Delta_{n-1}\wedge t_1} \psi^1 dB \right) dB_{t_1} \right] \geq$$

$$\geq -2G \left(\int\limits_0^T \widehat{\mathbb{E}} \left[- \int\limits_{\Delta_{n-1}\wedge t_1} \varphi^1 dB \int\limits_{\Delta_{n-1}\wedge t_1} \psi^1 dB \right] dt_1 \right).$$

The formula (6.1) with $t_1, -\varphi^1, \psi^1, l-1$ instead of T, φ, ψ, l, respectively, takes the form

$$\widehat{\mathbb{E}}\Big[- \int\limits_{\Delta_{n-1}\wedge t_1} \varphi^1 dB \int\limits_{\Delta_{n-1}\wedge t_1} \psi^1 dB \Big] \leq 2^{l-1} G^{l-1} \Big(\int\limits_{\Delta_{l-1}\wedge t_1} \widehat{\mathbb{E}}\Big[- \int\limits_{\Delta_{n-l}\wedge t_l} \varphi^l dB \int\limits_{\Delta_{n+p-l}\wedge t_l} \psi^l dB \Big] dt_2 \dots dt_l \Big),$$

hence

$$I_{n,p}(\varphi,\psi) \geq -2^l G^l \left(\int\limits_{\Delta_l} \widehat{\mathbb{E}} \left[- \int\limits_{\Delta_{n-l}\wedge t_l} \varphi^l dB \int\limits_{\Delta_{n+p-l}\wedge t_l} \psi^l dB \right] dt_1 \dots dt_l \right).$$

7. PROOF OF THEOREM 4.3

Let Π_T be the set of polynomials in $B_{t_1}, \dots, B_{t_p}$ $(t_1 \leq \dots \leq t_p \leq T)$. We first prove that Π_T is dense in $L_G^2(\Omega_T)$. In fact we have to approximate any $X = \varphi\left(B_{t_1}, \dots, B_{t_p}\right)$ by a sequence of Π_T, where $\varphi \in C_{l.Lip}(\mathbb{R}^p)$.

Since φ is Lipschitz on every closed ball $B\left(0, R\sqrt{p}\right) = \left\{x \in \mathbb{R}^p : |x| \leq R\sqrt{p}\right\}$ $(R > 0)$, then it is continuous on $B\left(0, R\sqrt{p}\right)$. Thus, by the Stone-Weierstrass theorem, there exists a sequence (P_n) of polynomials which converges uniformly to φ. Let T_R be the random variable defined by $T_R = \inf\{t \leq T : |B_t| > R\}$. Since $\left(B_{t_1\wedge T_R}, \dots, B_{t_p\wedge T_R}\right) \in B\left(0, R\sqrt{p}\right)$, then

$$\lim_{n\to\infty} P_n\left(B_{t_1\wedge T_R}, \dots, B_{t_p\wedge T_R}\right) = \varphi\left(B_{t_1\wedge T_R}, \dots, B_{t_p\wedge T_R}\right)$$

in $L_G^2(\Omega_T)$. The continuity of P_n, φ, B_t and $||\cdot||_2$, and the fact that T_R converges to T when R goes to infinity imply that

$$\lim_{R\to\infty} P_n\left(B_{t_1\wedge T_R}, \dots, B_{t_p\wedge T_R}\right) = P_n\left(B_{t_1}, \dots, B_{t_p}\right)$$

and

$$\lim_{R\to\infty} \varphi\left(B_{t_1\wedge T_R},\ldots,B_{t_p\wedge T_R}\right) = X \text{ in } L_G^2\left(\Omega_T\right).$$

Thus, using the commutativity between $\lim_{R\to\infty}$ and $\lim_{n\to\infty}$, we obtain

$$X = \lim_{n\to\infty} P_n\left(B_{t_1},\ldots,B_{t_p}\right) \text{ in } L_G^2\left(\Omega_T\right).$$

Let $\mathbb{H}_n$ be the linear space generated by $H_n\big(\int_0^T \varphi(t)dB_t\big)$ such that $\int_0^T \varphi^2(t)d\langle B\rangle_t = 1$ $\mathcal{P}-q.s.$ if $n \geq 1$ and $\mathbb{H}_0 = \{constants\}$. For each $\varphi \in L^2\left([0,T], d\langle B\rangle_t\right)$ such that

$$\int_0^T \varphi^2(t)d\langle B\rangle_t = 1 \quad \mathcal{P}-q.s.,$$

we claim that for every $p \in \mathbb{N}$ there exist (not necessarily unique) $(\varphi_i)_{i=1}^p \subset L^2\left([0,T], d\langle B\rangle_t\right), (\lambda_k)_{k=1}^p$ such that $\sum_{k=1}^p \lambda_k^2 = 1$ and $\varphi = \sum_{k=1}^p \lambda_k\varphi_k$ $\mathcal{P}-q.s.$ For example, if $0 = t_0 < \ldots < t_p = T$ is a subdivision of $[0,T]$ such that

$$\lambda_k = \left(\int_{t_{k-1}}^{t_k} \varphi^2(t)d\langle B\rangle_t\right)^{\frac{1}{2}} \neq 0,$$

then we have

$$\sum_{k=1}^p \lambda_k^2 = \sum_{k=1}^p \left(\int_{t_{k-1}}^{t_k} \varphi^2(t)d\langle B\rangle_t\right) = \int_0^T \varphi^2(t)d\langle B\rangle_t = 1 \quad \mathcal{P}-q.s.$$

We set

$$\varphi_k = \frac{\varphi}{\lambda_k} I_{[t_{k-1},t_k)}.$$

This implies that

$$\sum_{k=1}^p \lambda_k\varphi_k = \varphi.$$

According to Lemma 5.2 we have

$$H_n\left(\int_0^T \varphi\left(s\right)dB_s\right) = \sum_{\alpha_1+\alpha_2+\ldots+\alpha_p=n} C^{\alpha_1,\alpha_2,\ldots,\alpha_p} \prod_{k=1}^p \lambda^{\alpha_k} H_{\alpha_k}\left(\int_0^T \varphi_k(s)dB_s\right),$$

thus $\mathbb{H}_n \subset \mathcal{U}_n$, where $\mathcal{U}_n$ is the linear space generated by

$$H_{\alpha_1}\left(\int_0^T \varphi_1\left(s\right)dB_s\right) H_{\alpha_2}\left(\int_0^T \varphi_2(s)dB_s\right)\ldots H_{\alpha_p}\left(\int_0^T \varphi_p(s)dB_s\right)$$

such that

$$\varphi_1, \ldots, \varphi_p \in L^2\left([0,T], d\langle B\rangle_t\right) \text{ and } \alpha_1 + \ldots + \alpha_p = n.$$

We have

$$\Pi_T \subset \bigvee_{n\in\mathbb{N}} LS\left\{\left(\sum_{k=1}^{p}\mu_k B_{t_k}\right)^n : \mu_1, \ldots, \mu_p \in \mathbb{R}\right\},$$

where LS means "the linear space generated by".

Note that the double inequality $\underline{\sigma}^2 dt \leq d\langle B\rangle_t \leq \overline{\sigma}^2 dt$ implies that the integral of a point under the measure $d\langle B\rangle_t$ is zero. This allows us to modify, for a given $\nu_1, \ldots, \nu_p \in \mathbb{R}$, any $\varphi \in L^2\left([0,T], d\langle B\rangle_t\right)$ such that $\int_0^T \varphi^2(t)d\langle B\rangle_t = 1$ $\mathcal{P}-q.s.$ as a function $\widetilde{\varphi}$ such that $\widetilde{\varphi}(t_k) = \nu_k$ $(k = 1, \ldots, p)$ and $\int_0^T \widetilde{\varphi}^2(t)d\langle B\rangle_t = 1$ $\mathcal{P}-q.s.$, thus

$$\sum_{k=1}^{p}\mu_k B_{t_k} = \sum_{k=0}^{p}\nu_k\left(B_{t_{k+1}} - B_{t_k}\right) = \int_0^T \widetilde{\varphi}(s)dB_s,$$

which yields

$$LS\left\{\left(\sum_{k=1}^{p}\mu_k B_{t_k}\right)^n : \mu_1, \ldots, \mu_p \in \mathbb{R}\right\} \subset$$
$$\subset LS\left\{\left(\int_0^T \varphi(s)\, dB_s\right)^n : \varphi \in L^2\left([0,T], d\langle B\rangle_t\right), \int_0^T \varphi^2(s)d\langle B\rangle_s = 1\ \mathcal{P}-q.s.\right\}.$$

Thus

$$\Pi_T \subset \bigvee_{n\in\mathbb{N}} \mathbb{H}_n,$$

and according to the density of Π_T in $L_G^2(\Omega_T)$, we have

$$L_G^2(\Omega_T) = \overline{\bigvee_{n\in\mathbb{N}} \mathbb{H}_n} = \overline{\bigvee_{n\in\mathbb{N}} \mathcal{U}_n},$$

where $\overline{A}$ is the closure of A in $L_G^2(\Omega_T)$. On the other hand, if $\int_0^T \varphi^2(t)d\langle B\rangle_t = 1$ $\mathcal{P}-q.s.$ for φ in $L^2\left([0,T], d\langle B\rangle_t\right)$, then

$$H_n\left(\int_0^T \varphi(t)dB_t\right) = \int_{\Delta_n} \overset{n}{\varphi} dB \in \mathcal{Z}_n,$$

hence

$$\mathbb{H}_n \subset \mathcal{Z}_n \text{ for each } n \in \mathbb{N}.$$

Therefore, $L_G^2(\Omega_T) = \bigoplus_{n\in\mathbb{N}} Z_n$ $\mathcal{P}-q.s.$ and $\mathbb{H}_n = \mathcal{Z}_n$ for each $n \in \mathbb{N}$.

Note that the equality $L_G^2(\Omega_T) = \overline{\bigvee_{n\in\mathbb{N}}\mathcal{U}_n}$ gives us another decomposition of $L_G^2(\Omega_T)$. □

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