Extremal Problems for Infinite Order Parabolic Systems with Boundary Conditions Involving Integral Time Lags

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Abstract: Extremal problems for integral time lag infinite order parabolic systems are studied in the paper. An optimal boundary control problem for distributed infinite order parabolic systems in which integral time lags appear in the Neumann boundary conditions is solved. Such equations constitute in a linear approximation a universal mathematical model for many diffusion processes (e.g., modeling and control of heat transfer processes). The time horizon is fixed. Using the Dubovicki-Milutin framework, the necessary and sufficient conditions of optimality for the Neumann problem with the quadratic performance indexes and constrained control are derived.

Keywords: Boundary control; infinite order parabolic systems; integral time lags

1. Introduction

Extremal problems play an increasing role in applications of control theory [20]. Despite a great variety of these problems, they can be converted by a unified functional-analytic framework, first suggested by Dubovicki and Milutin.

In 1962 Dubovicki and Milutin found a necessary condition for an extremum in the form of an equation set down in the language of functional analysis. They were able to derive, as special cases of this condition, almost all previously known necessary extremum conditions and thus to recover the lost theoretical unity of the calculus of variations.

In particular, in the paper [6], the Dubovicki-Milutin approach was adopted for solving optimal control problems for parabolic-hyperbolic systems. The existence and uniqueness of solutions of such parabolic-hyperbolic systems with the Dirichlet boundary conditions are discussed. Using the Dubovicki-Milutin framework, the necessary and sufficient conditions of optimality for the Dirichlet problem with the quadratic performance indexes and constrained control are derived.

In the papers [9–13], the Dubovicki-Milutin approach was used for solving boundary optimal control problems for the case of time lag parabolic equations [9] and for the case of parabolic equations involving time-varying lags [10, 11], multiple time-varying lags [12], and integral time lags [13] respectively.

The sufficient conditions for the existence of a unique solution of such parabolic equations [9–13] are presented. Such equations with deviating arguments are a well-known mathematical tool for representing many physical phenomena.

Consequently, in the papers [9–13], the linear quadratic problems of parabolic systems with time lags given in various forms (constant time lags [9], time-varying lags [10, 11], multiple time-varying lags [12], integral time lags [13], etc.) were solved.

Extremal problems for integral time lag infinite order parabolic systems are investigated. The purpose of this paper is to show the use of the Dubovicki-Milutin theorem [11] in solving optimal control problems for distributed parabolic systems.

As an example, an optimal boundary control problem for a system described by a linear infinite order partial differential equation of parabolic type in which integral time lag appears in the Neumann boundary condition is considered. Such equation constitutes in a linear approximation universal mathematical model for many diffusion processes (e.g., modeling and control of heat transfer processes). The right-hand side of this equation and the initial condition are not continuous functions usually, but they are measurable functions belonging to L^2 or L^∞ spaces. Therefore, the solution of this equation is given in a certain Sobolev space [17]. The performance indexes have the quadratic form. Finally, we impose some constraints on the boundary control. Using the Dubovicki-Milutin theorem, the necessary and sufficient conditions of optimality with the quadratic performance indexes and constrained control are derived for the Neumann problem.

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2. Preliminares

Let Ω be a bounded set of R^n with smooth boundary Γ.

We define the infinite order Sobolev space $H^\infty\{a_\alpha, 2\}(\Omega)$ of functions $\Phi(x)$ defined on Ω [1, 2] as follows

$$H^\infty\{a_\alpha,2\}(\Omega)=\left\{\Phi(x)\in C^\infty(\Omega):\sum_{|\alpha|=0}^{\infty}a_\alpha\left\|D^\alpha\Phi\right\|_2^2<\infty\right\} \quad (1)$$

where $C^\infty(\Omega)$ is a space of infinite differentiable functions, $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is a norm in the space $L^2(\Omega)$, and

$$D^\alpha=\frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}\dots(\partial x_n)^{\alpha_n}} \quad (2)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index for differentiation, $|\alpha|=\sum_{i=1}^{n}\alpha_i$.

The space $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ [1, 2] is defined as the formal conjugate space to the space $H^\infty\{a_\alpha, 2\}(\Omega)$, namely

$$H^{-\infty}\{a_\alpha,2\}(\Omega)=\left\{\Psi(x):\Psi(x)=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^\alpha\Psi_\alpha(x)\right\} \quad (3)$$

where $\Psi_\alpha\in L^2(\Omega)$ and $\sum_{|\alpha|=0}^{\infty}a_\alpha\left\|\Psi_\alpha\right\|_2^2<\infty$.

The duality pairing of the spaces $H^\infty\{a_\alpha, 2\}(\Omega)$ and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ is postulated by the formula

$$\langle\Phi,\Psi\rangle=\sum_{|\alpha|=0}^{\infty}a_\alpha\int_\Omega\Psi_\alpha(x)D^\alpha\Phi(x)\,dx \quad (4)$$

where $\Phi\in H^\infty\{a_\alpha,2\}(\Omega)$, $\Psi\in H^{-\infty}\{a_\alpha,2\}(\Omega)$.

From above, $H^\infty\{a_\alpha, 2\}(\Omega)$ is everywhere dense in $L^2(\Omega)$ with topological inclusion and $H^{-\infty}\{a_\alpha, 2\}(\Omega)$ denoted the topological dual space with respect to $L^2(\Omega)$ so we have the following chain

$$H^\infty\{a_\alpha,2\}(\Omega)\subseteq L^2(\Omega)\subseteq H^{-\infty}\{a_\alpha,2\}(\Omega)$$

3. Problem Formulation. Optimality Conditions

Now we formulate the control problem for the system described by the following parabolic equation:

$$\frac{\partial y}{\partial t}+A(t)y=u \quad (x,t)\in\Omega\times(0,T) \quad (5)$$

$$y(x,0)=y_p(x) \quad x\in\Omega \quad (6)$$

$$\frac{\partial y}{\partial\gamma_A}=\int_a^b y(x,t-h)\,dh+v \quad (x,t)\in\Gamma\times(0,T), \quad h\in(a,b) \quad (7)$$

$$y(x,t')=\Psi_0(x,t') \quad (x,t')\in\Gamma\times[-b,0) \quad (8)$$

where Ω has the same properties as Section 2.

$$y\equiv y(x,t;v),\quad u\equiv u(x,t),\quad v\equiv v(x,t),$$

$$Q=\Omega\times(0,T),\quad \bar{Q}=\Omega\times(0,T),$$

$$\Sigma=\Gamma\times(0,T),\quad \Sigma_0=\Gamma\times[-b,0)$$

h is an integral time lag such that $h\in(a,b)$, Ψ_0 is an initial function defined on Σ_0.

The parabolic operator $\frac{\partial}{\partial t}+A(t)$ in the state equation (1) satisfies the hypothesis of Lions and Magenes [17] and $A(t)$ is given by

$$Ay=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2\alpha}y(x,t)$$

and

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2\alpha} \quad (9)$$

is an infinite order elliptic partial differential operator [3].

$$\frac{\partial y}{\partial\gamma_A}(x,t)=\sum_{|w|=0}^{\infty}\left(D^w y(v)\right)\cos(n,x_i)=q(x,t) \quad x\in\Gamma,\ t\in(0,T) \quad (10)$$

where: $\frac{\partial y}{\partial\gamma_A}$ is a normal derivative at Γ, directed towards the exterior of Ω, $\cos(n, x_i)$ is an i-th direction cosine of n, n-being the normal at Γ exterior to Ω and

$$q(x,t)=\int_a^b y(x,t-h)\,dh+v(x,t) \quad (11)$$

Now we shall present the sufficient conditions for the existence of a unique solution of the mixed initial-boundary value problem (5)–(8) for the case where the boundary control $v\in L^2(\Sigma)$. For this purpose we introduce the Sobolev space $H^{\infty,1}(Q)$ ([17], vol. 2, p. 6) defined by

$$H^{\infty,1}(Q)=H^0\left(0,T;H^\infty\{a_\alpha,2\}(\Omega)\right)\cap H^1\left(0,T;H^0(\Omega)\right) \quad (12)$$

which is the Hilbert space normed by

$$\left(\int_0^T\|y(t)\|^2_{H^\infty\{a_\alpha,2\}(\Omega)}dt+\|y(t)\|^2_{H^1(0,T;H^0(\Omega))}\right)^{1/2} \quad (13)$$

where the space $H^1(0,T; H^0(\Omega))$ is defined in Chapter 1 of [17], vol. 1, respectively.

The existence of a unique solution for the mixed initial--boundary value problem (5)–(8) on the cylinder Q can be proved using a constructive method, i.e., first, solving (5)–(8) on the subcylinder Q_1 and in turn on Q_2, etc. until the procedure covers the whole cylinder Q. In this way the solution in the previous step determines the next one.

Then the following result is fulfilled [15]:

Theorem 1: Let y_0, Ψ_0, v, and u be given with $y_0\in\overset{\infty}{H}\{a_\alpha,2\}(\Omega)$, $\Phi_0\in H^{\infty,1}(Q_0)$, $\Psi_0\in L^2(\Sigma_0)$, $v\in L^2(\Sigma)$ and $u\in\left(H^{\infty,1}(Q)\right)'$.

Then, there exists a unique solution $y\in H^{\infty,1}(Q)$ for the mixed initial-boundary value problem (5)–(8). Moreover, $y(\cdot,ja)\in H^\infty\{a_\alpha,2\}(\Omega)$ for $j = 1, \dots, K$.

In the sequel, we shall fix $u\in\left(H^{\infty,1}(Q)\right)'$.

In this paper we shall consider the optimal boundary control problem i.e., $v\in L^2(\Sigma)$.

Let us denote by $Y\in H^{\infty,1}(Q)$ the space of states and by $U\in L^2(\Sigma)$ the space of controls. The time horizon T is fixed in our problem.

The performance index is given by

$$I(y,v) = \lambda_1 \int_Q \left| y(x,t;v) - z_d \right|^2 dxdt + \lambda_2 \int_0^T \int_\Gamma (Nv) v d\Gamma dt \qquad (14)$$

where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 > 0$; z_d is a given element in $L^2(Q)$ and N is a strictly positive linear operator on $L^2(\Sigma)$ into $L^2(\Sigma)$.

From the Theorem 1 [15] it follows that for any $v \in U_{ad}$ the cost function (14) is well-defined since $y(v) \in H^{\infty,1}(Q) \subset L^2(Q)$. We assume the following constraints on control: $v \in U_{ad}$ is a closed, convex set with non-empty interior, a subset of U. (15).

The optimal control problem (5)–(8), (14), (15) will be solved as the optimization one in which the function v is the unknown function.

Taking advantage of the Dubovicki-Milutin theorem [11] we shall derive the necessary and sufficient conditions of optimality for the optimization problem (5)–(8), (14), (15).

The solution of the stated optimal control problem is equivalent to seeking a pair $(y^0, v^0) \in E = H^{\infty,1}(Q) \times L^2(\Sigma)$ which satisfies the equation (5)–(8) and minimizing the performance index (14) with the constraints on control (15).

We formulate the necessary and sufficient conditions of the optimality in the form of Theorem 2.

Theorem 2: The solution of the optimization problem (5)–(8) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:

State equation

$$\left.\begin{array}{ll} \frac{\partial y^0}{\partial t} + A(t) y^0 = u & (x,t) \in \Omega \times (0,T) \\ y^0(x,0) = y_p(x) & x \in \Omega \\ \frac{\partial y^0}{\partial \gamma_A} = \int_a^b y^0(x,t-h)dh + v^0 & (x,t) \in \Gamma \times (0,T), \quad h \in (a,b) \\ y^0(x,t') = \Psi_0(x,t') & (x,t') \in \Gamma \times [-b,0) \end{array}\right\} \qquad (16)$$

Adjoint equations

$$\left.\begin{array}{ll} -\frac{\partial p}{\partial t} + A^*(t) p = \lambda_1 (y^0 - z_d) & (x,t) \in \Omega \times (0,T) \\ \frac{\partial p}{\partial \gamma_{A^*}} = \int_a^b p(x,t+h)dh & (x,t) \in \Gamma \times (0,T-b) \\ \frac{\partial p}{\partial \gamma_{A^*}} = \int_a^{T-t} p(x,t+h)dh & (x,t) \in \Gamma \times (T-b,T-a) \\ \frac{\partial p}{\partial \gamma_{A^*}} = 0 & (x,t) \in \Gamma \times (T-a,T) \\ p(x,T) = 0 & x \in \Omega \end{array}\right\} \qquad (17)$$

Maximum condition

$$\int_0^T \int_\Gamma (p + \lambda_2 N v^0)(v - v^0) d\Gamma dt \geq 0 \quad \forall v \in U_{ad} \qquad (18)$$

Moreover

$$\left.\begin{array}{l} \frac{\partial p(v)}{\partial \gamma_{A^*}}(x,t) = \sum_{|w|=0}^{\infty} \left(D^w p(v)\right) \cos(n, x_i) \\ A^* p = -\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} p(x,t) \end{array}\right\} \qquad (19)$$

Outline of the proof

Due to the Dubovicki-Milutin theorem [11], we approximate the set representing the inequality constraints by the regular admissible cone (RAC), the equality constraint by the regular tangent cone (RTC), and the performance index by the regular improvement cone (RFC).

a) Analysis of the equality constraint

The set Q_1 representing the equality constraint has the form

$$Q_1 = \left\{\begin{array}{ll} \frac{\partial}{\partial t} + A(t) y = u & (x,t) \in \Omega \times (0,T) \\ y(x,0) = y\ (x) & x \in \Omega \\ \frac{\partial y}{\partial \gamma} = \int y(x,t-h)dh + v & (x,t) \in \Gamma \times (0,T) \\ & h \in (a,b) \\ y(x,t') = \Psi_0(x,t') & (x,t') \in \Gamma \times [-b,0) \\ & (y,v) \in E \end{array}\right\} \qquad (20)$$

We construct the regular tangent cone (RTC) of the set Q_1 using the Lusternik theorem (Theorem 9.1 [5]). For this purpose, we define the operator P in the form

$$P(y,v) = \left(\frac{\partial y}{\partial t} + Ay - u,\ y(x,0) - y_p(x), \right.$$
$$\left. \frac{\partial y}{\partial \gamma_A} - \int_a^b y(x,t-h)dh - v,\ y(x,t') - \Psi_0(x,t') \right) \qquad (21)$$

The operator P is the mapping from the space $H^{\infty e}(Q) \times L(\Sigma)$ into the space

$$\left(H^{\infty,1}(Q)\right)' \times H^{\infty,1}(Q_0) \times H^\infty\{a_\alpha, 2\}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0).$$

The Fréchet differential of the operator P can be written in the following from:

$$P'(y^0,v^0)(\bar{y},\bar{v}) = \left(\frac{\partial \bar{y}}{\partial t} + A\bar{y},\ \bar{y}(x,0),\ \frac{\partial \bar{y}}{\partial \gamma_A} - \right.$$
$$\left. \int_a^b \bar{y}(x,t-h)dh - \bar{v},\ \bar{y}|_{\Sigma_0}(x,t') \right) \qquad (22)$$

In fact, $\frac{\partial}{\partial t}$ (Theorem 2.8 [18]), $A(t)$ (Theorem 2.1 [16]) and $\frac{\partial}{\partial \gamma_A}$ (Theorem 2.1 [17]) are linear and bounded mappings.

By virtue of the Theorem 1 [15], we can prove that P' is the operator "one to one" from the space $H^{\infty,1}(Q) \times L^2(\Sigma)$ onto $\left(H^{\infty,1}(Q)\right)' \times H^{\infty,1}(Q_0) \times H^\infty\{a_\alpha, 2\}(\Omega) \times L^2(\Sigma) \times L^2(\Sigma_0)$.

Considering that the assumptions of Lusternik's theorem are fulfilled, we can write down the regular tangent cone (RTC) for the set in a point (y^0,v^0) in the form

$$RTC\left(Q_1,(y^0,v^0)\right) = \left\{(\bar{y},\bar{v}) \in E;\ P'(y^0,v^0)(\bar{y},\bar{v}) = 0\right\} \qquad (23)$$

where the cone (23) is the subspace (Theorem 9.1 [5]).

Therefore, using Theorem 10.1 [5] we know the form of the functional belonging to the adjoint cone

$$f_1(\bar{y},\bar{v}) = 0 \quad \forall (\bar{y},\bar{v}) \in RTC\left(Q_1,(y^0,v^0)\right) \qquad (24)$$

b) Analysis of the constraint on controls

The set $Q_2 = Y \times U_{ad}$ representing the inequality constraints is a closed and convex one with non-empty interior in the space E.

By virtue of Theorem 10.5 [5] we find the functional belonging to the adjoint regular admissible cone (RAC), i.e.

$$f_2(\overline{y},\overline{v}) \in \left[RAC\left(Q_2,\left(y^0,v^0\right)\right)\right]^*$$

We notice that if E_1, E_2 are two linear topological spaces, then the adjoint space to $E = E_1 \times E_2$ has the form

$$E^* = \left\{f = (f_1, f_2); \quad f_1 \in E_1^*, \quad f_2 \in E_2^*\right\}$$

and

$$f(x) = f_1(x_1) + f_2(x_2)$$

So we note the functional $f_2(\overline{y},\overline{v})$ as follows

$$f_2(\overline{y},\overline{v}) = f_1'(\overline{y}) + f_2'(\overline{v}) \tag{25}$$

where

$f_1'(\overline{y}) = 0 \ \forall\, y \in Y$ (Theorem 10.1 [5]),

$f_2'(\overline{v})$ is a support functional to the set U_{ad} in a point v^0 (Theorem 10.5 [5]).

c) Analysis of the performance functional

Taking advantage of Theorem 7.5 [5], we find the regular improvement cone (RFC) of the performance index (14)

$$RFC\left(I,\left(y^0,v^0\right)\right) = \left\{(\overline{y},\overline{v}) \in E;\, I'\left(y^0,v^0\right)(\overline{y},\overline{v}) < 0\right\} \tag{26}$$

where: $I'\left(y^0,v^0\right)(\overline{y},\overline{v})$ is the Fréchet differential of the performance index (14) and it can be written as

$$I'\left(y^0,v^0\right)(\overline{y},\overline{v}) = 2\lambda_0\lambda_1\int_Q\left(y^0 - z_d\right)\overline{y}dxdt + 2\lambda_0\lambda_2\int_0^T\int_\Gamma\left(Nv^0\right)\overline{v}d\Gamma dt$$

On the basis of Theorem 10.2 [5] we find the functional belonging to the adjoint regular improvement cone (RFC), which has the form

$$f_3(\overline{y},\overline{v}) = -\lambda_0\lambda_1\int_Q\left(y^0 - z_d\right)\overline{y}dxdt - \lambda_0\lambda_2\int_0^T\int_\Gamma\left(Nv^0\right)\overline{v}d\Gamma dt \tag{27}$$

where: $\lambda_0 > 0$.

d) Analysis of Euler-Lagrange's equation

The Euler-Lagrange equation for our optimization problem has the form

$$\sum_{i=1}^{3} f_i = 0 \tag{28}$$

Let $p(x,t)$ be the solution of (14) for (y^0, v^0).

Let us denote by $\overline{y}$ the solution of $P'(\overline{y},\overline{v}) = 0$ for any fixed $\overline{v}$. Then taking into account (24), (25) and (27) we can express (28) in the form

$$f_2'(\overline{v}) = \lambda_0\lambda_1\int_Q\left(y^0 - z_d\right)\overline{y}dxdt + \lambda_0\lambda_2\int_0^T\int_\Gamma\left(Nv^0\right)\overline{v}d\Gamma dt \quad \forall(\overline{y},\overline{v}) \in RTC\left(Q_1,\left(y^0,v^0\right)\right). \tag{29}$$

We transform the first component of the right-hand side of (29) introducing the adjoint variable by adjoint equations (17).

After transformations we get

$$\lambda_0\lambda_1\int_Q\left(y^0 - z_d\right)\overline{y}dxdt = \ldots\lambda_0\int_0^T\int_\Gamma p\overline{v}d\Gamma dt \tag{30}$$

Substituting (30) into (29) gives

$$f_2'(\overline{v}) = \lambda_0\int_0^T\int_\Gamma\left(p + \lambda_2 Nv^0\right)\overline{v}d\Gamma dt \tag{31}$$

Using the definition of the support functional [5] and dividing both members of the obtained inequality by λ_0, we finally get

$$\int_0^T\int_\Gamma\left(p + \lambda_2 Nv^0\right)\left(v - v^0\right)d\Gamma dt \quad \forall v \in U_{ad} \tag{32}$$

The last inequality is equivalent to the maximum condition (18).

In order to prove the sufficiency of the derived conditions of the optimality, we use the fact that constraints and the performance index are convex and that the Slater's condition is satisfied (Theorem 15.3 [5]). In fact, there exists a point $(\tilde{y},\tilde{v}) \in \operatorname{int} Q_2$ such that $(\tilde{y},\tilde{v}) \in Q_1$. This fact follows immediately from the existence of non-empty interior of the set Q_2 and from the existence of the solution of the equation (5)–(8) as well. The uniqueness of the optimal control follows from the strict convexity of the performance index (14).

The above remarks complete the proof of Theorem 2.

One may also consider the similar optimal control problem with the performance index

$$\hat{I}(y,v) = \lambda_1\int_\Sigma\left|y(v)\right|_\Sigma - z_{\Sigma d}\right|^2 d\Gamma dt + \lambda_2\int_0^T\int_\Gamma(Nv)v\,d\Gamma dt \tag{33}$$

where $z_{\Sigma d}$ is a given element in $L^2(\Sigma)$.

From the Theorem 1 [15] and the trace theorem [17] for each $v \in L^2(\Sigma)$, there exists a unique solution $y \in H^{\infty,1}(Q)$ with $y|_\Sigma \in L^2(\Sigma)$. Thus $\hat{I}(y,v)$ is well-defined. Then the solution of the formulated optimal control problem is equivalent to seeking a pair $\left(y^0,v^0\right) \in E = H^{\infty,1}(Q) \times L^2(\Sigma)$ which satisfies the equation (5)–(8) and minimizing the cost function (33) with the constraints on control (15).

We can prove the following theorem:

Theorem 3: The solution of the optimization problem (5)–(8), (33), (15) exists and it is unique with the assumptions mentioned above; the necessary and sufficient conditions of the optimality are characterized by the following system of partial differential equations and inequalities:

State equation (16)

Adjoint equations

$$\left.\begin{array}{ll}
-\dfrac{\partial p}{\partial t} + A^*(t)p = 0 & (x,t) \in \Omega \times (0,T) \\
\dfrac{\partial p}{\partial \gamma_{A^*}} = \int_a^b p(x,t+h)\,dh + \lambda_1\left(y^0 - z_{\Sigma d}\right) & (x,t) \in \Gamma \times (0,T-b) \\
\dfrac{\partial p}{\partial \gamma_{A^*}} = \int_a^{T-t} p(x,t+h)\,dh + \lambda_1\left(y^0 - z_{\Sigma d}\right) & (x,t) \in \Gamma \times (T-b,T-a) \\
\dfrac{\partial p}{\partial \gamma_{A^*}} = \lambda_1\left(y^0 - z_{\Sigma d}\right) & (x,t) \in \Gamma \times (T-a,T) \\
p(x,T) = 0 & x \in \Omega
\end{array}\right\} \tag{34}$$

Maximum condition

$$\int_0^T\int_\Gamma\left(p + \lambda_2 Nv^0\right)\left(v - v^0\right)d\Gamma dt \geq 0 \quad \forall v \in U_{ad} \tag{35}$$

The idea of the proof of the Theorem 3 is the same as in the case of the Theorem 2.

In the case of performance indexes (14) and (33) with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programming one [7, 8, 14] which can be solved by the use of the well-known algorithms, e.g., Gilbert's [4, 7, 8, 14] ones. The practical application of Gilbert's algorithm to optimal control problem for a parabolic system with boundary condition involving a time lag is presented in [14]. Using the Gilbert's algorithm, a one-dimensional numerical example of the plasma control process is solved [14].

4. Final Remarks

The derived conditions of the optimality (Theorems 2 and 3) are original from the point of view of application of the Dubovicki-Milutin theorem [11] in solving optimal control problems for infinite order parabolic systems in which integral time lags appear in the Neumann boundary conditions. The proved optimization results (Theorems 2 and 3) constitute a novelty of the paper with respect to the references [7, 8, 14] concerning application of the Lions framework [16] for solving linear quadratic problems of optimal control for the case of the Neumann problem. The results presented in the paper can be treated as a generalization of the results obtained in [9–13] to the case of integral time lags appearing in the Neumann boundary conditions.

The obtained optimization theorems (Theorems 2 and 3) demand the assumption dealing with the non-empty interior of the set Q_2 representing the inequality constraints. Therefore, we approximate the set Q_2 by the regular admissible cone (RAC) (if $\operatorname{int} Q_2 = \phi$ then this cone does not exist) [11]. The obtained results can be reinforced by omitting the assumption concerning the non-empty interior of the set Q_2 and utilizing the fact that the equality constraints in the form of the parabolic equations are "decoupling". The optimal control problem reduces to seeking $v^0 \in Q_2'$ and minimizing the performance index $I(v)$. Then, we approximate the set Q_2' representing the inequality constraints by the regular tangent cone (RTC), and for the performance index $I(v)$ we construct the regular improvement cone (RFC) [11].

The proposed methodology based on the Dubovicki-Milutin approach can be presented on a specific case study concerning hyperbolic systems described by infinite order partial differential equations of the hyperbolic type in which time lags appear in the integral form both in the state equations and in the Neumann boundary conditions.

Moreover, the Dubovicki-Milutin framework can be applied to solving optimization problems for a sophisticated case of infinite order parabolic systems with deviating arguments given in the integral form such that $h \in (0, b)$ with $a = 0$.

Another direction of research would be the analysis of case studies with numerical examples concerning the determination of optimal boundary control with constraints for infinite order parabolic systems with integral time lags.

An interesting possible future research direction may consist in formulation of extremal problems for advanced modern control strategies, for example, event-based control [19].

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Problemy ekstremalne dla parabolicznych systemów nieskończonego rzędu z warunkami brzegowymi, w których występują całkowe opóźnienia czasowe

Streszczenie: Zaprezentowano ekstremalne problemy dla systemów parabolicznych nieskończonego rzędu z całkowymi opóźnieniami czasowymi. Rozwiązano problem optymalnego sterowania brzegowego dla systemów parabolicznych nieskończonego rzędu, w których całkowe opóźnienia czasowe występują w warunkach brzegowych Neumanna. Tego rodzaju równania stanowią w liniowym przybliżeniu uniwersalny model matematyczny dla procesów dyfuzyjnych. Korzystając z metody Dubowickiego-Milutina wyprowadzono warunki konieczne i wystarczające optymalności dla problemu liniowo-kwadratowego.

Słowa kluczowe: sterowanie brzegowe, systemy paraboliczne nieskończonego rzędu, całkowe opóźnienia czasowe

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