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Modelling sequential events for risk, safety and maintenance assessments

Keywords

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Abstract

Assessing the Occurrence Probability of a given sequence of events in a determined order is necessary in many scientific fields. That is the case in the following fields: nucleation and microstructure growth in materials, Narrow-Band process, financial risk analysis, Sequential detection theory, rainfall modelling, in optics to model the sequences of photoelectrons under detection, population biology, software reliability, queuing in network traffic exhibiting long-range dependence behaviour, and DNA sequences and gene time expression modelling. However, the topic has a particular interest in the field of risk, safety and maintenance assessments. The lecture will focus on sequences composed of Double Stochastic Poisson Processes.

1. Introduction

For events involved in Poisson Stochastic Processes, analysts may often be interested in calculating, within a determined interval of time, the probability of the k^{th} occurrence of a sequence of double-events. This problem is generally identified under the title "Double Stochastic Poisson Process-DSPP" or "Modulated Poisson Process-MPP".

The DSPP are used in many fields such as: nucleation and microstructure growth in materials, [1], Narrow-Band process, [2], financial risk analysis, [3] [4], Sequential detection theory, [5] [6] [7], rainfall modelling, [8], in optics to model the sequences of photoelectrons under detection, [9], population biology, [10] and software reliability, [11]. The subject is of high interest such that many researchers develop software and numerical methods to fit observations with DSPP covering a wide spectrum of applications. An almost exhaustive account, of the most popular methods, is given in [12] and methods are benchmarked. P. Salvador, R. Valadas, and A. Pacheco, [13], developed an efficient algorithm of fitting in order to estimate accurately queuing behaviour for network traffic exhibiting long-range dependence behaviour.

Beyond the DSPP, higher order Stochastic Poisson Processes can be involved in varying fields such as: DNA sequences and gene time expression modelling, [14][15].

We are mainly interested in developing an analytical solution for the Probability Distribution Function (PDF) of the occurrence of a given sequence of n-ordered events k-times within a defined interval of time. Here, events have constant occurrence rates (Poisson Stochastic Processes). In the case of two-event sequences, we demonstrate that the PDF of the k^{th} occurrence can be generated using some generating functions. As these generating functions allow determining the PDF of the k^{th} occurrence, $P_k(t)$, as a function of the number of repetition of the sequence. Subsequently, it may be called Spectral Probability Generating Functions. The PDF can alos be called Spectral probability Functions (SPF). The SPF's do, in some way, determine the relative weight of each number of repetitions of a given sequence.

We are going to demonstrate that the $P_k(t)$, can be expressed on the following form:

$$P_k(t) = \Psi_k(t).e^{-\lambda t} - \Phi_k(t).e^{-\mu t} \quad (1)$$

Where, $\Psi_k(t)$ and $\Phi_k(t)$ are polynomials of order k, λ and μ are the constant occurrence rate of the 1st and the 2nd events, respectively.

A more general analytical solution covering sequences of k-independent events (Multi-Stochastic Poisson Process) is under investigations.

The idea of decomposing the solution of Birth-Death problems (with/without killing) with the help of polynomials has appeared early, [16][17]. Since, the subject continue to receive the interest of many researchers, [18][19][20]. In a very synthetic and interesting paper, [21], Langovoy gives a general count of the use of the special algebraic polynomials in solving stochastic integrals.

Some researchers may describe this "decomposing potential" in different terms. Peccati, [22], presents the DSPP rather as a polynomial of degree 2 in m variables (Poisson-Charlier polynomials). He was trying to characterize the convergence of the distribution towards a bi-variable Gaussian law. This a-priori assumption (a bi-variable Gaussian Law) did certainly limit his essay and did not allow investigating other options.

Bartlomiej Blaszczyszyn and René Schott, [23], have also proved that the intensity measure of the Voronoi tessellation of Euclidian space which is generated by an inhomogeneous Poisson point process admits an approximate "decompositions formula". Many similar aspects can be underlined between the "approximating decomposition formula" given in [23] and the exact solution given her for the k-succession of the DSPP.

It is not the aim of the paper to present an historical and complete count of the use of polynomials to solve the stochastic integral describing the DSPP. The unique subject of the paper is: to demonstrate the existence of an analytical solution to the stochastic integral describing the k^{th} occurrence of a DSPP and to demonstrate that the solution may be generated using some-well defined polynomials of k^{th} order. The solution of the Stochastic Integral using the formula we present will be called "Spectral Probability Functions, SPF".

2. General description

The stochastic events "a" and "b" obey to a homogeneous Poison Process and are characterised by their occurrence rate, respectively, λ h^{-1} for "a" and μ h^{-1} for "b". An initiating event T occurs when a and b occur in the given order "a then b". The PDF, $P_k(t)$, determines the probability of having the sequence "a then b" occurring only "k" times within the interval $[0,t]$. It is described by following stochastic integral:

$$P_k(t) = \int_{\xi=0}^{t}\int_{\eta=\xi}^{t} P_{k-1}(\xi).\lambda d\xi.e^{-\mu(\eta-\xi)}.\mu d\eta.e^{-\lambda(t-\eta)}, \qquad (2)$$

where, $k = 1,2,3,...$ and $P_0(t) = e^{-\lambda t}$

In order to generate the PDF $P_k(t)$, we propose using another function, $F_k(t)$, defined as:

$$F_k(t) = \left(P_k(t).e^{\lambda t}\right), F_0(t) = 1.$$

3. Solution of the stochastic integral

We demonstrate in [24] that the stochastic integral, Eq.(2), may have the following solution:

$$P_k(t) = \left(\frac{\lambda\mu}{\sigma^2}\right)^k.\left[e^{-\lambda t}.\sum_{j=0}^{k}(-1)^j.C_j^k\frac{(\sigma t)^{k-j}}{k-j!} - (-1)^k.e^{-\mu t}.\sum_{j=0}^{k}B_j^k\frac{(\sigma t)^{k-j}}{k-j!}\right] \qquad (3)$$

where

$$\sigma = (\mu - \lambda).$$

We can then distinguish two polynomials $\Psi_k(\sigma t)$ and $\Phi_k(\sigma t)$, defined as following:

$$\Psi_k(\sigma t) = \left(\frac{\lambda\mu}{\sigma^2}\right)^k.\left[\sum_{j=0}^{k}(-1)^j.C_j^k\frac{(\sigma t)^{k-j}}{k-j!}\right],$$

$$\Phi_k(\sigma t) = \left(\frac{\lambda\mu}{\sigma^2}\right)^k.\left[(-1)^k.\sum_{j=0}^{k}B_j^k\frac{(\sigma t)^{k-j}}{k-j!}\right]$$

Where,

$C_0^k = 1,$

$B_0^k = 0 \quad k \ge 0$

$C_k^k = B_k^k,$

$B_k^k = C_{k-1}^k + B_{k-1}^k, k \ge 1$

$$C^{k}_{j-1} = C^{k}_{j-2} + C^{k-1}_{j-1} \,,$$

$$B^{k}_{j-1} = B^{k}_{j-2} + B^{k-1}_{j-1} \,, k \geq j \geq 2$$

We illustrate some numerical values of the coefficients C^k_j and B^k_j in *Table 1* and *Table 2* below.

Table 1. some numerical values of Coefficient C^k_j

J=	0	1	2	3	4	5	6
C^0_j	1						
C^1_j	1	1					
C^2_j	1	2	3				
C^3_j	1	3	6	10			
C^4_j	1	4	10	20	35		
C^5_j	1	5	15	35	70	126	
C^6_j	1	6	21	56	126	252	462

Table 2. some numerical values of Coefficient B^k_j

J=	0	1	2	3	4	5	6
B^0_j	0						
B^1_j	0	1					
B^2_j	0	1	3				
B^3_j	0	1	4	10			
B^4_j	0	1	5	15	35		
B^5_j	0	1	6	21	56	126	
B^6_j	0	1	7	28	84	210	462

4. Application

In some future industrial power system, an undesired initiating event would occur if two events a and b occur in the given order a then b. The event a (pressure higher than 160 bars for few minutes) is characterised by an occurrence rate equal to 10^{-5}/h. The event b (temperature higher than 750 °C for few minutes) is characterised by an occurrence rate equal to 10^{-3}/h. Both basic events are independent. The safety level of the power system can tolerate a limited occurrence of this initiating event within a given interval of time. Still, we should determine it. The detection of this event is difficult to carry out. We should asses the safety of the power system as a function of the probability of the k^{th} occurrence of the initiating event over an interval of 6 months (full nominal operation). The 2[nd] aim is to optimize the detection rate of the detection system to be designed.
The probability of the k^{th} occurrence of this initiating event over an interval of 6 months (full nominal operation) is calculated using the spectral probability function P_k given in Eq.(3) and the figures are given in *Table 3*, below.

Table 3. probability of the k^{th} occurrence over an interval of 6 months

k	1	2	3	4	5
p_k	$3.3*10^{-2}$	$3.7*10^{-4}$	$1.9*10^{-6}$	$6.0*10^{-9}$	$1.2*10^{-11}$

The profile of the Spectral Probability P_k with the time is given in *Figure 1*, as well. We can evaluate some interesting characteristics such as the most probable time interval for each occurrence order k.

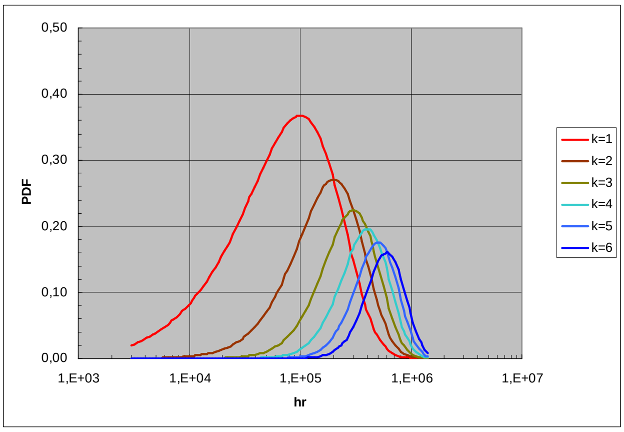

Figure 1. The Spectral Probability Functions up to $k = 6$

5. Conclusions

The stochastic integral describing the probability of the k^{th} occurrence of a double Poisson process has an analytical solution on the form of Spectral Polynomials. That is of great interest in Spectral Analysis & Optimisation activities in Risk & System Safety.
In future industrial systems, the concept of "tolerance to failures" requires the establishment of models allowing the determination of the occurrence

probabilities of the initiating events as functions of the repetition order of the undesired sequences. We do “Spectral Analysis”.

The author is attempting to extend these spectral models to sequences comprising more than double stochastic Poisson events. However, attempts are not successful up to now.

Admitting the possibility that an “Initiating Events” may be repetitive is equal to “admitting the system tolerance to some sets of undesired sequences”.

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