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The approximate location of imperfections in an ellipse using the spectral theory

Abstract

In the paper, the authors describe the analysis and results of research on the approximate location of a defect with the help of the spectral theory for the area which is a geometrical ellipse. A computer simulation was conducted in Matlab. At the end of the paper, the authors give an example to illustrate the method of determining the areas in which the ellipse may be damaged.

Keywords: defect location, Laplace operator, eigenvalues.

1. Introduction

This paper is a continuation of our research on the issue of fault location (discontinuities in the material) for various figures using spectral theory. The theoretical basis for our discussion is based on the spectral theorem which says that all eigenvalues of the Laplace operator on the bounded $\Omega \subseteq \mathbb{R}^2$ domain are positive, have finite multiplicities and $+\infty$ is the limit point of eigenvalues. The solutions of a similar problem for a circle and square are described in [1], [2]. The present paper shows the results for the geometric area in the form of an ellipse. We conducted a computer simulation in Matlab using Partial Differential Equation Toolbox (pdetool). Based on the results (measurements), isochore maps were created with the help of which we give the approximate location of the damage.

2. The definition of the problem for the domain without the imperfection

Let the domain $\Omega \subset \mathbb{R}^2$ be given. For the Ω domain and for the k=1,2,.. we define the problem

$$\begin{cases} Au_k^0(x,y) = \lambda_k^0 u_k^0(x,y) \ in \ \Omega \\ u_k^0(x,y) = 0 \qquad on \ \partial\Omega \end{cases} \tag{1}$$

where A is such a linear operator A: $H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ that the operator $-A$ is an elliptic operator and an operator inverse A^{-1}:$L^2(\Omega) \rightarrow L^2(\Omega)$, is a compact operator. Eigenvalues and eigenfunctions $(\lambda_k^0, u_k^0(x,y))$ of the problem (1) refer to the domain without deformation.

3. The definition of the problem for the domain with the imperfection

Let us consider the Ω domain with the D imperfection. The imperfection is a circle with radius r = 0.01. Let us mark the Ω domain with the described imperfection as Ω_D. Then we get $\Omega_D = \Omega \backslash D$ and we consider the following spectral problem for the Ω_D domain and k=1, 2,…

$$\begin{cases} Au_k(x,y) = \lambda_k u_k(x,y) \ in \ \Omega_D \\ u_k(x,y) = 0 \qquad on \ \partial\Omega_D \end{cases} \tag{2}$$

The solution to the problem above is an infinite sequence of pairs $(\lambda_k, u_k(x,y))$ for k=1,2,..., where λ_k are eigenvalues and $u_k(x,y)$ are eigenfunctions. These eigenvalues are dependent on the location and size of the imperfection D.

4. The definition of the inverse problem

The main aim of our study is to solve the inverse task, which consists in localizing the imperfection (coordinates (x_0, y_0)) in the ellipse basing on the spectrum of the Laplace operator for this ellipse

$$\sigma(A) \rightarrow (x_0, y_0) \tag{3}$$

5. Computer simulation

The computer simulation was carried out in MATLAB program using the PDE Tool package. In the mathematical model, the Laplace equation defined for the ellipse domain was taken as an elliptic operator

$$Au(x,y) := \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y). \tag{4}$$

The operator is elliptic if and only if for every (x,y) in $\Omega \subset \mathbb{R}^2$ and every non-zero $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$

$$\sum_{i,j=1,2} a_{ij}(x,y) \xi_i \xi_j \neq 0. \tag{5}$$

In our case, we have $a_{11} = a_{22} = 1, a_{12} = a_{21} = 0$. So it is clear that $\xi_1^2 + \xi_2^2 \neq 0$ for every (x,y) in $\Omega \subset \mathbb{R}^2$ and every non-zero $\xi = (\xi_1, \xi_2)$ in $\mathbb{R}^2$.

The main aim of the numerical simulation is to observe the behavior of the spectrum of the Laplace operator for the ellipse when the deformation is located at different points of this ellipse. To do this, we chose a grid of equidistant points (Fig. 1) to create the isochore maps for the minimum and maximum values of this spectrum (Figs. 4 and 7). These maps will now serve to determine the approximate areas with defects.

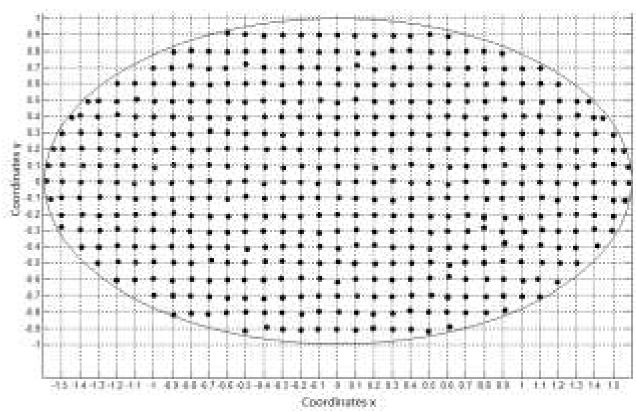

Fig. 1. The grid of points which the deformation was placed at

More precisely, we analyzed the spectrum for every subsequent point, which we placed the deformation at (Fig. 1). For every point (deformation) we used the undermentioned algorithm to get the maximum and minimum values of the eigenvalues for it.

The algorithm:
Step 1. An ellipse is used as domain Ω.
Step 2. The deformation is placed at the fixed point of this ellipse.
Step 3. The area containing defects is covered with grid points (triangulation of the domain with deformation).
Step 4. The type of the equation and its specification is chosen (Eigenmodes, a = 1:0, c = 0:0, d = 1:0).
Step 5. Eigenvalues (λ_{min} and λ_{max}) are obtained.

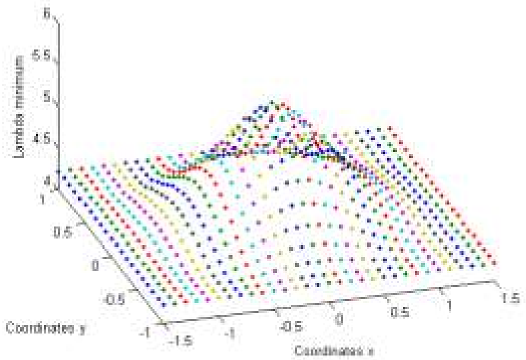

Fig. 2. `Scattered data obtained for the minimum values of eigenvalues λ_{min}

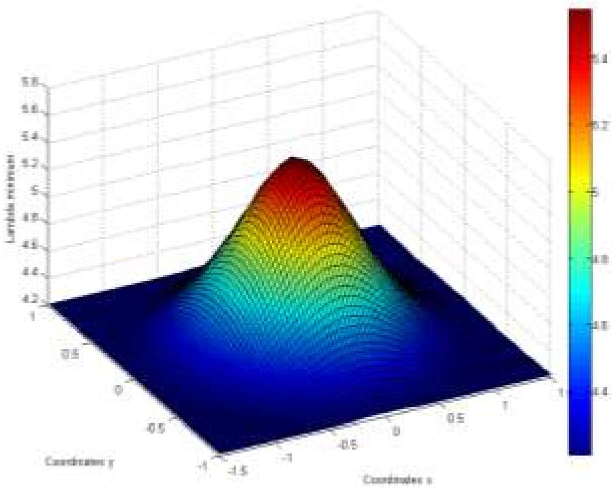

Fig. 3. Three-dimensional surface made from the scattered data of λ_{min}

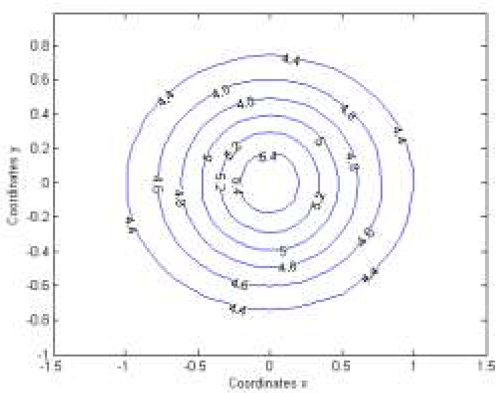

Fig. 4. Isochor map I_{min}

Fig. 2 presents scattered data which we get for minimum values of eigenvalues λ_{min}. From this data we make a three-dimensional surface (Fig. 3). Now that the data is in a gridded format, we compute and plot the contours of isochors (lines connecting points of the same value) (Fig. 4). We will use this map to read approximate location of the imperfection.

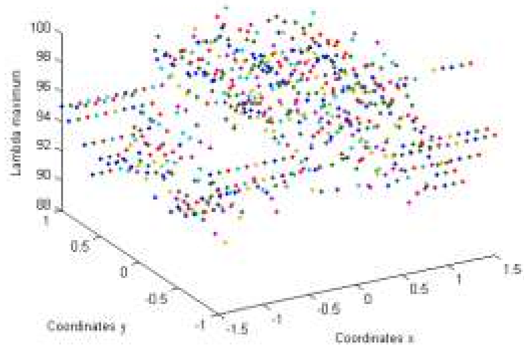

Fig. 5. Scattered data obtained for the maximum values of eigenvalues λ_{max}

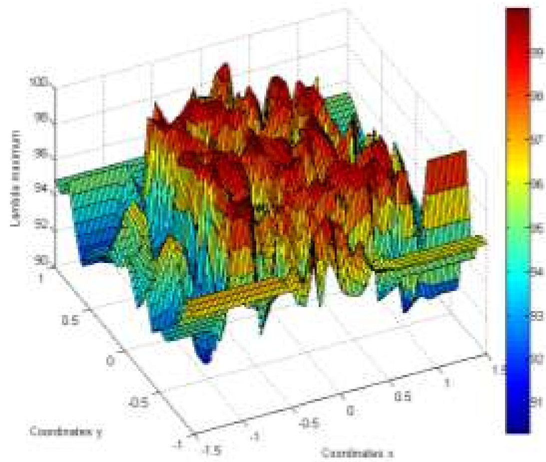

Fig. 6. Three-dimensional surface made from the scattered data of λ_{max}

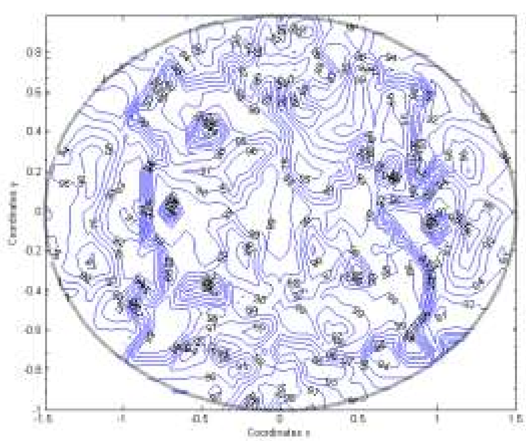

Fig. 7. Isochor map I_{max}

The same analysis will be carried out for the maximum values λ_{max} of the spectrum of the Laplace operator. Fig. 5 presents scattered data for maximum values of eigenvalues λ_{max}. From this data we make a three-dimensional surface (Fig. 6). Fig. 7 presents the projection of the three-dimensional surface of minimum values of eigenvalues λ_{max} on XY plane.

Now, when we have the isochors of the minimum and maximum values of eigenvalues, we are able to determine the domains, where the deformation will be located in. These domains will be the common parts of the isochors maps I_{min} and I_{max} which correspond to the measurement λ_{min} and λ_{max} .

6. Example

As an illustrative example let us now consider the spectrum of form [4.94, 10.07, 13.13, 15.46, 19.98, 24.86, 28.37, 28.88, 36.35, 39.93, 40.72, 49.26, 50.27, 51.57, 54.42, 63.97, 65.52, 68.52, 68.55, 71.07, 78.33, 80.43, 85.39, 89.72, 91.71, 98.41, 99.88]. For this measurement, we have the minimum value λ_{min}= 4.94, and maximum value λ_{max}= 99.88. From the isochore maps, I_{min} and I_{max}, we select the appropriate areas which after merging together give the approximate areas where the defect is located. For the minimum value, the range is defined with lines $4.8 < \lambda_{min} < 5$. For the maximum values, the inequality takes the form $99 < \lambda_{max}$. In Fig. 8, the authors indicated the approximate areas (gray) with the defect.

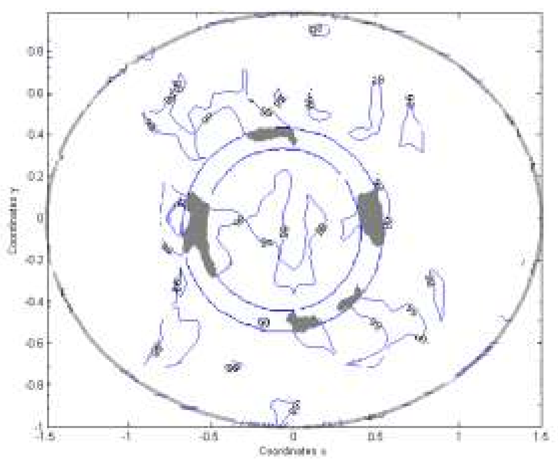

Fig. 8. Approximate area where the defect can be found (gray)

7. Concluding remarks

As the numerical simulation shows, basing on the spectrum of the Laplace operator for the ellipse with the deformation, we are able to define the domains, where deformations are located in. This method does not solve the problem, unfortunately, a clear inverse searching damage, but greatly simplifies the search for further damage.

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8. References

[1] Brzęk M., Mitkowski W.: Lokalizacja uszkodzeń w zadanym obszarze z wykorzystaniem teorii spektralnej. Pomiary Automatyka Kontrola, 2014, vol. 60, no. 1, pp. 53-55.

[2] Brzęk M., Zagórowska M., Mitkowski W.: The approximate location of imperfection in a unit circle using spectrum of Laplace operator as a research tool. Automatyka/Automatics, 2015, vol.19, no. 1.

[3] Lipnicka M.: Przybliżona lokalizacja uszkodzeń w zadanym obszarze. Biuletyn Polskiego Towarzystwa Matematycznego Tom, 2011, 13/54, s. 47-78.

[4] Jackowska-Strumiłło L., Sokołowski J., Żochowski A., Henrot A.: On Numerical Solution of Shape Inverse Problem. Computational Optimization and Application, 2002, vol. 23, s. 231-255.

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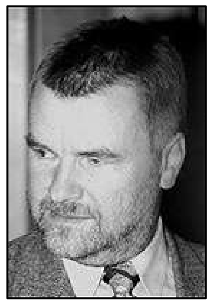

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