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REMARKS ABOUT DISCRETE YOUNG MEASURES AND THEIR MONTE CARLO SIMULATION

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Abstract. This article is devoted to the problem of simulation of random variables distributed according to Young measures associated with piecewise affine functions determined on bounded intervals. We start with simple functions which can take on a finite number of different values with inverse images being the intervals or their unions. We present some formal results connected with related discrete Young measures and propose an algorithm for generating random variables having such distributions. Next, based on these results we introduce an algorithm designed for approximation of Young measures in various, more general situations. We also present an example where a Young measure associated with a piecewise affine function is approximated with the help of computer simulations. In this benchmarking problem the theoretical results are compared with the ones obtained in the Monte Carlo experiment.

Keywords: *Young measures, Monte Carlo simulation, simple function, piecewise affine function*

1. Introduction

Young measures occur while investigating limits of sequences of bounded fast oscillating functions, for example the minimizing sequences of energy functionals that do not attain their infima. Such sequences are divergent in the strong topology, but they are weakly* convergent to a function, which does not minimize the energy functional. Laurence Chisolm Young proposed in [1] enlarging the space of functions to the measure spaces and to consider generalized limits of the minimizing sequences there (in fact, the generalized limits of the composition of the elements of the minimizing sequence with a continuous function satisfying certain growth conditions). This allows one to analyse the oscillatory nature of the sequences and therefore the underlying microstructure arising for example in phase transition in certain elastic crystals. Young called these measure limits "generalized trajectories"; today they are called "Young measures" (sometimes also "relaxed trajectories" or "transition probabilities").

The main problem concerning Young measures is that obtaining their explicit form, i.e. calculating the weak* limit of an appropriate sequence of functions is rather difficult. A somewhat simpler approach has been proposed in [2]; it uses the notion of a *quasi-Young measure*, which is easier to calculate and is quite often equal to the "original" Young measure. In particular, the (quasi-)Young measure associated with a constant function is merely a Dirac measure concentrated at the point being the value of that function. On the other hand, the generalized limits of the sequences minimizing multi-well potentials are exactly the same as Young measures concentrated on the wells (see for instance [3, 4]). Thus discrete Young measures play important role both in the theory and applications.

So, the theoretical analysis of Young measures in many important cases can be very difficult or even impossible to perform. Usually in such a case when the formal analysis of a probabilistic model is extremely complex, the Monte Carlo simulation approach can be adopted, see e.g. [5-7]. Here the term "Monte Carlo simulation" refers to the analysis of a stochastic phenomenon through the generation of sample realizations (observations) of the random variable under study with the help of computer codes involving random-number generators. The generation of random numbers distributed according to specific Young measure (or its good approximation) has two important possible applications. First, it makes it possible to analyse the "empirical" distribution related to this probability measure in a classical statistical manner. Second, such Young-measure generators can be used for simulation analysis of complex stochastic systems with only some factors (parameters and/or variables) distributed according to this probability distribution.

In this paper we propose an algorithm for performing such simulations in a case where the Young measure is associated with a simple function. Next, the introduced generator is used for *approximation* of Young measures in some more general situations.

The article is organized as follows. In Section 2 we recall the definition and basic properties of the discrete Young measures and give an example of such a measure associated with a simple function. The reader who wants go deeper into the subject is referred to [8, 9]. A very concise introduction based entirely on [9] may also be found in the section 2 in [2]. In Section 3, the Young measures' generator is described and some benchmarking experiments are discussed. In these experiments we generate Young measures which have a known explicit form, so we can compare the simulation and the formal results. Section 4 is devoted to the approximation of Young measures in some more general cases. Here we present some examples of statistical analysis of an empirical distribution that approximates the underlying (true) Young measure associated with specific piecewise functions.

2. Quasi-Young measures associated with affine functions

Let Ω be an open, bounded subset of $\mathbb{R}$, dx - a Lebesgue measure on Ω, $M > 0$ - Lebesgue measure of Ω. Denote $d\mu(x) := \frac{1}{M} dx$. Let $u\colon \Omega \to \mathbb{R}$ be an affine function, that is, a function of the form $u(x) = ax + b, a, b \in \mathbb{R}$. Observe that the range of u is a subset of a certain compact subset K of $\mathbb{R}$. The space of continuous functions from K to $\mathbb{R}$ will be denoted by $C(K, \mathbb{R})$.

Definition 2.1. We say that a family of probability measures $(\nu_x)_{x\in\Omega}$ is a quasi-Young measure associated with an affine function u, if for every $\beta \in C(K, \mathbb{R})$ there holds an equality

$$\int_K \beta(k) d\nu_x(k) = \int_\Omega \beta\big(u(x)\big) d\mu.$$

Proposition 2.2. (a) let $u: \Omega \to \mathbb{R}$ be a constant function: $\forall\, x \in \Omega\; u(x) = p$. Then the quasi-Young measure associated with u is the Dirac measure δ_p. **(b)** let $u: \Omega \to \mathbb{R}$ be an affine function: $\forall\, x \in \Omega\; u(x) = ax + b$, with $a, b \in \mathbb{R}$ and $a \neq 0$. Then the quasi-Young measure associated with u is the measure absolutely continuous with respect to the Lebesgue measure dy on K. Its density is equal to $\frac{1}{M}|\frac{1}{a}|$.

Proof. Using the change of variable theorem we get

$$\int_K \beta(k)\, d\nu_x(k) = \int_\Omega \beta\big(u(x)\big)\, d\mu(x) = \beta(p) = \int_K \beta(y)\, d\delta_p,$$

which proves (a). Analogously,

$$\int_K \beta(k)\, d\nu_x(k) = \int_\Omega \beta(ax + b)\, d\mu(x) = \frac{1}{M}\int_K \beta(y)\left|\frac{1}{a}\right| dy,$$

which is (b). ■

Corollary 2.3. Denote by $\{I_i\}_{i=1}^n$ an open partition of Ω, such that $\bigcup_{i=1}^n \mathrm{cl} I_i = \mathrm{cl}\Omega$, where "cl" stands for "closure". Denote by m_i the Lebesgue measure of the set I_i, $i = 1,2, \dots, n$. Let u be a simple function on Ω. We can write $u(x) = \sum_{i=1}^n p_i \chi_{I_i}(x)$, $p_i \in \mathbb{R}, i = 1,2, \dots, n$. Here χ_A denotes, as usual in such context, the characteristic function of the set A. Using the mathematical induction we can prove that the quasi-Young measure associated with u of the form

$$\nu_x = \frac{1}{M}\sum_{i=1}^n m_i\, \delta_{p_i}.$$

Further, let for each $i = 1,2, \dots, n$, $u_i(x) = a_i x + b$, that is u is affine on I_i, with $a_i \neq 0$ and $\bigcup_{i=1}^n \mathrm{cl} u_i(I_i) = K$, and such that $u(x) := \sum_{i=1}^n u_i(x)\chi_{I_i}(x)$ is continuous on Ω. Let $\{B_1, B_2\}$ be a partition of the set of indices $\{1,2, \dots, n\}$ such that if $i \in B_1$ then the function u_i is strictly increasing; otherwise - strictly decreasing. Choose and fix such $l \in B_1, m \in B_2$, that $K_s := J_{lm} := u_l(I_l) \cap u_m(I_m) \neq \emptyset$. Then we have $\bigcup_{s=1}^r K_s = K$ and the quasi-Young measure associated with u is absolutely continuous with respect to the Lebesgue measure on K with density

$$g(y)=\begin{cases}\frac{1}{M}\sum_{s_1}\left|\frac{1}{a_{s_1}}\right|, & \text{if } y\in K_1\\ \vdots & \\ \frac{1}{M}\sum_{s_r}\left|\frac{1}{a_{s_r}}\right|, & \text{if } y\in K_r\end{cases}.$$

Remark 2.4. Observe that in the above cases the quasi-Young measure does not depend on the variable x. Such a (quasi-)Young measure is called *homogeneus*. Therefore in what follows we will omit the subscript x in the symbol ν_x.

Remark 2.5. By the theorem 6.1 of [2], the quasi-Young measures associated with the functions of the above proposition and its corollary are equal to the Young measures associated with these functions.

Example 2.6. Let $\Omega := (-2,2)$ and define

$$u(x):=\begin{cases}-2, \text{ if } x\in(-2,-1]\\ 1, \text{ if } x\in(-1,0]\\ -1, \text{ if } x\in(0,1]\\ 0, \text{ if } x\in(1,2]\end{cases}.$$

Then the Young measure associated with u is of the form

$$\nu=\frac{1}{4}\delta_{-2}+\frac{1}{4}\delta_1+\frac{1}{4}\delta_{-1}+\frac{1}{4}\delta_0.$$

Example 2.7. Define

$$u(x):=\begin{cases}4x, \text{ if } x\in(0,\frac{1}{4}]\\ x+\frac{3}{4}, \text{ if } x\in(\frac{1}{4},\frac{5}{4}]\\ -2x+\frac{9}{2}, \text{ if } x\in\left(\frac{5}{4},2\right]\\ -\frac{1}{4}x+1, \text{ if } x\in(2,4)\end{cases}. \tag{1}$$

Here $\Omega=(0,4)$, $K=\mathrm{cl}g(\Omega)=[0,2]$, $d\mu(x):=\frac{1}{4}dx$.

Then we have

$$\int_0^2\beta(k)d\nu(k)=\int_0^{\frac{1}{2}}\beta(y)\frac{17}{16}dy+\int_{\frac{1}{2}}^1\beta(y)\frac{3}{16}dy+\int_1^2\beta(y)\frac{3}{8}dy.$$

Therefore $\nu=g(y)dy$, where

$$g(y)=\begin{cases}\frac{17}{16}, \text{ if y}\in[0,\frac{1}{2}]\\ \frac{3}{16}, \text{ if y}\in(\frac{1}{2},1].\\ \frac{3}{8}, \text{ if y}\in(1,2]\end{cases} \tag{2}$$

3. Generating of Young measures associated with piecewise functions

The idea of the Monte Carlo simulation is to draw a sample i.e. a realization of the stochastic process $\{Z_1, Z_2, \ldots Z_m\}$ composed of independent random variables with the same distribution as the random phenomenon under study. Based on this sample, important information concerning stochastic characteristics of the examined distribution can be derived with the help of statistical-inference tools. Indeed, by the strong law of large numbers, for any Borel function f for which the expected value E$f(Z)$ exists, the average $\bar{f}_m = \frac{1}{m}\sum_{i=1}^{m} f(Z_i)$ will almost surely (*a.s.*) converge to E$f(Z)$. In particular, when the sample size m tends to infinity, we can quite precisely evaluate all moments of the investigated distribution (e.g. expected value, variance) as well as probabilities of random events. The latter can be used for evaluating the theoretical frequencies of various intervals, so we can also obtain a histogram that approximates the distribution density function. The approximation of the density function is the better, the larger is the observation sequence, but in the Monte Carlo experiment we can usually receive as many observations as we need.

Now let us consider simple functions f defined on a bounded interval I, and such that their values have inverse images being the intervals or their unions. By Proposition 2.2, Corollary 2.3 and Remark 2.5, the Young measures associated with such functions are the discrete probability distributions of a very simple form which can be easily simulated by computer procedures. In Monte Carlo simulations, the sample of random variables having such distributions can be generated according the following routine DYM(f , I, N).

```
Set k = 1;
While n ≤ N Do Step 1 to Step 3
   Step 1. Set t = Random(I)
   Step 2. Set z[k]=f(t)
   Step 3. Set k = k + 1
Set sample = (z[1],...,z[N])
Return sample
```

The procedure DYM is called with three arguments:

– the formula f that defines the simple function and
– its domain, i.e. the interval I,
– the sample size N.

The subroutine `Random(I)` returns a pseudorandom number generated according to the uniform probability distribution defined on I.

The routine DYM can also be used in order to approximate Young measures in cases that are a bit more sophisticated. It is well known from the measure theory that any Borel function can be approximated with the simple function (more precisely, it is a limit of a proper sequence of simple functions) Thus it can be expected that for a large class of functions, we can approximate related Young measures by a properly chosen *simple* function. In this paper, such an approximation will be addressed to as *simple approximation*, meaning the simple function that is used for the approximation. Here we propose the following construction of simple approximation for any piecewise function f determined on the interval I.

– Split the interval $I = (a,b)$ into n subintervals equal in length $I_1,...,I_n$,
– Choose the sequence $y_i = f(x_i)$, $i = 1,...,n$, where x_i is the centre of the subinterval I_i,
– As simple approximation of f choose the following simple function:

$$u(x) := \sum_{k=1}^{n} y_k \chi_{I_k}(x), \quad x \in I$$

Below we present routine AppYM(f, a, b, N, n) that realizes the above approximation and returns a sample from the (approximated) Young measure related to any piecewise function.

```
Set k = 1;
Set jump = (b-a)/n
While k ≤ n Do Step 1 to Step 3
   Step 1. Set t = a+(k-1/2)*jump
   Step 2. Set y[k] = f(t)
   Step 3. Set k = k+1
Set i=1;
While i ≤ N Do Step 4 to Step 6
   Step 4. Set k = RandomI({1,..,n})
   Step 5. Set z[i] = y(k)
   Step 6. Set i = i+1
Set sample = (z[1],...,z[N])
Return sample
```

The arguments for the AppYM are the following:

– the formula f that defines the simple function,
– the simple-function's domain, i.e. the endpoints of the interval $I = (a,b)$,
– the number n of subintervals $I_1,...,I_n$,
– the sample size N.

The subroutine `RandomI(A)` returns a pseudorandom integer number generated according to the uniform probability distribution defined on A - a finite subset of integers.

4. Examples of Monte Carlo approximation of Young measures

In this section we present examples of possible applications of the AppYM routine. Let us start with Example 2.7. The graph of the function f considered in that example (see Eq. (1)) is presented in Figure 1.

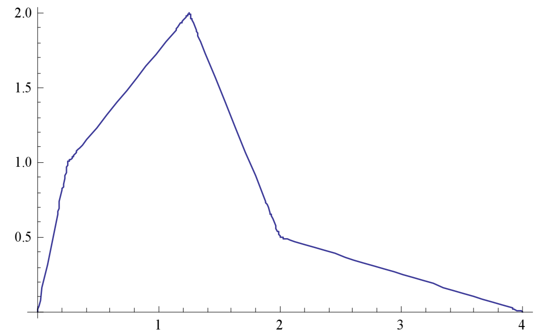

Fig. 1. Graph of piecewise function f given by (1)

The probability density function g of the Young measure associated with this function is given by formula (2) and its graph is presented in Figure 2.

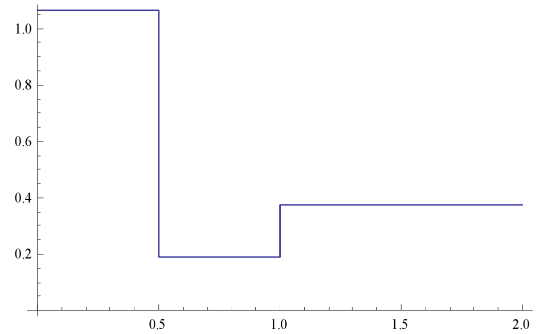

Fig. 2. Graph of the PDF g given by (2)

Now we perform a benchmarking experiment with the help of the routine AppYM. We assume the following arguments in our simulation: $N = 1\,000\,000$ and $n = 20\,000$. As a result we obtain the following histogram that approximates the function g, see (2).

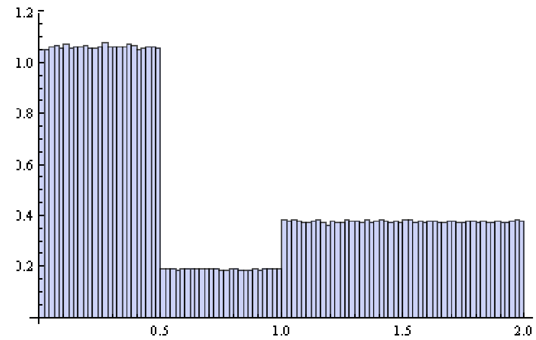

Fig. 3. The histogram that approximates the function g given by (2). The data were generated with the help of the routine AppYM

We see that the approximation is really good. In order to verify this impression more precisely, we formally state the hypothesis that the data was obtained from random variable distributed according to the Young measure given by g, see (2). To verify this hypothesis, we adopt the Pearson goodness-of-fit chi-square test. For that purpose we split the observation region into 200 separate classes and compute the usual Pearson χ^2 statistics. We receive χ^2 = 198.2, a value that is close to the mean of the χ^2 distribution with 199 degrees of freedom. This fact confirms that our generator works really well.

5. Conclusions

The formal analysis of Young measures is usually a very difficult task. On the other hand, in various engineering problems it is very important to know at least some probabilistic characteristics of these measures. One of the possible solutions in such a case is to make use of the Monte Carlo simulation. We show that with the help of a rather simple computer routine, we can generate random numbers distributed according to the Young measure associated with any piecewise function. The benchmarking example presented in Section 4 confirmed that the proposed method is quite effective and may be very useful. It should also be emphasized that the described simulations are very fast - it takes seconds to receive data containing 1 000 000 numbers. Statistical analysis of such data set may be a very important source of information about the considered Young measure.

References

[1] Young L.C., Generalized curves and the existence of an attained absolute minimum in the calculus of variations, Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, classe III, 1937, 212-234.

[2] Puchała P., An elementary method of calculating an explicit form of Young measures in some special cases, Optimization 2014, 63, 9, 1419-1430.

[3] Bielski W., Kruglenko E., Telega J.J., Miary Younga i ich zastosowania w mikromechanice i optymalizacji. Część I. Podstawy matematyczne, Matematyka Stosowana 2003, 4, 90-138.

[4] Nicolaides R.A., Walkington N.J., Computation of microstructure utilizing Young measure representations, J. Intelligent Materials Systems and Structures 1993, 4, 795-800.

[5] Spall J.C., Introduction to Stochastic Search and Optimization; Estimation, Simulation, and Control, A John Wiley & Sons. Inc. Publication, 2003.

[6] Grzybowski A.Z., Simulation approach to optimal stopping in some blackjack type problems, Scientific Research of the Institute of Mathematics and Computer Science 2011, 10, 2, 75-86.

[7] Bartłomiejczyk K., Chain-like structures' motion - metamodeling of the translocation time in some specific situations, Journal of Applied Mathematics and Computational Mechanics 2014, 13, 3, 5-12.

[8] Pedregal P., Variational Methods in Nonlinear Elasticity, SIAM, Philadelphia 2000.

[9] Roubíček T., Relaxation in Optimization Theory and Variational Calculus, Walter de Gruyter, Berlin, New York 1997.