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STANDARD AND FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS WITH ZERO TRANSFER MATRICES

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Abstract: The transfer matrix of the standard and fractional linear discrete-time linear systems is investigated. Necessary and sufficient conditions for zeroing of the transfer matrix of the linear discrete-time systems are established. The considerations are illustrated by examples of the standard and fractional linear discrete-time systems.

Key words: fractional, discrete-time, linear system, observability, reachability, zero transfer matrix

1. INTRODUCTION

The notion of controllability and observability and the decomposition of linear systems have been introduced by Kalman [13, 14]. This theory was developed in the following years (e.g. Kailath [12], Klamka [15], Rosenbrock [22]), and became the basic concepts of the modern control theory (e.g. Antsaklis [2], Farina and Rinaldi [5], Poldermann [21]) and modern data-driven system theory and control (see e.g. Dörfler et al. [4], Markovsky and Dörfler [16] and other works cited in this paper). The notion of controllability and observability have been also extended to positive linear systems [5, 9] and electrical circuits [7, 11]. A dynamical system is called positive if its trajectory starting from any non-negative initial state remains forever in the positive orthant for all non-negative inputs. Variety of positive models can be found in electrical engineering, economics, social sciences, biology and medicine, etc.

Fractional calculus is the branch of mathematics that studies integrals and derivatives of non-integer order. Mathematical fundamentals of the fractional calculus are given in various monographs (e.g. Oldham and Spanier [18], Ostalczyk [19], Podlubny [20]). The fractional calculus and its application in many fields of science and engineering have been recently investigated. Numerous applications have been found in mechanics, electricity, chemistry, signal processing, etc. [19, 24, 27]. Fractional-order models of real world phenomena have become more accurate than classical integer order ones. Theory of fractional systems is a rapidly growing field and it concerns properties of processes and control systems, including stability, controllability, observability, realisability, etc. [1, 3, 6, 7, 17, 23, 26, 28]. The standard and positive fractional linear systems have been investigated in monographs by Kaczorek [9] and Kaczorek and Rogowski [11] and the positive linear systems with different fractional orders have been analysed by Kaczorek [8] and Sajewski [25].

Transfer functions (matrices) are very popular in modelling physical phenomena and represent the relation between input and output signals. They are commonly used in the analysis of dynamical systems. In this paper the standard and fractional linear discrete-time systems with zero transfer matrices will be investigated. To the authors' knowledge, this problem for the fractional discrete-time linear systems has not been considered yet. This paper extends the theory of fractional-order systems on this topic.

The remainder of this paper is organised as follows. In Section 2 the basic definitions and theorems concerning the linear discrete-time are given and a class of standard linear discrete-time with zero transfer matrices is investigated. The basic definitions and theorems concerning the fractional linear discrete-time systems and an extension of the results of Section 2 are presented in Section 3. The considerations have been illustrated by linear discrete-time systems. Concluding remarks are given in Section 4.

The following notation will be used: $\Re$ is the set of real numbers; $\Re^{n\times m}$ represents the set of $n\times m$ real matrices; $\Re_+^{n\times m}$ denotes the set of $n\times m$ matrices with non-negative and $\Re_+^n=\Re_+^{n\times 1}$; and I_n is the $n\times n$ identity matrix.

2. STANDARD LINEAR DISCRETE-TIME SYSTEMS

Consider a linear discrete-time system described by the following equations:

$$x_{i+1}=Ax_i+Bu_i,\ i\in Z_+=\{0,\ 1,\ \dots\},\tag{2.1a}$$

$$y_i=Cx_i,\tag{2.1b}$$

with the initial condition x_0, where $x_i\in\Re^n$, $u_i\in\Re^m$ and $y_i\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$ and $C\in\Re^{p\times n}$.

The transfer matrix of the linear system (2.1) is given by the following equations:

$$T(z)=C[I_nz-A]^{-1}B.\tag{2.2}$$

Definition 2.1. [11, 15] The linear system (2.1) is called reachable in $q\le n$ steps if there exists an input $u_i\in\Re^m$ for $i=0,1,\dots,q\le n-1$ that transfers the state of the system from the initial state $x_0\in\Re^n$ to the given final state $x_f=x_q$ in the q steps.

Theorem 2.1. [11, 15] The linear system (2.1) is reachable in q steps if and only if one of the following equivalent conditions is satisfied:

$$rank[B,\ AB,\ \ldots,\ A^{n-1}B] = n, \tag{2.3}$$

$rank[I_n z - A,\ B] = n$ *for all* $z \in$ C (the field of complex numbers). (2.4)

Definition 2.2. [11] The linear system (2.1) is called observable in q steps if knowing the input u_i and the output y_i in the $q \le n - 1$ steps it is possible to find its unique initial state x_0.

Theorem 2.2. [11] The linear system (2.1) is observable in q steps if and only if one of the following equivalent conditions is satisfied:

$$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n, \tag{2.5}$$

$$rank\begin{bmatrix} I_n z - A \\ C \end{bmatrix} = n \text{ for all } z \in \mathrm{C}. \tag{2.6}$$

It is well-known [11] that if the linear system (2.1) is unreachable or unobservable then some cancelation of common factors in the numerator and denominator of the transfer matrix (2.2) occurs.

Theorem 2.3. Let the transfer matrix (2.2) of the linear system (2.1) be a zero matrix. Then:

1. if the matrix C is non-zero, then the pair $(A,\ B)$ of the linear system (2.1) is unreachable; and
2. if the matrix B is non-zero, then the pair $(A,\ C)$ of the linear system (2.1) is unobservable.

Proof. It is well-known that the impulse response matrix g_i of the linear system satisfies the condition

$$g_i = CA^{i-1}B = 0 \text{ for } i = 1,\ \ldots, n. \tag{2.7}$$

if and only if the transfer matrix (2.2) is a zero matrix.

From Eq. (2.7), we have

$$C[B,\ AB,\ \ldots,\ A^{n-1}B] = 0. \tag{2.8}$$

Therefore, if $C \neq 0$ then

$$rank[B,\ AB,\ \ldots,\ A^{n-1}B] < n \tag{2.9}$$

and the pair $(A,\ B)$ is unreachable.

Similarly, if $B \neq 0$ then

$$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} < n \tag{2.10}$$

and the pair $(A,\ C)$ is unobservable.

The following theorem gives the necessary and sufficient conditions for zeroing of the transfer matrix (2.2) of the linear system (2.1).

Theorem 2.4. The transfer matrix (2.2) of the unreachable and unobservable linear system (2.1) is a zero matrix if and only if $n \ge m + p$ and

$$CB = 0. \tag{2.11}$$

Proof. It is well known [2, 5, 9, 11] that if the pair $(A,\ B)$ is unreachable then there exists a non-singular matrix $P \in \Re^{n\times n}$ such that

$$PAP^{-1} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, CP^{-1} = [C_1 \quad C_2] \tag{2.12a}$$

where $A_1 \in \Re^{n_1\times n_1}$, $A_2 \in \Re^{n_1\times n_2}$, $A_3 \in \Re^{n_2\times n_2}$, $B_1 \in \Re^{n_1\times m}$, $C_1 \in \Re^{p\times n_1}$, $C_2 \in \Re^{p\times n_2}$, $n_1 + n_2 = n$,

$$rank[B,\ AB,\ \ldots,\ A^{n-1}B] = n_1 \tag{2.12b}$$

and the pair $(A_1,\ B_1)$ is reachable, i.e. $rank[B_1,\ A_1B_1,\ \ldots,\ {A_1}^{n_1-1}B_1] = n_1$.

Note that

$$CB = CP^{-1}PB = [C_1 \quad C_2]\begin{bmatrix} B_1 \\ 0 \end{bmatrix} = C_1B_1 = 0, \tag{2.13}$$

since the pair $(A,\ C)$ is unobservable and $C_1 = 0$.

Using (2.2) and (2.12a) we obtain

$$\begin{aligned} T(z) &= C[I_n z - A]^{-1}B = CP^{-1}[PP^{-1}z - PAP^{-1}]^{-1}PB \\ &= [C_1 \quad C_2]\begin{bmatrix} [I_{n_1}z - A_1] & -A_2 \\ 0 & [I_{n_2}z - A_3] \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ &= [C_1 \quad C_2]\begin{bmatrix} [I_{n_1}z - A_1]^{-1} & * \\ 0 & [I_{n_2}z - A_3]^{-1} \end{bmatrix}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ &= C_1[I_{n_1}z - A_1]^{-1}B_1 = 0 \end{aligned} \tag{2.14}$$

if and only if the condition (2.11) is satisfied, where * denotes a matrix unimportant in these considerations. Therefore, the transfer matrix (2.2) of the unreachable and unobservable linear system (2.1) is a zero matrix if and only if $n \ge m + p$ and the condition (2.11) is satisfied.

Example 2.1. Consider the linear system (2.1) with the matrices

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, C = [-0.5 \quad 1 \quad 0]. \tag{2.15}$$

The pair $(A,\ B)$ with (2.15) is unreachable and the pair $(A,\ C)$ is unobservable, since

$$rank[B,\ AB,\ A^2B] = rank\begin{bmatrix} 4 & 8 & 16 \\ 2 & 4 & 8 \\ 1 & 3 & 5 \end{bmatrix} = 2 < n = 3 \tag{2.16a}$$

and

$$rank\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank\begin{bmatrix} -0.5 & 1 & 0 \\ 0.5 & -1 & 0 \\ -0.5 & 1 & 0 \end{bmatrix} = 1 < n = 3. \tag{2.16b}$$

The condition (2.11) is also satisfied, since

$$CB = [-0.5 \quad 1 \quad 0]\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = 0. \tag{2.17}$$

In this case the matrix P has the form

$$P = \begin{bmatrix} 1 & -2 & 0 \\ 0.5 & 0 & 0 \\ -0.5 & 1 & 0 \end{bmatrix} \tag{2.18}$$

and

$$\bar{A} = PAP^{-1} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \bar{B} = PB = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},$$

$$\bar{C} = [0 \quad 0 \quad 1]. \tag{2.19}$$

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The transfer function of the linear system with (2.15) has the form

$$T(z) = C[I_3 z - A]^{-1}B =$$

$$[-0.5 \quad 1 \quad 0]\begin{bmatrix} z-1 & -2 & 0 \\ -1 & z & 0 \\ -1 & 0 & z+1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = 0 \tag{2.20}$$

and

$$\bar{T}(z) = \bar{C}[I_3 z - \bar{A}]^{-1}\bar{B} =$$

$$[0 \quad 0 \quad 1]\begin{bmatrix} z+1 & -2 & 0 \\ 0 & z-2 & -1 \\ 0 & 0 & z+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 0. \tag{2.21}$$

This confirms Theorem 2.4.

3. FRACTIONAL LINEAR DISCRETE-TIME SYSTEMS

Consider the fractional discrete-time linear system described by the equations

$$\Delta^\alpha x_{i+1} = Ax_i + Bu_i,\ 0 < \alpha < 1,\ i \in Z_+, \tag{3.1a}$$

$$y_i = Cx_i, \tag{3.1b}$$

where

$$\Delta^\alpha x_i = \sum_{j=1}^{i} c_j x_{i-j} \tag{3.2a}$$

$$c_j = (-1)^j \binom{\alpha}{j},\quad \binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for} \quad j = 1,2,\dots \end{cases} \tag{3.2b}$$

is the fractional α-order difference of x_i and $x_i \in \Re^n$, $u_i \in \Re^m$ and $y_i \in \Re^p$ are the state, input and output vectors, $x_0$is the initial condition and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and $C \in \Re^{p\times n}$. Eq. (3.2a) is the definition of the Grünwald–Letnikov fractional derivatives.

Substitution of Eq. (3.2) into Eq. (3.1a) yields

$$x_{i+1} = A_\alpha x_i - \sum_{j=2}^{i+1} c_j x_{i-j+1} + Bu_i, i \in Z_+, \tag{3.3a}$$

where

$$A_\alpha = A + I_n\alpha. \tag{3.3b}$$

The solution of Eq. (3.1a) is given by

$$x_i = \Phi_i x_0 + \sum_{j=0}^{i-1} \Phi_{i-j-1} Bu_j, \tag{3.4a}$$

where the matrices Φ_i are defined by

$$\Phi_{i+1} = \Phi_i A_\alpha + \sum_{j=2}^{i+1}(-1)^{j+1}\binom{\alpha}{j}\Phi_{i-j+1}, \qquad \Phi_0 = I_n,$$
$$i = 0, 1, \dots \tag{3.4b}$$

It is well-known [9] that if $0 < \alpha < 1$ then

1) $-c_j > 0$ for $j = 1,2,\dots$ (3.5a)

2) $\sum_{j=1}^{n} c_j = -1$ (3.5b)

The transfer matrix of the fractional linear discrete-time system is given by

$$T(\bar{z}) = C[I_n\bar{z} - A_\alpha]^{-1}B, \tag{3.6a}$$

where

$$\bar{z} = z - c_\alpha,\ c_\alpha = \sum_{j=2}^{i+1}(-1)^{j-1}\binom{\alpha}{j} z^{1-j}. \tag{3.6b}$$

Definition 3.1. [9] The fractional linear discrete-time system (3.1) is called reachable in q steps if for any given final state $x_f \in \Re^n$ there exists an input sequence u_i for $i \in [0,\ q]$ that steers the state of the system from $x_0 = 0$ to the given final state $x_q = x_f$.

Theorem 3.1. [9] The fractional linear discrete-time system (3.1) is reachable if and only if one of the equivalent conditions is satisfied:

1) $rank[B,\ A_\alpha B,\ \dots,\ {A_\alpha}^{q-1}B] = n,$ (3.7a)

2) $rank[I_n z - A_\alpha,\ B] = n$ *for all* $z \in$ C
(the field of complex numbers). (3.7b)

Definition 3.2. [9] The fractional linear discrete-time system (3.1) is called observable in q steps if knowing the input $u_i \in \Re^m$ and the output $y_i \in \Re^p$ in the q steps it is possible to find its unique initial state $x_0 \in \Re^n$.

Theorem 3.2. [9] The fractional linear discrete-time system (3.1) is observable in q steps if and only if one of the following equivalent conditions is satisfied:

$$1)\ rank\begin{bmatrix} C \\ CA_\alpha \\ \vdots \\ CA_\alpha^{q-1} \end{bmatrix} = n, \tag{3.8a}$$

$$2)\ rank\begin{bmatrix} I_n z - A_\alpha \\ C \end{bmatrix} = n \text{ for all } z \in \mathrm{C}. \tag{3.8b}$$

Theorem 3.3. The transfer matrix (3.6) of the unreachable and unobservable linear system (3.1) is a zero matrix if and only if $n \geq m + p$ and

$$CB = 0. \tag{3.9}$$

Proof. It is well-known that the transfer matrix $T(z) = 0$ if and only if the corresponding matrix of impulse responses $g_i = 0$. If the pair $(A_\alpha,\ B)$ is unreachable then the pair by similarity transformation can reduced to the form

$$A_\alpha = \begin{bmatrix} A_{1\alpha} & A_{2\alpha} \\ 0 & A_{3\alpha} \end{bmatrix}, A_{1\alpha} \in \Re^{n_1\times n_1}, A_{3\alpha} \in \Re^{n_2\times n_2},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$
$$B_1 \in \Re^{n_1\times m}, n_1 + n_2 = n \tag{3.10a}$$

and

$$C = [C_1 \quad C_2], C_1 \in \Re^{p\times n_1}, C_2 \in \Re^{p\times n_2}. \tag{3.10b}$$

In this case, from Eq. (3.4b), it follows that

$$\Phi_q = \begin{bmatrix} \Phi_{1\alpha} & \Phi_{2\alpha} \\ 0 & \Phi_{3\alpha} \end{bmatrix}, \Phi_{1\alpha} \in \Re^{n_1\times n_1}, \Phi_{3\alpha} \in \Re^{n_2\times n_2}, \tag{3.11}$$

$$g(q) = C\Phi_q B = [C_1 \quad C_2]\begin{bmatrix} \Phi_{1\alpha} & \Phi_{2\alpha} \\ 0 & \Phi_{3\alpha} \end{bmatrix}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} = C_1\Phi_{1q}B_1 = 0, \tag{3.12}$$

since the system is unobservable and $C_1 = 0$.

Therefore, the transfer matrix is a zero matrix if the system is an unreachable and unobservable system and the condition (3.9) is satisfied.

Note that the Theorem 3.3 can be also proved in a manner similar to Theorem 2.4.

Example 3.1. Consider the fractional discrete-time linear system (3.1) for $\alpha = 0.4$ with the matrices

$$A = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.4 \end{bmatrix},\quad B = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\quad C = [0 \quad 0 \quad 1]. \tag{3.13}$$

In this case we have

$$A_\alpha = A + I_3\alpha = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} = \begin{bmatrix} A_{1\alpha} & A_{2\alpha} \\ 0 & A_{3\alpha} \end{bmatrix},$$

$$A_{1\alpha} = \begin{bmatrix} 0.6 & 0.1 \\ 0 & 0.7 \end{bmatrix}, A_{3\alpha} = [0.8]. \tag{3.14}$$

Note that the pair $(A_\alpha,\ B)$

$$rank[B, \quad A_\alpha B] = rank\begin{bmatrix} 0.4 & 0 & 0.24 & 0.1 \\ 0 & 1 & 0 & 0.7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 2 < n = 3 \tag{3.15}$$

is unreachable and the pair $(A_\alpha,\ C)$

$$rank\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = rank\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0.4 \\ 0 & 0 & 0.32 \end{bmatrix} = 1 < n = 3 \tag{3.16}$$

is unobservable.

The transfer matrix of the system with Eq. (3.13) can be given as

$$T(\bar{z}) = C[I_3\bar{z} - A_\alpha]^{-1}B$$
$$= [0 \quad 0 \quad 1]\begin{bmatrix} \bar{z}-0.6 & -0.1 & -0.2 \\ 0 & \bar{z}=0.7 & 0 \\ 0 & 0 & \bar{z}-0.8 \end{bmatrix}^{-1}\begin{bmatrix} 0.4 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= [0 \quad 0]. \tag{3.17}$$

This confirms Theorem 3.3.

From a comparison of the considerations presented in Sections 2 and 3, we have the following important conclusion.

Conclusion 3.1. The zeroing of the transfer matrix of the linear systems is invariant under the order of the differential equations describing the linear discrete-time systems.

The above considerations can be extended to normal linear discrete-time systems [10].

4. CONCLUDING REMARKS

The problem of zeroing of the transfer matrix of standard and fractional linear discrete-times has been investigated. Necessary and sufficient conditions for the zeroing of the transfer matrices of linear discrete-time systems have been established (Theorem 2.3 and 2.4).These conditions have been extended to fractional linear discrete-time systems (Theorem 3.3). It has been shown that the necessary and sufficient conditions are invariant under the fractional orders of the linear discrete-time systems. The considerations have been illustrated by standard and fractional examples of linear discrete-time systems. The considerations can be extended to linear discrete-time systems characterised by different orders.

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