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MINKOWSKI SET OPERATORS

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Abstract. Paper brings few ideas about a concept of Minkowski combinations of point sets, which can be defined as generalisation of Minkowski set operations in the Euclidean space E^n. Minkowski sum and product of two point sets are used for definition of Minkowski set operators as special mappings in E^n, which can be used for modelling of manifolds in this space. Three distinguished types of Minkowski operators are introduced based on different combinations of Minkowski sum and product of point sets.

Keywords: Minkowski point set operations, Minkowski linear, product and arithmetic operator

1 Introduction

Minkowski point set operations – Minkowski sum and Minkowski product of two point sets are known concepts appearing in various resources and used in numerous applications in different fields of computer graphics, geometric modeling, and other related applied disciplines, see [1], [2]. Definitions presented in [3], [4] are based in vector representations and operations with vectors. Considering n-dimensional Euclidean space E^n with the usual orthogonal Cartesian coordinate system $\langle O, x_1, x_2, \ldots, x_n\rangle$ and its associated vector space V^n with the orthonormal basis $\langle O, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\rangle$, relation between this two algebraic structures can be determined by the following mapping π. Any point $m \in E^n$ given by Cartesian coordinates $m = [x_1^m, x_2^m, \ldots, x_n^m]$ can be associated with its position vector $\mathbf{m}$ in a unique way

$$\begin{gathered}\pi : E^n \to V^n, m \in E^n \mapsto \mathbf{m} \in V^n \\ m = [x_1^m, x_2^m, \ldots, x_n^m] \mapsto \pi(m) = \mathbf{m} = x_1^m\mathbf{e}_1 + x_2^m\mathbf{e}_2 + \ldots + x_n^m\mathbf{e}_n = (x_1^m, x_2^m, \ldots, x_n^m)\end{gathered} \quad (1)$$

Denoting by $V(A)$, $V(B)$ sets of all positioning vectors of points in A, B respectively, definition of Minkowski sum and product of point sets A and B can be precisely formulated.

Definition 1. Minkowski sum of two point sets A and B in E^n is a point set, which is the sum of all points from the set A with all points from the set B, i.e. set of points

$$\begin{aligned}A \oplus B &= \{a + b; a \in A, b \in B\} = \\ &= \{m \in E^n : m = \pi^{-1}(\mathbf{m}), \mathbf{m} = \mathbf{a} + \mathbf{b}; \mathbf{a} \in V(A), \mathbf{b} \in V(B)\}\end{aligned} \quad (2)$$

Definition 2. Minkowski product of two point sets A and B in E^n is a point set, which is the outer product of all points from the set A with all points from the set B, i.e. set of points

$$\begin{aligned}A \otimes B &= \{a \wedge b; a \in A, b \in B\} = \\ &= \{m \in E^n : m = \pi^{-1}(\mathbf{m}), \mathbf{m} = \mathbf{a} \wedge \mathbf{b}; \mathbf{a} \in V(A), \mathbf{b} \in V(B)\}\end{aligned} \quad (3)$$

Based on the operation of multiplication of a vector by any scalar, the concept of scalar multiplication of a point set can be determined.

Definition 3. Multiple of a point set A by a scalar $k \in R$ is the set A_k in E^n of points defined as

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$$A_k = k.A = \{k.a : a \in A, k \in R\} = \{m \in E^n : m = \pi^1(\mathbf{m}), \mathbf{m} = k.\mathbf{a}, \mathbf{a} \in V(A), k \in R\}. \quad (4)$$

Let us now focus our considerations to the linear combination of vectors, which is a well defined abstract concept of vector space algebra and can be extended to a Minkowski linear combination of point sets. Outer (wedge) product of scalar multiples of two vectors can be extended to Minkowski product combination of point sets, while combinations of these two vector operations leads to concepts of Minkowski arithmetic combination of point sets.

2 Minkowski linear set operator

Concept of Minkowski linear combination of two point sets can be generally determined in the form of the structure of Minkowski linear operator defined on the pairs of point sets in the Euclidean space. There exists more than one possible applicable forms of such definition, while the most natural way is used here.

Definition 4. Minkowski linear operator $L_{k,l}$ is a mapping defined on $2^{En} \times 2^{En}$, in which any pair of point sets A, B in E^n is attached a point set C in E^n in the following way

$$\begin{aligned} &L_{k,l} : 2^{E^n} \times 2^{E^n} \to 2^{E^n} \\ &(A,B) \mapsto C = L_{k,l}(A,B) = k.A \oplus l.B, A, B, C \subset E^n, k, l \in R \end{aligned}. \quad (5)$$

Set C is Minkowski linear combination of point sets A and B.

Visualisation of Minkowski linear combinations of two planar curves are presented in Fig. 1., while surface patches Fig. 2. are Minkowski linear combinations of curves in space.

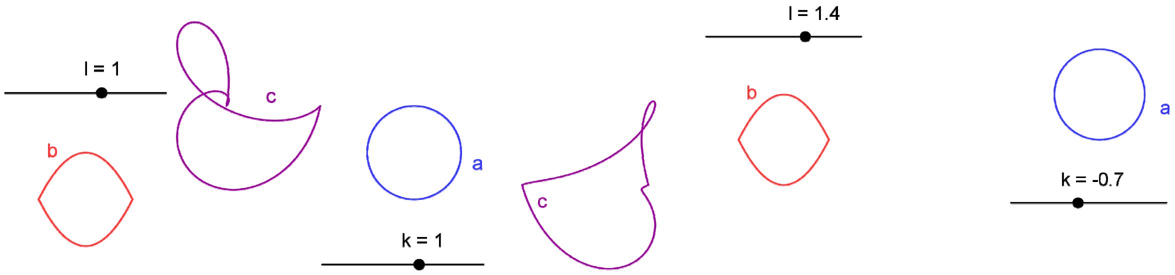

Figure 1: Minkowski linear combinations of planar curves

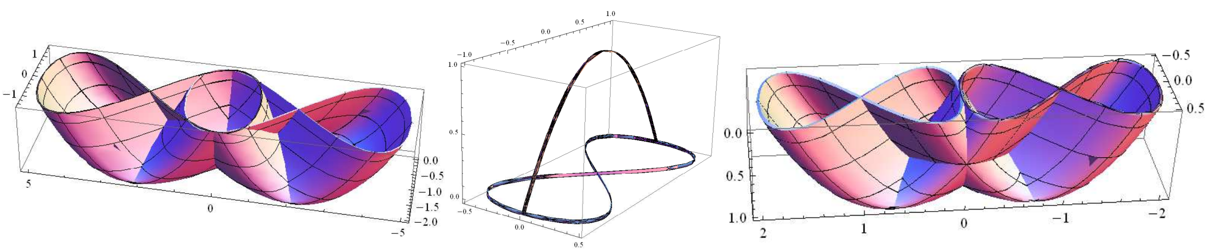

Figure 2: Minkowski linear combinations of curves in space

3 Minkowski product set operator

Operation of outer (wedge) vector product can be used to determine other form of point set combinations denoted as Minkowski product combinations.

Definition 5. Minkowski product operator $LP_{k,l}$ is a mapping defined on $2^{En} \times 2^{En}$, in which any pair of point sets A, B in E^n is attached a point set C in E^n in the following way

$$\begin{aligned} &LP_{k,l}: 2^{E^n} \times 2^{E^n} \to 2^{E^n} \\ &(A,B) \mapsto C = LP_{k,l}(A,B) = k.A \otimes l.B,\, A,B,C \subset E^n, k,l \in R \end{aligned} . \tag{6}$$

Set C is Minkowski product combination of point sets A and B.

Minkowski product combinations of curve segments in Fig. 2. are presented in Fig. 3.

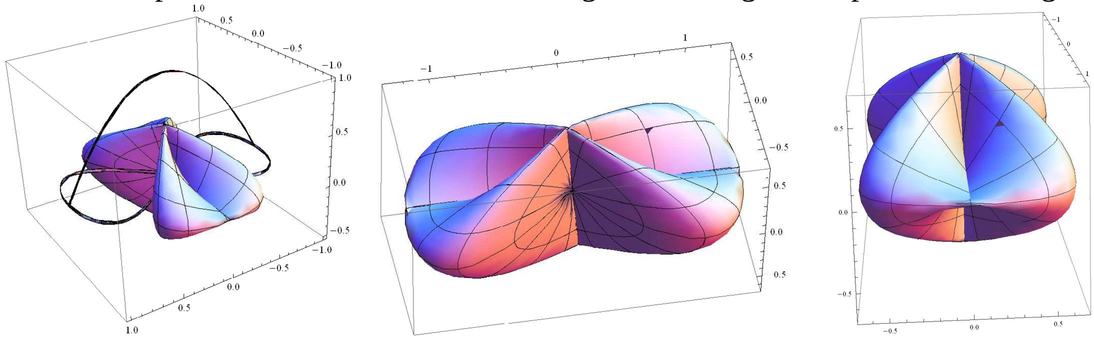

Figure 3: Views of Minkowski product combination of two curve segments in 3D

Consequence 1. Minkowski product combination $k.A \otimes l.B$ of two point sets A and B in $\boldsymbol{E}^n$ is the $k.l$ multiple of Minkowski product $A \otimes B$ of point sets A and B

$$k.A \otimes l.B = A_k \otimes B_l = (A \otimes B)_{k.l}, k,l \in R. \tag{7}$$

Proof: $k.A \otimes l.B = A_k \otimes B_l = \{ k.a \wedge l.b : a \in A, b \in B \} =$

$$= \{ k.l(a \wedge b) : a \in A, b \otimes B \} = k.l.(A \otimes B) = (A \otimes B)_{k.l}.$$

4 Minkowski arithmetic set operator

Concept of Minkowski arithmetic combination of three point sets can be introduced based on Minkowski sum and Minkowski product.

Definition 6. Let k, l, h be real numbers. Minkowski arithmetic operator $LA_{k,l,h}$ is a mapping defined on $2^{En} \times 2^{En} \times 2^{En}$, in which any triple of point sets A, B, C in E^n is attached a point set W in E^n in the following way

$$\begin{aligned} &LA_{k,l,h}: 2^{E^n} \times 2^{E^n} \times 2^{E^n} \to 2^{E^n} \\ &(A,B,C) \mapsto W = LA_{k,l,h}(A,B,C) = (k.A \oplus l.B) \otimes h.C,\, A,B,C,W \subset E^n \end{aligned} . \tag{8}$$

Set W is Minkowski arithmetic combination of point sets A, B, C.

Minkowski arithmetic combination of point sets A, B, C can be determined as

$$\begin{aligned} W &= (k.A \oplus l.B) \otimes h.C = (A_k \oplus B_l) \otimes C_h = \\ &= \{ (k.a + l.b) \wedge h.C : a \in A, b \in B, c \in C \} \end{aligned} . \tag{9}$$

Consequence 2. Minkowski arithmetic combination of three point sets A, B and C in $\boldsymbol{E}^n$ can be represented as the h-multiple of Minkowski sum of Minkowski product combinations of sets A, C and sets B, C

$$(k.A \oplus l.B) \otimes h.C = (A_k \oplus B_l) \otimes C_h = ((A \otimes C)_k \oplus (B \otimes C)_l)_h, h,k,l \in R. \tag{10}$$

Proof: $(k.A \oplus l.B) \otimes h.C = \{ (k.a + l.b) \wedge h.c : a \in A, b \in B, c \in C \} =$

$$\begin{aligned} &= \{ (k.a \wedge h.c) + (l.b \wedge h.c) : a \in A, b \in B, c \in C \} = \\ &= (k.A \otimes h.C) \oplus (l.B \otimes h.C) = k.h(A \otimes C) \oplus l.h(B \otimes C) = \\ &= (A \otimes C)_{k.h} \oplus (B \otimes C)_{l.h} = ((A \otimes C)_k \oplus (B \otimes C)_l)_h \end{aligned}$$

Consequence 3. Minkowski arithmetic combination of three point sets A, B and C in $\boldsymbol{E}^n$ for constants $k = l = h$ is k^2 multiple of Minkowski product $A \otimes B$ of point sets A and B

$$(k.A \oplus k.B) \otimes k.C = ((A \otimes C) \oplus (B \otimes C))_{k^2} \quad \textbf{(11)}$$

Proof: $(k.A \oplus k.B) \otimes k.C = \{(k.a + k.b) \wedge k.c, a \in A, b \in B, c \in C\} =$

$$= \{k.(a+b) \wedge k.c, a \in A, b \in B, c \in C\} =$$
$$= \{k^2.((a+b) \wedge c), a \in A, b \in B, c \in C\} =$$
$$= \{k^2.((a \wedge c) + (b \wedge c)), a \in A, b \in B, c \in C\} =$$
$$= k^2.((A \otimes C) \oplus (B \otimes C)) = ((A \otimes C) \oplus (B \otimes C))_{k^2}$$

Minkowski arithmetic combinations of three curve segments in space are surface patches, while some illustrations are presented in Fig. 4.

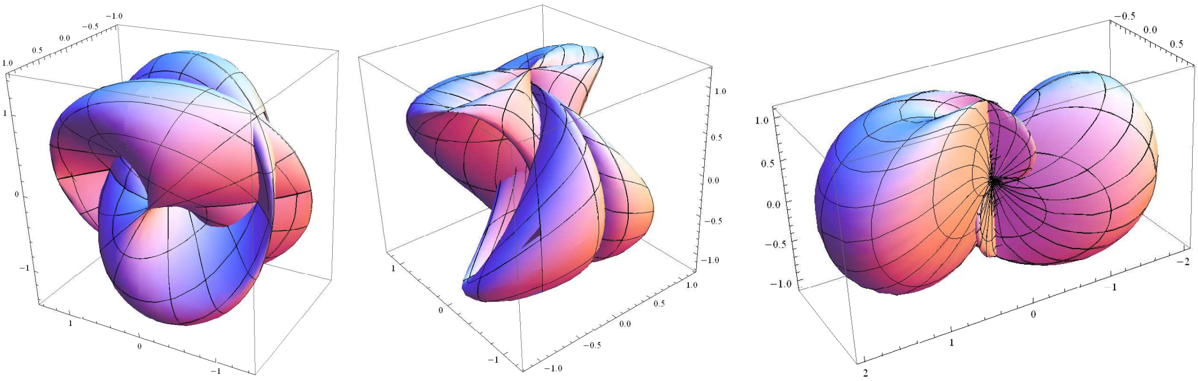

Figure 4: Minkowski arithmetic combinations of three curve segments in 3D

All presented illustrations were generated by author using free graphical program GeoGebra for examples of Minkowski operations on plane sets, software package Mathematica has been used in the case of modeling surfaces as Minkowski combinations of curves in space.

References

[1] Ghosh P., K.: *A Unified Computational Framework for Minkowski Operations*. In Comput. & Graphics, Vol. 17, No. 4, 1994.

[2] Smukler, M.: *Geometry, Topology and applications of the Minkowski Product and Action*. Harvey Mudd College. Senior thesis, 2003.

[3] Velichová D.: *Minkowski Set Operations in Modelling of Manifolds.* Proceedings of the GeoGra 2012 International Conference, Budapest 2012, Hungary, ISBN 978-963-08-3162-8, CD-rom, 4 ppr.

[4] Velichová D.: *Minkowski Product in Surface Modelling*. Aplimat - Journal of Applied Mathematics, N°1/2010, Volume 3, Slovak University of Technology in Bratislava, 2010, ISSN 1337-6365, pp. 277-286.

OPERATORY MINKOWSKIEGO NA ZBIORACH PUNKTÓW

W pracy wprowadzono pojęcia operatorów Minkowskiego, jako kombinacji zbiorów w przestrzeni euklidesowej E^n, opartych na znanych operacjach Minkowskiego (suma i produkt) na zbiorach punktów i będących zarazem ich uogólnieniem. Omówiono trzy typy operatorów Minkowskiego otrzymanych na podstawie różnych kombinacji Minkowskiego operacji sumy i iloczynu zbiorów punktowych. Jako przykłady zastosowania operatorów Minkowskiego w modelowaniu graficznym przedstawiono krzywe płaskie wygenerowane za

pomocą programu GeoGebra oraz powierzchnie w przestrzeni trójwymiarowej otrzymane jako kombinacje krzywych bazowych w środowisku pakietu Mathematica.

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