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Stability Analysis of Fractional Order Prey-Predator Model with Disease in Prey

Abstract This paper discusses the formulation and study of a fractional-order eco-epidemiological model in the context of infectious diseases in the prey population. The purpose of fractional order analysis is to investigate the effect of time memory on the growth rate of the three populations. The fractional-order prey-predator model's local and global stability are then discussed. The fractional order appears to have a stabilizing effect, which could aid in the control of co-existence between susceptible prey, infected prey, and predator populations.

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1. Introduction Fractional calculus has the ability to transform the way we think about models and influence our surroundings (v. Arafa et al. (2014)). The main motivation for employing fractional is that they are inextricably linked to memory processes (v. El-Sayed et al. (2004), El-Sayed et al. (2010)). Ahmed et al. (2007) showed that the stability of fractional-order differential equations is comparable to that of their integer-order counterparts. The ability of fractional-order differential equations to describe nonlinear occurrence almost accurately has made it an important research subject in science and engineering.

The major reason for this is that the fractional differential equation can depict the current state as a process that includes the record of past states (v. Li et al. (2017), Huda and Mukerjee (2017)). As a result, most researchers are optimistic about the fractional-order differential equation, particularly in biological modelling. Fractional calculus is the study of non-integer order differential and integral operators, as well as differential equations containing such operators (v. Diethelm (2010)). Leibniz and Guillaume de l'Hospital proposed the fractional calculus for the first time in 1695 (v. Petráš (2011))[1]. The

[1] Its first appearance is in a letter written to Guillaume de l'Hôpital by Gottfried Wilhelm Leibniz in 1695 (v. Katugampola (2014)).

presence of time memory or long-range space interaction affects the systems using fractional-order differential equations (v. Rihan et al. (2015), Suryanto et al. (2017)).

The population ecosystem is the most important aspect of modeling because it allows the mathematical model to demonstrate the dynamic response of the system (v. Sharmila and Gunasundari (2022)). Communicable diseases have become an integral part of the human and animal population monitoring and disease in the eco system has become a substantial investigation topic (v. Chandrasekar and Murugaiah (2021)).

The following is a breakdown of the current contribution: Section 2 introduces the assumptions of the model with disease in the prey. Section 3 provides an overview of the theory of fractional calculus. We show that the model's solution exists and is unique in sections 3 and 4. We also demonstrate that the solutions are non-negative and uniformly bounded. Section 5 delves into the concepts of equilibrium points. The local and global behaviour of the fractional-order preypredator model are examined in sections 6 and 7.

2. Construction of our Model In this section, we consider a fractional order predator system with disease in prey as shown below.

$$\begin{aligned} {}^{C}\mathbb{D}_{*}^{\alpha} a_1 &= s_1 a_1 \left(1 - \frac{a_1}{k}\right) - \lambda a_1 a_2 - \phi a_1 b, \\ {}^{C}\mathbb{D}_{*}^{\alpha} a_2 &= s_2 a_2 \left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 a_2 - \eta a_2 b - \rho a_2, \\ {}^{C}\mathbb{D}_{*}^{\alpha} b &= \omega_1 a_1 b + \omega_2 a_2 b - \mu b. \end{aligned} \tag{1}$$

The Caputo fractional derivative of order α is denoted by ${}^{C}\mathbb{D}_{*}^{\alpha}$ and $\alpha \in (0, 1]$. The population sizes of the susceptible prey species, infected prey species, and predator species are denoted by a_1, a_2, and b, respectively. Logistically, the prey is increasing with the growth rate s_1 and the carrying capacity k. Here λ is the prey infection rate, ρ is the death rate of the infected prey, ϕ is the predation rate on susceptible prey, η is the predation rate on infected prey, ω_1, ω_2 are the biomass conversion ratios of susceptible and infected prey, and μ is the predator death rate.

3. Preliminaries This section provides basic definitions, main findings, and properties of fractional differential equations that can be used to prove theorems.

Definition 3.1 For a function $f : R^+ \to R$, the fractional integral of order $\alpha > 0$ is defined as,

$$\mathbb{I}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} f(w)\, dw.$$

Likewise the Caputo fractional derivative of order α is defined as

$$ {}^{C}\mathbb{D}_{*}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-v)^{n-\alpha-1} f(v)\, dv. $$

where $\Gamma(\alpha)$ is the Euler's Gamma function and $\mathbb{D} = \frac{d}{dt}$.

LEMMA 3.2 *If* $f(t) \in \mathbb{C}[c,d]$, ${}^{C}\mathbb{D}_{*}^{\alpha} f(t) \in \mathbb{C}(c,d]$ *and* $0 < \alpha \leq 1$*, then* $f(t) = f(c) + \frac{1}{\Gamma(c)} \left({}^{C}\mathbb{D}_{*}^{\alpha}\right)(v)(t-c)^{\alpha}, c \leq v \leq x, x \in (c,d]$.

LEMMA 3.3 *Consider the system* ${}^{C}\mathbb{D}_{*}^{\alpha} x(t) = f(t,x)$*,* $t > t_0$ *with initial conditions* x_{t_0} *where* $0 < \alpha \leq 1$*,* $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n, \Omega \in \mathbb{R}^n$*. If* $f(t,x)$ *satisfies locally Lipschitz condition with respect to* x*, then there exists a unique solution of the above system on* $[t_0, \infty) \times \Omega$*.*

4. Non-negativity and boundedness

Let $\mathbb{R}_{+}^{3} = \left\{x \in \mathbb{R}^3 | x \geq 0\right\}$ and $Y(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ b(t) \end{bmatrix}$

THEOREM 4.1 *Solution of* (1) *are positive and uniformly bounded.*

PROOF To show that $a_1(t)$ is positive for all $t \in \mathbb{R}^{+} \cup \{0\}$. Based on the first equation of (1),

$$ {}^{C}\mathbb{D}_{*}^{\alpha} a_1(t)\,|_{t=t_1} \,. $$

From Lemma 3.2, we have $a_1\left(t_1^{+}\right) = 0$, which contradicts to the fact $a_1\left(t_1^{+}\right) \leq 0$ for $t \geq t_1$.Therefore we have $a_1(t) \geq 0$, $t \in \mathbb{R}^{+} \cup \{0\}$. Similarly, it can be proved that $a_2(t)$, $b(t)$ are all non-negative.
Let us define a function

$$ \mathbb{B}(t) = a_1(t) + a_2(t) + b(t) \,. $$

Taking the fractional derivative of Caputo on both sides, we get

$$ \begin{aligned} {}^{C}\mathbb{D}_{*}^{\alpha}\mathbb{B}(t) &= {}^{C}\mathbb{D}_{*}^{\alpha} a_1(t) + {}^{C}\mathbb{D}_{*}^{\alpha} a_2(t) + {}^{C}\mathbb{D}_{*}^{\alpha} b(t) \\ &= s_1 a_1 - \frac{s_1 a_1^2}{k} - \lambda a_1 a_2 - \phi a_1 b + s_2 a_2 - \frac{s_2 a_1 a_2}{k} - \frac{s_1 a_2^2}{k} + \lambda a_1 a_2 - \eta a_2 b \\ &\quad - \rho a_2 + \omega_1 a_1 b + \omega_2 a_2 b - \mu b \\ &= s_1 a_1 - \frac{s_1 a_1^2}{k} - a_1 b(\phi - \omega_1) + s_2 a_2 - \frac{s_2 a_1 a_2}{k} - \frac{s_1 a_2^2}{k} - a_2 b(\eta - \omega_2) \\ &\quad - \rho a_2 - \mu b \end{aligned} \tag{2} $$

Choosing $\phi < \omega_1$, $\eta < \omega_2$,

$$
\begin{aligned}
{}^{C}\mathbb{D}_{*}^{\alpha}\mathbb{B}(t) + \varepsilon\mathbb{B}(t) &= s_1 a_1 - \frac{s_1 a_1^2}{k} + s_2 a_2 - \frac{s_2 a_1 a_2}{k} - \frac{s_1 a_2^2}{k} - \rho a_2 \\
&\quad - \mu b + \varepsilon a_1 + \varepsilon a_2 + \varepsilon b \\
&\leq (s_1 + \varepsilon) a_1 - \frac{s_1 a_1^2}{k} + (s_2 + \varepsilon) a_2 - \frac{s_2 a_2^2}{k} - \rho a_2 + (\varepsilon - \mu) b \\
&\leq \frac{k}{4s_1}(s_1 + \varepsilon)^2 + \frac{k}{4s_2}(s_2 + \varepsilon - \rho)^2 + (\varepsilon - \mu) b
\end{aligned} \tag{3}
$$

Taking $\varepsilon < min\mu$ and from 3.3, we get

$$
\begin{aligned}
\mathbb{B}(t) &\leq \left(\mathbb{B}(0) - \frac{l}{\varepsilon}\right)\mathbb{E}_{\alpha}[-\varepsilon t^{\alpha}] + \frac{l}{\varepsilon} \\
&\leq \mathbb{B}(0)\mathbb{E}_{\alpha}[-\varepsilon t^{\alpha}] - \frac{l}{\varepsilon}[1 - \mathbb{E}_{\alpha}[-\varepsilon t^{\alpha}]]
\end{aligned} \tag{4}
$$

where $l = \frac{k}{4s_1}(s_1 + \varepsilon)^2 + \frac{k}{4s_2}(s_2 + \varepsilon - \rho)^2 > 0$.
Thus, $\mathbb{B}(t) \to \frac{l}{\varepsilon}$ as $t \to \infty \& \ 0 < \mathbb{B}(t) \leq \frac{l}{\varepsilon}$.
As a result, all solutions of the system (1) beginning with $\mathbb{R}_{+}^{3}$ are confined to the region $\Omega = \left\{(a_1, a_2, b) \in \mathbb{R}_{+}^{3} | B(t) \leq \frac{l}{\varepsilon} + \varepsilon\right\}$. ∎

5. Existence and Uniqueness We examine the existence and uniqueness conditions of (1) in this section. The following is the solution to the system:

$$
\mathbb{H}(Y) = Y = Y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha - 1} F(X(\theta)) d\theta
$$

and we have

$$
\mathbb{H}(Y_1) - \mathbb{H}(Y_2) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \theta)^{\alpha - 1} \big[F(Y_1(\theta)) - F(Y_2(\theta))\big] d\theta
$$

If the norm of N is defined as $\|N\| = \sup_{\tau \in (0,t]} |N(\tau)|$, then the norm of the matrix $\mathbb{M}$ is defined as

$$
\|M\| = \sum_{i,j} \sup_{\tau \in (0,t]} |m_{i,j}[\tau]|.
$$

Denote

$$
\begin{aligned}
\mathbb{L} = \frac{\mathbb{T}^{\alpha}}{\Gamma(\alpha + 1)} \max \Big\{ & s_1 + 2\mathbb{M}\left(\frac{s_1}{k} + \lambda + \phi\right), \\
& s_2 + 2\mathbb{M}\left(\frac{2s_2}{k} + \lambda + \eta\right) + \rho, 2\mathbb{M}(\omega_1 + \omega_2 + \mu) \Big\}.
\end{aligned} \tag{5}
$$

THEOREM 5.1 *The sufficient condition for the existence and uniqueness of solution of system* (1) *with initial conditions* $Y(0) = Y_0$ *and* $\tau \in (0, t]$ *is* $\mathbb{L} < 1$.

PROOF The fractional-order dynamical system (1) is denoted by

$$^{C}\mathbb{D}_{*}^{\alpha}x(\tau) = \mathbb{F}\left(Y\left(\tau\right)\right), \ \tau \in \big(0, t\big], Y\big(0\big) = Y_0$$

where

$$Y = \begin{bmatrix} a_1 \\ a_2 \\ b \end{bmatrix}, Y_0 = \begin{bmatrix} a_{1_0} \\ a_{2_0} \\ b_0 \end{bmatrix}.$$

$$F\left(Y\right) = \begin{bmatrix} s_1 a_1 \left(1 - \frac{a_1}{k}\right) - \lambda a_1 a_2 - \phi a_1 b \\ s_2 a_2 \left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 a_2 - \eta a_2 b - \rho a_2 \\ \omega_1 a_1 b + \omega_2 a_2 b - \mu b \end{bmatrix}.$$

Now

$$\left\|\mathbb{H}\left(Y_1\right) - \mathbb{H}\left(Y_2\right)\right\| \leq \mathbb{L}\left\|Y_1 - Y_2\right\|.$$

where $\mathbb{L}$ is given by (5). If $\mathbb{L} < 1$, then the mapping $Y = \mathbb{H}(Y)$ becomes a contraction mapping. This ends the proof. ■

6. Local Stability. The stability of the fractional order prey-predator model is examined in this section. First, we identify all of the system's possible equilibrium points (1),

$$\begin{aligned} s_1 a_1 \left(1 - \frac{a_1}{k}\right) - \lambda a_1 a_2 - \phi a_1 b = 0, \\ s_2 a_2 \left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 a_2 - \eta a_2 b - \rho a_2 = 0, \\ \omega_1 a_1 b + \omega_2 a_2 b - \mu b = 0. \end{aligned} \tag{6}$$

The equilibrium points are,

$$
\begin{aligned}
E_1 &= \{0,0,0\},\\
E_2 &= \{k,0,0\},\\
E_3 &= \left\{0,-\frac{k(\rho-s_1)}{s_2},0\right\},\\
E_4 &= \left\{0,\frac{\mu}{\omega_2},0\right\},\\
E_5 &= \left\{\frac{\mu}{\omega_1},0,\frac{s_1(-\mu+k\omega_1)}{k\phi\omega_1}\right\},\\
E_6 &= \left\{\frac{-ks_1s_2-\rho k^2\lambda+k^2s_2\lambda}{s_1s_2-ks_2\lambda+k^2\lambda^2},\frac{-s_1\left(\rho k-k^2\lambda\right)}{s_1s_2-ks_2\lambda+k^2\lambda^2},0\right\},\\
E_7 &= \frac{-\phi s_2\mu+\rho k\phi\omega_2-k\phi s_2\omega_2-k\eta\mu\lambda+k\eta s_1\omega_1}{-\eta s_1\omega_2+\phi s_2\omega_2-\phi s_2\omega_1-k\phi\omega_2\lambda+k\eta\omega_1\lambda},\\
&\frac{-\eta s_1\mu-\phi s_2\mu-\rho k\phi\omega_1-k\eta s_1\omega_1+k\phi s_2\omega_1+k\phi\mu\lambda}{-\eta s_1\omega_2+\phi s_2\omega_2-\phi s_2\omega_1-k\phi\omega_2\lambda+k\eta\omega_1\lambda},\\
&\frac{-s_1s_2\mu+\rho ks_1\omega_2-ks_1s_2w_1-ks_2\mu\lambda-\rho k^2\omega_1\lambda-k^2s_1\omega_2\lambda+k^2s_2\omega_1\lambda+k^2\mu\lambda^2}{-\eta s_1\omega_2+\phi s_2\omega_2-\phi s_2\omega_1-k\phi\omega_2\lambda+k\eta\omega_1\lambda}.
\end{aligned}
$$

The Jacobian matrix $J(a_1,a_2,b)$ is equal

$$
\left[\begin{array}{c|c|c}
s_1-\frac{2s_1a_1}{k}-\lambda a_2-\phi b & -\lambda a_1 & -\phi a_1\\
\hline
-\frac{s_2a_2+\lambda a_2}{k} & \begin{array}{c} s_2-\frac{s_2a_1}{k}-\frac{2s_2a_2}{k}\\ +\lambda a_1-\eta b-\rho\end{array} & -\eta a_2\\
\hline
\omega_1 b & \omega_2 b & \omega_1a_1+\omega_2a_2-\mu
\end{array}\right]. \tag{7}
$$

THEOREM 6.1 *$E_1=\{a_1\to 0,a_2\to 0,b\to 0\}$ of system (1) is always a saddle point.*

PROOF The Jacobian matrix of system (1) at $E_1=\{a_1\to 0,a_2\to 0,b\to 0\}$ is given by

$$
J(E_1)=\begin{bmatrix} s_1 & 0 & 0\\ 0 & s_2-\rho & 0\\ 0 & 0 & -\mu\end{bmatrix}.
$$

Eigen values are $\Lambda_1=s_1,\Lambda_2=s_2-\rho,\Lambda_3=-\mu$. Here $\Lambda_1>0,\Lambda_3<0$. As a result, system (1)'s equilibrium point is a saddle point. ■

THEOREM 6.2 *$E_2=\{a_1\to k,a_2\to 0,b\to 0\}$ of system (1) is asymptotically stable if $\lambda k>\rho$ then E_2 is saddle point.*

PROOF The Jacobain matrix at E_2 is

$$J(E_2) = \begin{bmatrix} -s_1 & -\lambda k & -\phi k \\ 0 & \lambda k-\rho & 0 \\ 0 & 0 & k\omega_1-\mu \end{bmatrix}.$$

Eigen values are $\Lambda_1 = -s_1, \Lambda_2 = \lambda k - \rho, \Lambda_3 = k\omega_1 - \mu$.Here $\Lambda_1 < 0$. If $\lambda k < \rho$ then $\lambda_2 < 0$, $k\omega_1 < \mu$ then $\Lambda_3 < 0$.
Therefore E_2 is asymptotically stable. If $\lambda k > \rho$ then E_2 is saddle point. ■

THEOREM 6.3 *$E_3 = \left\{a_1 \to 0, a_2 \to -\frac{k(\rho-s_1)}{s_2}, b \to 0\right\}$ of system (1) is stable if$s_1 < \frac{\lambda k(s_2-\rho)}{s_2}$ and $\frac{k\omega_2(s_2-\rho)}{s_2} < \mu$. Otherwise it is saddle point.*

PROOF The Jacobain matrix at E_3 is

$$J(E_3) = \begin{bmatrix} s_1 - \frac{\lambda k(s_2-\rho)}{s_2} & 0 & 0 \\ -\frac{s_2}{k}\cdot\frac{k(s_2-\rho)}{s_2} + \frac{\lambda k(s_2-\rho)}{s_2} & s_2 - \frac{2s_2}{k}\left(\frac{k(s_2-\rho)}{s_2}\right) & -\frac{k\eta(s_2-\rho)}{s_2} \\ 0 & 0 & \frac{k\omega_2(s_2-\rho)}{s_2} - \mu \end{bmatrix}$$

$$= \begin{bmatrix} s_1 - \frac{\lambda k(s_2-\rho)}{s_2} & 0 & 0 \\ s_2 - \rho + \frac{\lambda k(s_2-\rho)}{s_2} & \rho - s_2 & -\frac{k\eta(s_2-\rho)}{s_2} \\ 0 & 0 & \frac{k\omega_2(s_2-\rho)}{s_2} - \mu \end{bmatrix}.$$

Eigenvalues are,

$$\lambda_1 = s_1 - \frac{\lambda k\,(s_2-\rho)}{s_2},$$
$$\lambda_2 = \rho - s_2,$$
$$\lambda_3 = \frac{k\omega_2\,(s_2-\rho)}{s_2} - \mu.$$

E_3 is locally asymptotically stable if $s_1 < \frac{\lambda k(s_2-\rho)}{s_2}$ and $\frac{k\omega_2(s_2-\rho)}{s_2} < \mu$. Otherwise it is saddle point. ■

THEOREM 6.4 *$E_4 = \left\{a_1 \to 0, a_2 \to \frac{\mu}{\omega_2}, b \to 0\right\}$ of system (1) is stable if $\lambda\mu > s_1\omega_2$ when $\lambda\mu < s_1\omega_2$. Otherwise E_4 is a saddle point.*

PROOF The Jacobian matrix at $E_4 = \left\{a_1 \to 0, a_2 \to \frac{\mu}{\omega_2}, b \to 0\right\}$ of system (2.1) is

$$J(E_4) = \begin{bmatrix} s_1 - \frac{\lambda\mu}{\omega_2} & 0 & 0 \\ -\frac{s_2\mu}{k\omega_2} + \frac{\lambda\mu}{\omega_2} & \frac{2s_2\mu}{k\omega_2} - \rho & -\eta\frac{\mu}{\omega_2} \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $\lambda_1 = 0$, $\lambda_2 = \frac{-\lambda\mu + s_1\omega_2}{\omega_2}$, $\lambda_3 = -\frac{2s_2\mu + \rho\omega_2 k}{k\omega_2}$.

It is obvious that λ_3 is always negative. If $\lambda\mu > s_1\omega_2$ then equilibrium point E_3 is stable. Otherwise it is a saddle point. ■

THEOREM 6.5 *$E_5 = \left\{a_1 \to \frac{\mu}{\omega_1}, a_2 \to 0, b \to \frac{s_1(-\mu+k\omega_1)}{k\phi\omega_1}\right\}$ of system (1) is stable if $\lambda\mu > s_1\omega_2$ when $\lambda\mu < s_1\omega_2$, E_4 is a+ saddle point.*

PROOF The Jacobian matrix at $E_5 = \left\{a_1 \to \frac{\mu}{\omega_1}, a_2 \to 0, b \to \frac{s_1(-\mu+k\omega_1)}{k\phi\omega_1}\right\}$ of system (1) is

$$J(E_5) = \begin{bmatrix} s_1 - \frac{2s_1\mu}{k\omega_1} - \frac{s_1(-\mu+k\omega_1)}{k\omega_1} & -\frac{\lambda\mu}{\omega_1} & -\frac{\phi\mu}{\omega_1} \\ 0 & s_2 - \frac{s_2\mu}{k\omega_1} + \frac{\lambda\mu}{\omega_1} - \frac{\eta s_1(-\mu+k\omega_1)}{k\phi\omega_1} & 0 \\ \frac{s_1(-\mu+k\omega_1)}{k\phi} & \frac{\omega_2 s_1(-\mu+k\omega_1)}{k\phi\omega_1} & 0 \end{bmatrix}.$$

Here the eigenvalues are

$$\lambda_1 = \frac{-\phi k^2 s_1\mu\omega_1 - \sqrt{\phi^2k^4s_1^2\mu^2\omega_1^2 + 4\phi^2k^5s_1\mu^2\omega_1^3 - 4\phi^2k^6s_1\mu^2\omega_1^4}}{2k^3\phi\omega_1^2},$$
$$\lambda_2 = \frac{-\phi k^2 s_1\mu\omega_1 + \sqrt{\phi^2k^4s_1^2\mu^2\omega_1^2 + 4\phi^2k^5s_1\mu^2\omega_1^3 - 4\phi^2k^6s_1\mu^2\omega_1^4}}{2k^3\phi\omega_1^2},$$
$$\lambda_3 = \frac{\phi k\lambda\mu + \eta s_1\mu + \phi k\omega_1 s_2 - \phi k\rho\omega_1 - k\eta s_1\omega_1 - \phi\mu s_2}{\phi k\omega_1}.$$

The equilibrium point E_5 is stable if

$$\phi^2k^4s_1^2\mu^2\omega_1^2 + 4\phi^2k^5s_1\mu^2\omega_1^3 > 4\phi^2k^6s_1\mu^2\omega_1^4,$$
$$-\phi k^2s_1\mu\omega_1 > \sqrt{\phi^2k^4s_1^2\mu^2\omega_1^2 + 4\phi^2k^5s_1\mu^2\omega_1^3 - 4\phi^2k^6s_1\mu^2\omega_1^4}$$

and

$$\phi k\lambda\mu + \eta s_1\mu + \phi k\omega_1 s_2 < \phi k\rho\omega_1 - k\eta s_1\omega_1 - \phi\mu s_2.$$

Otherwise it is saddle point. ■

THEOREM 6.6 *$E_6 = \left\{a_1 \to \frac{-ks_1s_2-\rho k^2\lambda+k^2s_2\lambda}{s_1s_2-ks_2\lambda+k^2\lambda^2}, a_2 \to \frac{-s_1(\rho k-k^2\lambda)}{s_1s_2-ks_2\lambda+k^2\lambda^2}, b \to 0\right\}$ of system (1) is stable if $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$.*

PROOF Jacobian matrix at E_6 is,

$$J(E_6) = \begin{bmatrix} M_* & N_* & P_* \\ Q_* & R_* & S_* \\ 0 & 0 & T_* \end{bmatrix}$$

where

$$M_* = s_1 - \frac{2s_1}{k}\left(\frac{-ks_1s_2 - \rho k^2\lambda + k^2 s_2\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right) - \lambda\left(\frac{-s_1\rho k + k^2 s_1\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right),$$

$$N_* = -\lambda\left(\frac{-ks_1s_2 - \rho k^2\lambda + k^2 s_2\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right),$$

$$P_* = -\phi\left(\frac{-ks_1s_2 - \rho k^2\lambda + k^2 s_2\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right),$$

$$Q_* = -\frac{s_2}{k}\left(\frac{-s_1\rho k + k^2 s_1\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right),$$

$$R_* = s_2 - \frac{s_2}{k}\left(\frac{-ks_1s_2 - \rho k^2\lambda + k^2 s_2\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right) - \frac{2s_2}{k}\left(\frac{-s_1\rho k + k^2 s_1\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right)$$
$$+ \lambda\left(\frac{-ks_1s_2 - \rho k^2\lambda + k^2 s_2\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right) - \rho,$$

$$S_* = -\eta\left(\frac{-s_1\rho k + k^2 s_1\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right),$$

$$T_* = \omega_1\left(\frac{-ks_1s_2 - \rho k^2\lambda + k^2 s_2\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right) + \omega_2\left(\frac{-s_1\rho k + k^2 s_1\lambda}{s_1s_2 - ks_2\lambda + k^2\lambda^2}\right) - \mu.$$

The characteristic equation is,

$$\Lambda^3 - (M_* + R_* + T_*)\,\Lambda^2 + (R_*T_* + M_*T_* + M_*R_* - Q_*N_*)\,\Lambda - (M_*R_* - N_*Q_*)\,T_* = 0.$$

This can be written as,
$\Lambda^3 + p_1\Lambda^2 + p_2\Lambda + p_3 = 0.$
where $p_1 = -(M_* + R_* + T_*)$, $p_2 = (R_*T_* + M_*T_* + M_*R_* - Q_*N_*)$, $p_3 = (M_*R_*T_* - N_*(Q_*T_*))$.
Stable if $p_1 > 0$, $p_2 > 0$, $p_3 > 0$ and $p_1p_2 - p_3 > 0$ by Routh-Hurwitz criterion. ∎

THEOREM 6.7 *The equilibrium point* E_7

$$a_1 \to \frac{-\phi s_2\mu + \rho k\phi\omega_2 - k\phi s_2\omega_2 - k\eta\mu\lambda + k\eta s_1\omega_1}{-\eta s_1\omega_2 + \phi s_2\omega_2 - \phi s_2\omega_1 - k\phi\omega_2\lambda + k\eta\omega_1\lambda},$$

$$a_2 \to \frac{-\eta s_1\mu - \phi s_2\mu - \rho k\phi\omega_1 - k\eta s_1\omega_1 + k\phi s_2\omega_1 + k\phi\mu\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - \phi s_2\omega_1 - k\phi\omega_2\lambda + k\eta\omega_1\lambda},$$

$$b \to \frac{-s_1s_2\mu + \rho ks_1\omega_2 - ks_1s_2w_1 - ks_2\mu\lambda - \rho k^2\omega_1\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - \phi s_2\omega_1 - k\phi\omega_2\lambda + k\eta\omega_1\lambda} - \frac{k^2 s_1\omega_2\lambda - k^2 s_2\omega_1\lambda - k^2\mu\lambda^2}{-\eta s_1\omega_2 + \phi s_2\omega_2 - \phi s_2\omega_1 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}$$

of system (1) *is locally stable if* $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ *and* $b_1b_2 - b_3 > 0$.

Proof The Jacobian matrix at E_7 of (1) is

$$J(E_7) = \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix}.$$

where

$$\psi_{11} = s_1 - \frac{2s_1}{k}\left(\frac{-\phi s_2\mu + \rho k\phi\omega_2 - k\phi s_2\omega_2 - k\eta\mu\lambda + k\eta s_1\omega_2}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right)$$
$$- \lambda\left(\frac{-\eta s_1\mu - \phi s_2\mu - \rho k\phi\omega_1 - k\eta s_1\omega_1 + k\phi s_2\omega_1 + k\phi\mu\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right) -$$
$$\phi\left(\frac{-s_1s_2\mu + \rho k s_1\omega_2 - ks_2s_1\omega_1 - ks_2\mu\lambda - \rho k^2\omega_1\lambda - k^2s_2w_1\lambda + k^2\mu\lambda^2}{-k\left(-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda\right)}\right),$$
$$\psi_{12} = -\lambda\left(\frac{-\phi s_2\mu + \rho k\phi\omega_2 - k\phi s_2\omega_2 - k\eta\mu\lambda + k\eta s_1\omega_2}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right),$$
$$\psi_{13} = -\phi\left(\frac{-\phi s_2\mu + \rho k\phi\omega_2 - k\phi s_2\omega_2 - k\eta\mu\lambda + k\eta s_1\omega_2}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right),$$
$$\psi_{21} = -\frac{s_2}{k}\left(\frac{-\eta s_1\mu - \phi s_2\mu - \rho k\phi\omega_1 - k\eta s_1\omega_1 + k\phi s_2\omega_1 + k\phi\mu\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right)$$
$$+ \lambda\left(\frac{-\eta s_1\mu - \phi s_2\mu - \rho k\phi\omega_1 - k\eta s_1\omega_1 + k\phi s_2\omega_1 + k\phi\mu\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right),$$
$$\psi_{22} = s_2 - \frac{s_2}{k}\left(\frac{-\phi s_2\mu + \rho k\phi\omega_2 - k\phi s_2\omega_2 - k\eta\mu\lambda + k\eta s_1\omega_2}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right)$$
$$- \frac{2s_2}{k}\left(\frac{-\eta s_1\mu - \phi s_2\mu - \rho k\phi\omega_1 - k\eta s_1\omega_1 + k\phi s_2\omega_1 + k\phi\mu\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right)$$
$$+ \lambda\left(\frac{-\phi s_2\mu + \rho k\phi\omega_2 - k\phi s_2\omega_2 - k\eta\mu\lambda + k\eta s_1\omega_2}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right)$$
$$- \eta\left(\frac{-s_1s_2\mu + \rho k s_1\omega_2 - ks_2s_1\omega_1 - ks_2\mu\lambda - \rho k^2\omega_1\lambda - k^2s_2w_1\lambda + k^2\mu\lambda^2}{-k\left(-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda\right)}\right) - \rho,$$
$$\psi_{23} = -\eta\left(\frac{-\eta s_1\mu - \phi s_2\mu - \rho k\phi\omega_1 - k\eta s_1\omega_1 + k\phi s_2\omega_1 + k\phi\mu\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right),$$
$$\psi_{31} = \omega_1\left(\frac{-s_1s_2\mu + \rho k s_1\omega_2 - ks_2s_1\omega_1 - ks_2\mu\lambda - \rho k^2\omega_1\lambda - k^2s_2w_1\lambda + k^2\mu\lambda^2}{-k\left(-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda\right)}\right),$$
$$\psi_{32} = \omega_2\left(\frac{-s_1s_2\mu + \rho k s_1\omega_2 - ks_2s_1\omega_1 - ks_2\mu\lambda - \rho k^2\omega_1\lambda - k^2s_2w_1\lambda + k^2\mu\lambda^2}{-k\left(-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda\right)}\right),$$
$$\psi_{33} = \omega\left(\frac{-\phi s_2\mu + \rho k\phi\omega_2 - k\phi s_2\omega_2 - k\eta\mu\lambda + k\eta s_1\omega_2}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right)$$
$$+ \omega_2\left(\frac{-\eta s_1\mu - \phi s_2\mu - \rho k\phi\omega_1 - k\eta s_1\omega_1 + k\phi s_2\omega_1 + k\phi\mu\lambda}{-\eta s_1\omega_2 + \phi s_2\omega_2 - k\phi\omega_2\lambda + k\eta\omega_1\lambda}\right) - \mu.$$

The characteristic equation is,

$$\Lambda^3 - (\psi_{11} + \psi_{22} + \psi_{33})\Lambda^2 + (\psi_{22}\psi_{33} + \psi_{11}\psi_{33} + \psi_{11}\psi_{33} - \psi_{21}\psi_{12})\Lambda - (\psi_{11}\psi_{22}\psi_{33} - \psi_{12}(\psi_{21}\psi_{33})) = 0$$

This can be written as,
$\Lambda^3 + b_1\Lambda^2 + b_2\Lambda + b_3 = 0$. where $b_1 = -(\psi_1 1 + \psi_2 2 + \psi_3 3)$,
$b_2 = (\psi_2 2\psi_3 3 + \psi_1 1\psi_3 3 + \psi_1 1\psi_3 3 - \psi_2 1\psi_1 2)$,
$b_3 = (\psi_{11}\psi_{22}\psi_{33} - \psi_{12}(\psi_{21}\psi_{33}))$.
Stable if $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $b_1b_2 - b_3 > 0$ by Routh-Hurwitz criterion. ■

7. Global Stability Here we analyse the global stability properties of (2.1).

LEMMA 7.1 *Assume $a_1(t) \in R^+$ is a continuous and differentiable function. Then when $t > 0$*
$^C D_*^\alpha \left[a_1(t) - a_1^* - a_1^* \ln \frac{a_1(t)}{a_1^*}\right] \le \left(1 - \frac{a_1^*}{a_1(t)}\right) {}^C D_*^\alpha$, $a_1^* \in R^+, 0 < \alpha < 1$.

THEOREM 7.2 *The equilibrium point $E_2 = \{a_1 \to k, a_2 \to 0, b \to 0\}$ is globally stable if $\lambda k + s_2 < \rho$, $\omega_1 < \phi$, $\phi k < \mu$, $\omega_2 < \eta$.*

PROOF Consider the Lyapunov function as shown below,
$V(a_1, a_2, b) = \left(a_1 - k - k \ln \frac{a_1}{k}\right) + a_2(t) + b(t)$.
Here $V > 0$ for all $(a_1(t), a_2(t), b(t)) \neq (k, 0, 0)$ and at $(k, 0, 0)$ we get $V = 0$.

$$
\begin{aligned}
{}^C D_*^\alpha V(a_1, a_2, b) &\le \frac{(a_1 - k)}{a_1}{}^C D_*^\alpha a_1(t) + D_*^\alpha a_2(t) + D_*^\alpha b(t) \\
&= \frac{(a_1 - k)}{a_1}\left[s_1 a_1\left(1 - \frac{a_1}{k}\right) - \lambda a_1 a_2 - \phi a_1 b\right] \\
&\quad + \left[s_2 a_2\left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 a_2 - \eta a_2 b - \rho a_2\right] + (\omega_1 a_1 b + \omega_2 a_2 b - \mu b) \\
&= (a_1 - k)\left[s_1\left(1 - \frac{a_1}{k}\right) - \lambda a_2 - \phi b\right] \\
&\quad + \left[s_2 a_2\left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 a_2 - \eta a_2 b - \rho a_2\right] + [\omega_1 a_1 b + \omega_2 a_2 b - \mu b] \\
&\le -\frac{s_1}{k}(a_1 - k)^2 - \lambda a_1 a_2 - \phi a_1 b + \lambda a_2 k + \phi b k + s_2 a_2 - \frac{s_2 a_1 a_2}{k} - \frac{s_2 a_2^2}{k} \\
&\quad + \lambda a_1 a_2 - \eta a_2 b - \rho a_2 + \omega_1 a_1 b + \omega_2 a_2 b - \mu b \\
&\le -\frac{s_1}{k}(a_1 - k)^2 + a_2(\lambda k + s_2 - \rho) + a_1 b(\omega_1 - \phi) + b(\phi k - \mu) \\
&\quad + a_2 b(\omega_2 - \eta) - \frac{s_2 a_2^2}{k} \le 0.
\end{aligned}
$$

If $\lambda k + s_2 < \rho$, $\omega_1 < \phi$, $\phi k < \mu$, $\omega_2 < \eta$. then ${}^C D_*^\alpha V(a_1, a_2, b) \le 0$ and ${}^C D_*^\alpha V(a_1, a_2, b) = 0$ at E_2. Therefore E_2 is globally asymptotically stable if the above conditions are satisfied. ■

THEOREM 7.3 *(Equilibria E_3 and E_4)*

(i) The equilibrium points $E_3 = \{a_1 \to 0, a_2 \to \zeta_1, b \to 0\}$ where $\zeta_1 = -\frac{k(\rho - s_1)}{s_2}$ is globally asymptotically stable if $s_1 < \lambda\zeta_1$, $\omega_1 < \phi$, $s_2 < \rho$ and $\omega_2 < \eta$.

(ii) The equilibrium points $E_4 = \{a_1 \to 0, a_2 \to \zeta_2, b \to 0\}$ where $\zeta_2 = \frac{\mu}{\omega_2}$ is globally asymptotically stable if $s_1 < \lambda\zeta_2$, $\omega_1 < \phi$, $s_2 < \rho$ and $\omega_2 < \eta$.

PROOF Consider the Lyapunov function, $V(a_1, a_2, b) = a_1(t) + \left(a_2 - \zeta_1 - \zeta \ln \frac{a_2}{\zeta_1}\right) + b(t)$.
Here $V > 0$ for all $(a_1(t), a_2(t), b(t)) \neq (0, \zeta_1, 0)$ and at $(0, \zeta_1, 0)$ we get $V = 0$.

$$
\begin{aligned}
&{}^C D_*^\alpha V(a_1, a_2, b) \\
&\leq {}^C D_*^\alpha a_1(t) + \frac{(a_2 - \zeta_1)}{a_2} D_*^\alpha a_2(t) + {}^C D_*^\alpha b(t) \\
&= s_1 a_1 \left(1 - \frac{a_1}{k}\right) - \lambda a_1 a_2 - \phi a_1 b + (a_2 - \zeta_1)\left[s_2 a_2 \left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 a_2 - \eta a_2 b - \rho a_2\right] + \\
&\quad [\omega_1 a_1 b + \omega_2 a_2 b - \mu b] \\
&\leq s_1 a_1 - \lambda a_1 a_2 - \phi a_1 b + (a_2 - \zeta_1)\left[s_2 - \frac{s_2(a_1 + a_2)}{k} + \lambda a_1 - \eta b - \rho\right] + [\omega_1 a_1 b + \omega_2 a_2 b - \mu b] \\
&\leq s_1 a_1 - \lambda a_1 a_2 - \phi a_1 b + s_2 a_2 - \frac{s_2 a_2 (a_1 + a_2)}{k} + \lambda a_1 a_2 - \eta a_2 b - \rho a_2 - \zeta_1 s_2 \\
&\quad + \frac{s_2 \zeta_1 (a_1 + a_2)}{k} - \lambda a_1 \zeta_1 - \eta \zeta_1 b - \rho \zeta_1 + \omega_1 a_1 b + \omega_2 a_2 b - \mu b \\
&\leq a_1\left(s_1 - \frac{s_2}{k}(a_2 - \zeta_1) - \lambda \zeta_1\right) + a_1 b(\omega_1 - \phi) + a_2(s_2 - \rho) - \zeta_1(s_2 - \rho) + a_2 b(\omega_2 - \eta) - \mu b \leq 0.
\end{aligned}
$$

If $s_1 < \lambda \zeta_1$, $\omega_1 < \phi$, $s_2 < \rho$ and $\omega_2 < \eta$ then ${}^C D_*^\alpha V(a_1, a_2, b) \leq 0$ and ${}^C D_*^\alpha V(a_1, a_2, b) = 0$ at E_3. Therefore E_3 is globally asymptotically stable if the above conditions are satisfied.

Likewise, we can demonstrate (ii). ■

THEOREM 7.4 *(Equilibria E_5, E_6 and E_7)*

(i) The equilibrium point $E_7 = \{a_1 \to \zeta_7, a_2 \to \zeta_8, b \to \zeta_9\}$ where

$$
\begin{aligned}
\zeta_7 &= \frac{-\phi s_2 \mu + \rho k \phi \omega_2 - k \phi s_2 \omega_2 - k \eta \mu \lambda + k \eta s_1 \omega_1}{-\eta s_1 \omega_2 + \phi s_2 \omega_2 - \phi s_2 \omega_1 - k \phi \omega_2 \lambda + k \eta \omega_1 \lambda}, \\
\zeta_8 &= \frac{-\eta s_1 \mu - \phi s_2 \mu - \rho k \phi \omega_1 - k \eta s_1 \omega_1 + k \phi s_2 \omega_1 + k \phi \mu \lambda}{-\eta s_1 \omega_2 + \phi s_2 \omega_2 - \phi s_2 \omega_1 - k \phi \omega_2 \lambda + k \eta \omega_1 \lambda}, \\
\zeta_9 &= \frac{-s_1 s_2 \mu + \rho k s_1 \omega_2 - k s_1 s_2 w_1 - k s_2 \mu \lambda - \rho k^2 \omega_1 \lambda}{-\eta s_1 \omega_2 + \phi s_2 \omega_2 - \phi s_2 \omega_1 - k \phi \omega_2 \lambda + k \eta \omega_1 \lambda} \\
&\quad + \frac{-k^2 s_1 \omega_2 \lambda + k^2 s_2 \omega_1 \lambda + k^2 \mu \lambda^2}{-\eta s_1 \omega_2 + \phi s_2 \omega_2 - \phi s_2 \omega_1 - k \phi \omega_2 \lambda + k \eta \omega_1 \lambda}
\end{aligned}
$$

is globally stable if $\omega_1 \zeta_9 > \omega_1 b$, $\omega_2 b < \omega_2 \zeta_9$, $s_2 - \rho$ and $b - \zeta_9$.

(ii) The equilibrium point $E_5 = \{a_1 \to \zeta_3, a_2 \to 0, b \to \zeta_4\}$ where $\zeta_3 = \frac{\mu}{\omega_1}$, $\zeta_4 = \frac{s_1(-\mu + k\omega_1)}{k\phi\omega_1}$ if $\phi > \omega_1$, $\eta > \omega_2$, $s_2 < \frac{s_2 a_2}{k}$ and $\mu > \phi\zeta_3$.

(iii) The equilibrium point $E_6 = \{a_1 \to \zeta_5, a_2 \to \zeta_6, b \to 0\}$ where

$$
\zeta_5 = \frac{-k s_1 s_2 - \rho k^2 \lambda + k^2 s_2 \lambda}{s_1 s_2 - k s_2 \lambda + k^2 \lambda^2},
$$

$$\zeta_6 = \frac{-s_1\left(\rho k - k^2\lambda\right)}{s_1 s_2 - k s_2 \lambda + k^2\lambda^2}$$

is globally stable if $\phi > \omega_1$, $\eta > \omega_2$, $s_2 < \frac{a_2 s_2}{k}$, $\mu > \phi\zeta_5$ *and* $\lambda\zeta_6 > \frac{\zeta_6 s_2}{k}$.

PROOF Define the Lyapunov function as, $V\left(a_1, a_2, b\right) = \left(a_1 - \zeta_7 - \zeta_7 \ln \frac{a_1}{\zeta_7}\right) + \left(a_2 - \zeta_8 - \zeta_8 \ln \frac{a_2}{\zeta_8}\right) + \left(b - \zeta_9 - \zeta_9 \ln \frac{b}{\zeta_9}\right)$.
Here $V > 0$ for all $\left(a_1\left(t\right), a_2\left(t\right), b\left(t\right)\right) \neq \left(\zeta_7, \zeta_8, \zeta_9\right)$ and at $\left(\zeta_7, \zeta_8, \zeta_9\right)$ we get $V = 0$.

$$\begin{aligned}
&{}^C D_*^\alpha V\left(a_1, a_2, b\right)\\
&\leq \frac{\left(a_1 - \zeta_7\right)^C}{a_1} D_*^\alpha a_1\left(t\right) + \frac{\left(a_2 - \zeta_8\right)}{a_2} D_*^\alpha a_2\left(t\right) + \frac{\left(b - \zeta_9\right)}{b} D_*^\alpha b\left(t\right)\\
&= \frac{\left(a_1 - \zeta_7\right)}{a_1}\left[s_1 a_1\left(1 - \frac{a_1}{k}\right) - \lambda a_1 a_2 - \phi a_1 b\right] + \frac{\left(a_2 - \zeta_8\right)}{a_2}\left[s_2 a_2\left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 a_2 - \eta a_2 b - \rho a_2\right]\\
&+ \frac{\left(b - \zeta_9\right)}{b}\left[\omega_1 a_1 b + \omega_2 a_2 b - \mu b\right]\\
&= \left(a_1 - \zeta_7\right)\left[s_1\left(1 - \frac{a_1}{k}\right) - \lambda a_2 - \phi b\right] + \left(a_2 - \zeta_8\right)\left[s_2\left(1 - \frac{a_1 + a_2}{k}\right) + \lambda a_1 - \eta b - \rho\right]\\
&+ \left(b - \zeta_9\right)\left[\omega_1 a_1 + \omega_2 a_2 - \mu\right]\\
&\leq -\frac{s_1}{k}\left(a_1 - \zeta_7\right)^2 - a_1\left(\phi b + \lambda\zeta_8 + \frac{a_2 s_2}{k} + \omega_1\zeta_9 - \omega_1 b - \frac{\zeta_8 s_2}{k}\right)\\
&- a_2\left(-\lambda\zeta_7 - s_2 - \omega_2 b + \omega_2\zeta_9 + \eta b + \rho + \frac{a_2 s_2}{k} + \frac{\zeta_8 s_2}{k}\right) - \zeta_8\left(s_2 - \rho - \eta b\right) - \mu\left(b - \zeta_9\right) \leq 0.
\end{aligned}$$

If $\omega_1\zeta_9 > \omega_1 b$, $\omega_2 b < \omega_2\zeta_9$,$s_2 - \rho$ and $b - \zeta_9$ then ${}^C D_*^\alpha V\left(a_1, a_2, b\right) \leq 0$ and ${}^C D_*^\alpha V\left(a_1, a_2, b\right) = 0$ at E_7.
Therefore E_7 is globally asymptotically stable if the above conditions are satisfied. Similarly we can prove (ii) and (iii). ■

8. Conclusion We described an eco-epidemiological fractional-order predator prey model that explains the relationship between predator and prey inhabitants in the presence of infections in the prey. Here we developed a necessary conditions for the existence and uniqueness of the fractional-order system solution. We demonstrated that this model has at most seven equilibrium points where the local and global stability are thoroughly examined.

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Analiza stabilności modelu drapieżnik-ofiara z chorobami u ofiar.

N.B. Sharmila i Gunasundari Chandrasekar

Streszczenie W artykule omówiono formułowanie i badanie modelu eko–epidemiologicznego ułamkowego rzędu z chorobami zakaźnymi w populacji ofiar. Celem analizy tego modelu jest zbadanie wpływu pamięci czasu na tempo wzrostu trzech populacji. W pracy są kluczowe ustaleń, takie jak istnienie, niepowtarzalność, nieujemność i ograniczoność rozwiązań układów dynamicznych ułamkowego rzędu. Następnie omówiono lokalną i globalną stabilność modelu ofiara-drapieżnik w ułamkowego rzędu. Wydaje się, że model ułamkowego rzędu ma działanie stabilizujące, które może pomóc w sterowaniu współistnienia podatnej ofiary, zakażonej ofiary i populacji drapieżników.

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