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Hubert Wysocki*

SPIRA MIRABILIS IN THE SELECTED MODELS OF BITTNER OPERATIONAL CALCULUS

*** Nature loves logarithmic spirals ***
Mario Livio [22]

ABSTRACT

Parametric descriptions of spirals being analogues of the logarithmic spiral are determined using the concept of the exponential element in the non-classical Bittner operational calculus and applying the chosen models of it.

Key words:
logarithmic spiral, operational calculus, derivative, integral, limit condition, exponential element.

INTRODUCTION

One-parameter family of the *logarithmic spirals*

$$r(t) = c\, e^{a\, t}, \quad t \in \mathbb{R}_{\geqslant 0}, \tag{1}$$

where $c \in \mathbb{R}_{>0}$ is the family parameter and

$$a = \cot(\alpha) = \frac{r'(t)}{r(t)},$$

* Polish Naval Academy, Faculty of Mechanical and Electrical Engineering, Śmidowicza 69 Str., 81-127 Gdynia, Poland; e-mail: h.wysocki@amw.gdynia.pl

is the set of plane curves which cross the boundle of lines passing through the point O of the polar coordinate system (O, r, t) at a constant angle α.

The above-mentioned property substantiates the concept of the *equiangular spiral*[1] for any curve from the considered family.

From (1), for any $t \in \mathbb{R}_{\geqslant 0}$ we get

$$t = \frac{1}{a} \ln\left(\frac{r}{c}\right).$$

This identity substantiates the concept of the *logarithmic spiral*[2], as for any point $P = (r, t)$ of that curve its amplitude t is proportional to the logarithm of the part r/c of the radius r of the point.

If the ratio r/c, equal to the *golden number* (cf. (31))

$$\Phi = \frac{1 + \sqrt{5}}{2},$$

corresponds to the amplitude $t = \frac{\pi}{2}$, then the logarithmic spiral is called the *golden one.*

Thus, for the golden spiral we have

$$a = \frac{2 \ln(\Phi)}{\pi}.$$

The concept of a spiral (1) was introduced by a French philosopher, mathematician and physicist René Descartes (1596–1650) (Latin: Cartesius). He described its definition, based on equiangularity, in 1638 in a letter to his schoolmate of Jesuit College — Marin Mersenne (1588–1648), a mathematician and frater minimorum.

At the turn of the XVII and XVIII centuries the *Cartesian spiral* was also studied by Evangelista Torricelli, Pietro Nicolas, Pierre Varignon, John Wallis and Edmund Halley. However, it was Jacob Bernoulli who was extremely fascinated with algebraic and geometric properties of the Cartesian spiral, especially with its

1 The term was introduced in 1714 by Roger Cotes (1682–1716), English mathematician and astronomer.

2 The term was introduced in 1691 by Jacob Bernoulli (1654–1705), Swiss mathematician and physicist.

self-similarity. He called it a *Marvelous Spiral* (Latin: *Spira Mirabilis*) in 1692 in the German scientific journal *Acta Eruditorum* [5], [6], [16], [29].

Using the relation

$$x = r\cos(t), \quad y = r\sin(t)$$

between the point coordinates (r, t) in a polar system and its coordinates (x, y) in an orthocartesian system we conclude that the curves (1) have their parametric descriptions

$$x(t) = c\, e^{a\, t}\cos(t), \quad y(t) = c\, e^{a\, t}\sin(t). \tag{2}$$

Next, on the basis of the Euler formula and from (2) we obtain the complex form of the spiral family (1):

$$z(t) = c\, e^{(a+\mathrm{i})t}, \tag{3}$$

where ‘i’ means the imaginary unit.

A logarithmic spiral

$$z(t) = c\, e^{(a+b\,\mathrm{i})t}, \tag{4}$$

which is the hodograph of the vector function

$$\boldsymbol{r}(t) = [c\, e^{a\, t}\cos(b\, t), c\, e^{a\, t}\sin(b\, t)], \tag{5}$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 > 0$, is a generalization of (3) for a fixed $c \in \mathbb{R}_{>0}$.

If $a = 0$, then (5) is a circle, while when $b = 0$ it is a half-line ($a > 0$) or a segment ($a < 0$) on the abscissas axis. They are the degenerate cases, and we will not study them.

Increasing the parameter $t \in \mathbb{R}_{\geqslant 0}$, the movement of points on next spiral coils starts with its pole $B = (c, 0)$ and it is anticlockwise if $b > 0$, while it is clockwise when $b < 0$. These points draw from the pole $\boldsymbol{B}$ when $a > 0$ and go to the origin $O = (0, 0)$ of the spiral if $a < 0$. The sequence of points $\{z_k\} = \{z(t_k)\}$ $= \{c\, e^{(a+b\,\mathrm{i})t_k}\}, k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$[3] of the spiral on the complex plane, the radii of

[3] $\mathbb{N}$ means the set of positive integers.

which form the geometric sequence $\{r_k\} = \{c\, e^{a\,k\,t_0}\}, k \in \mathbb{N}_0$, corresponds to the parameters t forming the arithmetic sequence $\{t_k\} = \{kt_0\}, k \in \mathbb{N}_0$. In this connection, in 1693 the Italian mathematician Pietro Nicolas described this spiral as *geometric*, differentiating it from the arithmetic spiral of Archimedes.

The curve (5) is an equiangular spiral, as on the basis of[4]

$$\cos\alpha(t) = \cos|\measuredangle(\boldsymbol{r}(t), \boldsymbol{r}'(t))| = \frac{|\boldsymbol{r}(t) \circ \boldsymbol{r}'(t)|}{|\boldsymbol{r}(t)||\boldsymbol{r}'(t)|} \tag{6}$$

we get the measure of α:

$$\alpha(t) \equiv \alpha = \arccos\left[\frac{|a|}{\sqrt{a^2 + b^2}}\right],$$

which is independent of t.

The arc length of the Cartesian spiral from its pole B to any point P corresponding to a parameter $t > 0$ is calculated from the formula

$$\ell(t) = \int_0^t |\boldsymbol{r}'(\tau)| d\tau. \tag{7}$$

Thus, using (5) we obtain

$$\ell(t) = \frac{c\sqrt{a^2 + b^2}\,(e^{a\,t} - 1)}{a}. \tag{8}$$

Hence, it follows that the spiral 'coiling itself' to the origin O (i.e. when $a < 0$) has a finite length

$$\ell = \lim_{t \to +\infty} \ell(t) = -\frac{c\sqrt{a^2 + b^2}}{a}.$$

The curvature of the logarithmic spiral is determined on the basis of[5]

$$\kappa(t) = \frac{|\boldsymbol{r}'(t) \wedge \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}. \tag{9}$$

[4] '$\circ$' means the inner product operation.

[5] '$\wedge$' means the quasi-inner product operation.

Taking (5) under consideration, we get

$$\kappa(t) = \frac{|b|e^{-at}}{c\sqrt{a^2+b^2}}. \tag{10}$$

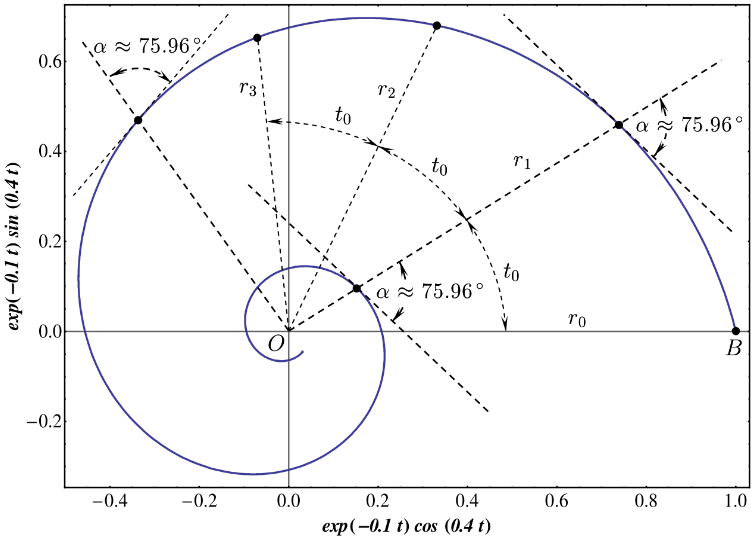

Fig. 1. Anticlockwise coiling itself logarithmic spiral:
$a = -0.1, b = 0.4, c = 1, \alpha \approx 75.9638°, \ell \approx 4.12311$[6]

Figure 2 presents the graph of the golden spiral

$$\boldsymbol{r}_{Au}(t) = [e^{\frac{2\ln(\Phi)t}{\pi}}\cos(t), e^{\frac{2\ln(\Phi)t}{\pi}}\sin(t)].$$

If $r_k := |\boldsymbol{r}_{Au}(k \cdot \frac{\pi}{2})|, k \in \mathbb{N}_0$, then

$$r_k = \Phi^k \quad \text{and} \quad \frac{r_{k+1}}{r_k} = \Phi, \quad k \in \mathbb{N}_0.$$

Moreover,

$$r_{k+1} = r_{k-1} + r_k,\ k \in \mathbb{N}, \quad \text{as} \quad \Phi^{k+1} = \Phi^{k-1} + \Phi^k,\ k \in \mathbb{N},$$

and

$$r_k = F_{k-1} + r_1 F_k,\ k \in \mathbb{N}, \quad \text{as} \quad \Phi^k = F_{k-1} + \Phi F_k,\ k \in \mathbb{N},$$

[6] The symbolic and numerical computations as well as the graphs used in this paper were made using the *Mathematica*® program.

where F_k is a general term of the *Fibonacci sequence* determined by the *Binet formula*

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right], \quad k \in \mathbb{N}_0 \tag{11}$$

(cf. (30)).

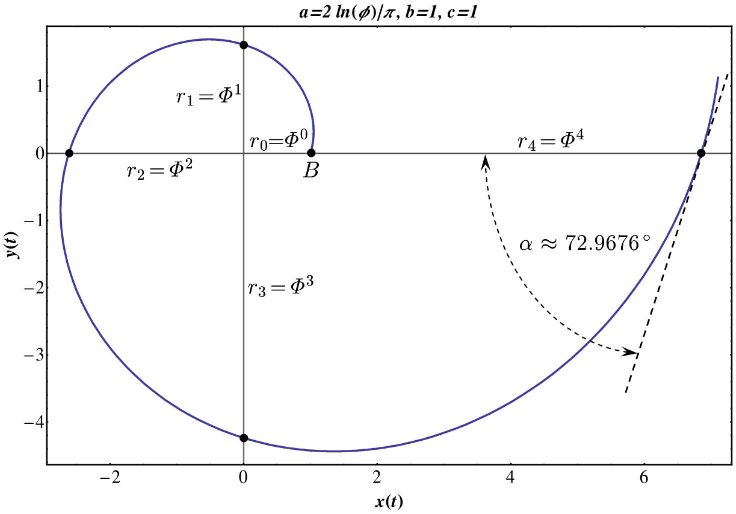

Fig. 2. The golden spiral

For the golden spiral we also have

$$\alpha = \arccos\left[\frac{2\ln(\Phi)}{\sqrt{\pi^2 + 4\ln^2(\Phi)}}\right] \approx 1.273525\,\mathrm{rad} \approx 72.96761°.$$

Let $\lambda := a + b\,\mathrm{i}$. Notice that the function $z = \{z(t)\}$[7] defining the logarithmic spiral (4) constitutes the solution of the Cauchy problem

$$\frac{dz}{dt} = \lambda z, \quad z(0) = c,$$

which can be presented in a form of

$$S z = \lambda z, \quad s_0 z = c, \tag{12}$$

[7] $\{z(t)\}$ means a symbol of a function z, whereas $z(t)$ means the value of $z = \{z(t)\}$ at the point t. In particular, $\{z(0)\}$ is a constant function equal to the value $z(0)$ in its domain. This notation derives from J. Mikusiński [23]. In what follows, we will omit the brackets { } whenever this does not cause ambiguity.

where

$$S z := \left\{\frac{dz(t)}{dt}\right\}, \quad s_0 z := \{z(0)\} \tag{13}$$

and $c \equiv \{c\}$.

From the fundamental theorems of the integral calculus it follows that S, s_0 and

$$T_0 f := \left\{\int_0^t f(\zeta)d\zeta\right\} \tag{14}$$

are such operations that

$$S T_0 f = f, \quad T_0 S z = z - s_0 z, \tag{15}$$

where $f = \{f(t)\}, z = \{z(t)\}$.

Considering (15), the operations (13), (14) belong to the so-called *classical continuous model* (*representation*) of the *Bittner operational calculus* [2–4] with the *ordinary derivative* $S = d/dt$.

There exist various continuous and discrete models of the Bittner operational calculus, in which the properly defined operations S, T_q, s_q have the properties of (15). We present here how the solutions of the problem (12) generate 'the logarithmic spirals' in the selected models of that calculus.

FOUNDATIONS OF THE NON-CLASSICAL BITTNER OPERATIONAL CALCULUS

The *Bittner operational calculus* is referred to as a system[8]

$$CO(L^0, L^1, S, T_q, s_q, Q), \tag{16}$$

where L^0 and L^1 are linear spaces (over the same field $\mathscr{F}$ of scalars[9]) such that $L^1 \subset L^0$. The linear operation $S : L^1 \longrightarrow L^0$ (denoted as $S \in \mathscr{L}(L^1, L^0)$) called the (abstract) *derivative*, is a surjection. Moreover, Q is a set of indices q for the operations $T_q \in \mathscr{L}(L^0, L^1)$ such that $S T_q f = f, f \in L^0$, called *integrals* and for the operations

[8] The abbreviation CO is derived from the French *calcul opératoire* (operational calculus).

[9] In this paper, we shall assume that $\mathscr{F}$ is a field $\mathbb{R}$ of reals.

$s_q \in \mathscr{L}(L^1, L^1)$ such that $s_q x = x - T_q S\,x,\ x \in L^1$, called *limit conditions*. The kernel of S, i.e. $\operatorname{Ker} S$ is called a set of *constants* for the derivative S.

The limit conditions $s_q, q \in Q$ are projections from L^1 onto the subspace $\operatorname{Ker} S$.

By induction we define a sequence of spaces $L^n, n \in \mathbb{N}$ such that

$$L^n := \{x \in L^{n-1} : S\,x \in L^{n-1}\}.$$

Then

$$\ldots \subset L^n \subset L^{n-1} \subset \ldots \subset L^1 \subset L^0$$

and

$$S^n(L^{m+n}) = L^m,$$

where

$$\mathscr{L}(L^n, L^0) \ni S^n := \underbrace{S \circ S \circ \ldots \circ S}_{n-\text{times}}, \quad m \in \mathbb{N}_0,\ n \in \mathbb{N}.$$

Let

$$Z^0 = L^0 + \mathrm{i}\,L^0 := \{h = f + g\,\mathrm{i} \ : \ \ f, g \in L^0\},$$
$$Z^1 = L^1 + \mathrm{i}\,L^1 := \{z = x + y\,\mathrm{i} \ : \ \ x, y \in L^1\}$$

and

$$S\,z := S\,x + \mathrm{i}\,S\,y, \quad s_q z := s_q x + \mathrm{i}\,s_q y, \quad z \in Z^1,$$
$$T_q h := T_q f + \mathrm{i}\,T_q g, \quad h \in Z^0.$$

Assume that $z = 0$ for $c = 0$ is the only solution of a problem

$$S\,z = \lambda\,z, \quad \lambda \in \mathbb{C}^{10} \tag{17}$$

$$s_q z = c, \quad c \in \operatorname{Ker} S\,. \tag{18}$$

If there exists an element $z \neq 0$ satisfying the equation (17) and the limit condition (18), then it is called an *exponential element* (with *exponent* $\lambda = a + b\,\mathrm{i}$). The exponential element is uniquely determined.

[10] $\mathbb{C}$ means the set of complexes.

SPIRALS IN CONTINUOUS MODELS

The model with the Euler derivative

Consider all real functions $f = \{f(t)\}$ determined on the interval $[1, +\infty)$ and integrable in such a way that

$$\int_{t_0}^{t} \frac{f(\zeta)}{\zeta} d\zeta < \infty$$

for any t_0, t and $t > t_0 \geqslant 1$.

The discussed set with common addition of functions and common multiplication of functions by reals is a linear space L^0. The space L^1 we define in the following way:

$$L^1 := \left\{ x = \{x(t)\} \in L^0 : \quad \left\{ t \frac{dx(t)}{dt} \right\} \in L^0 \right\}.$$

In the model with the *Euler derivative* [1], [3]

$$S\,x := \left\{ t \frac{dx}{dt} \right\}, \quad x \in L^1,$$

the limit conditions

$$s_{t_0} x := \{x(t_0)\},$$

where $t_0 \equiv q \in Q := [1, +\infty)$, correspond to the integrals

$$T_{t_0} f := \left\{ \int_{t_0}^{t} \frac{f(\zeta)}{\zeta} d\zeta \right\}, \quad f \in L^0.$$

In this case the exponential element

$$z(t) = \left(\frac{t}{t_0} \right)^{\lambda} c,$$

where $c \in \operatorname{Ker}(t\frac{d}{dt}) \simeq \mathbb{R}$, is the solution of the problem (17), (18).

Particularly, for $t_0 = 1$ we get the complex form

$$z(t) = c\, t^a(\cos(b\ \ln(t)) + \mathrm{i}\ \sin(b\ \ln(t)))$$

and the vector form of the corresponding spiral

$$\boldsymbol{r}_1(t) = [c\, t^a \cos(b\ \ln(t)), c\, t^a \sin(b\ \ln(t))].$$

It is the Cartesian spiral, as

$$\boldsymbol{r}_1(t) = \boldsymbol{r}(\ln(t)) \quad \text{for} \quad t \in [1, +\infty)$$

or

$$\boldsymbol{r}(t) = \boldsymbol{r}_1(e^t) \quad \text{for} \quad t \in [0, +\infty),$$

where $\boldsymbol{r}(t)$ is the vector function (5).

Therefore, the parametric presentations $\boldsymbol{r}(t)$ and $\boldsymbol{r}_1(t)$ are equivalent.

On the basis of (8) and (10), for $\boldsymbol{r}_1(t)$ we get

$$\ell(t) = \frac{c\sqrt{a^2+b^2}(t^a-1)}{a}, \quad \kappa(t) = \frac{|b|t^{-a}}{c\sqrt{a^2+b^2}}, \quad t \in [1, +\infty).$$

The model with the ordinary derivative of the second order

It is easy to verify that we have

$$T_q^2 S^2 x = x - (s_q x + T_q s_q S\, x), \quad q \in Q$$

for $x \in L^2$.

Therefore, the integrals $\widehat{T_q} := T_q^2, q \in Q$ and the limit conditions $\widehat{s_q} := s_q + T_q s_q S, q \in Q$ correspond to the derivative $\widehat{S} := S^2$.

In particular, the integrals

$$\widehat{T}_{t_0} f := \left\{ \int_{t_0}^{t} \int_{t_0}^{\eta} f(\zeta) d\zeta d\eta \right\}, \quad f = \{f(t)\} \in \widehat{L}^0 = L^0 := C^0(\mathbb{R}_{\geqslant 0}, \mathbb{R})$$

and the limit conditions

$$\widehat{s}_{t_0} x := \{x(t_0) + (t - t_0)x'(t_0)\}, \quad x \in \widehat{L}^1,$$

where $t_0 \equiv q \in Q := \mathbb{R}_{\geqslant 0}$, correspond to the ordinary derivative of the second order

$$\widehat{S}x := \left\{\frac{d^2x}{dt^2}\right\}, \quad x = \{x(t)\} \in \widehat{L}^1 = L^2 := C^2(\mathbb{R}_{\geqslant 0}, \mathbb{R}).$$

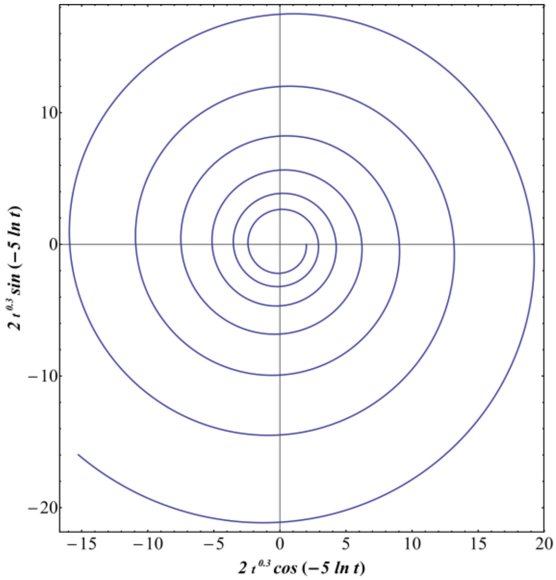

Fig. 3. Opening out, logarithmic clockwise spiral in the model with the Euler derivative: $a = 0.3, b = -5, c = 2, \alpha \approx 86.5664°$

In this case, the exponential element (cf. [25])

$$z(t) = c_0 \cosh[\sqrt{\lambda}(t - t_0)] + \frac{c_1}{\sqrt{\lambda}} \sinh[\sqrt{\lambda}(t - t_0)]$$

is the solution of (17), i.e.

$$z''(t) = \lambda z(t),$$

with the initial conditions

$$z(t_0) = c_0, z'(t_0) = c_1, \quad \text{where} \quad c_0, c_1 \in \mathbb{R}, c_0^2 + c_1^2 > 0,$$

that is, with the limit condition (18) of the form

$$c = c_0 + (t - t_0)c_1 .$$

The spiral

$$\boldsymbol{r}_2(t) = [x(t), y(t)],$$

where $x(t) = \mathrm{Re}[z(t)], y(t) = \mathrm{Im}[z(t)]$, corresponds to this exponential element.

Figure 4 shows the graphs of the spiral $\boldsymbol{r}_2(t)$ for various parameters a, b, c_0, c_1, when $t_0 = 0$.

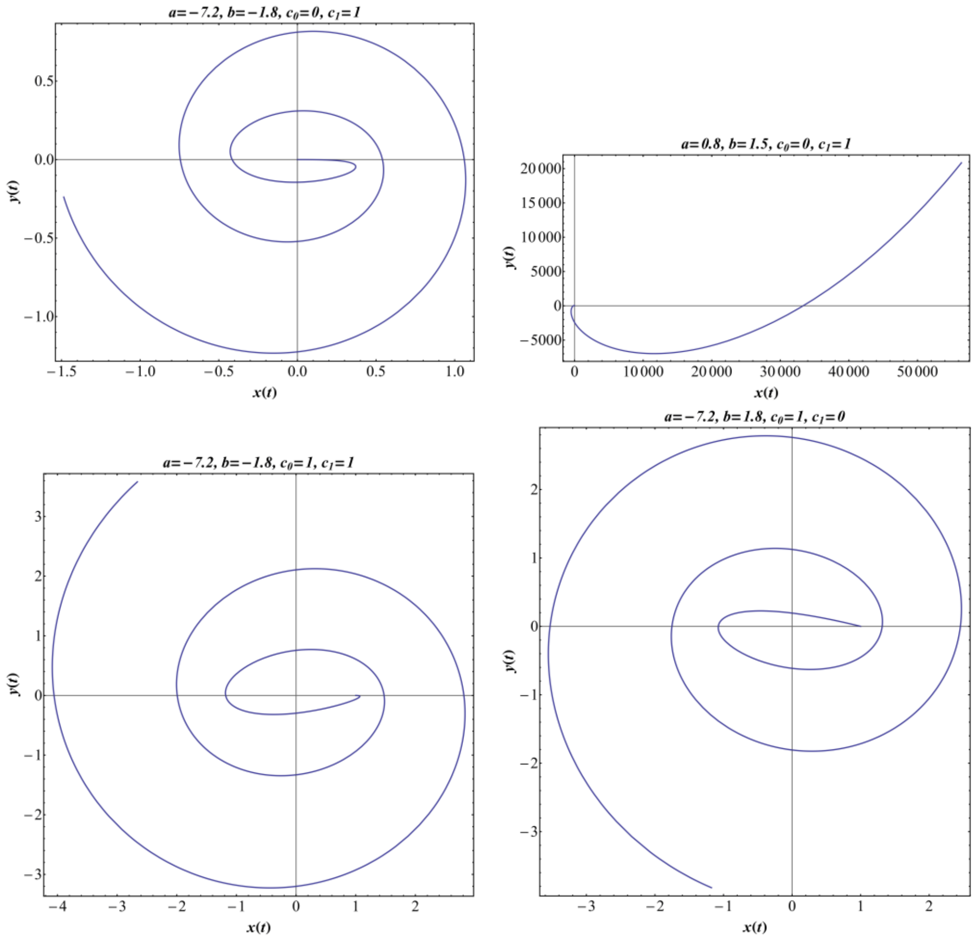

Fig. 4. The logarithmic spiral in the model with the ordinary derivative of the second order

The model with the Bessel derivative

The set of all real functions $f = \{f(t)\}$ determined on the interval $[1, +\infty)$ and integrable in such a way that

$$\int_{t_0}^{t} \frac{1}{\eta} \int_{t_0}^{\eta} f(\zeta) d\zeta d\eta < \infty$$

for any t_0, t and $t > t_0 \geqslant 1$, considered as an algebraic structure with common operations of function addition and of multiplication by real numbers, is the linear space L^0. The space L^1 we define as

$$L^1 := \left\{ x = \{x(t)\} \in L^0 : \quad \left\{ \frac{d}{dt}\left(t \frac{dx(t)}{dt} \right) \right\} \in L^0 \right\}.$$

The operation

$$Sx \equiv S_2 S_1 x := \left\{ \frac{d}{dt}\left(t \frac{dx}{dt} \right) \right\} = \{tx'' + x'\}, \quad x \in L^1 \tag{19}$$

is called the *Bessel derivative* (cf. [9]).

The operation (19) is a superposition of the Euler derivative S_1 and the ordinary derivative S_2, i.e.

$$S_1 x := \left\{ t \frac{dx}{dt} \right\}, \quad S_2 x := \left\{ \frac{dx}{dt} \right\},$$

to which the integrals

$$T_{1,t_0} f := \left\{ \int_{t_0}^{t} \frac{f(\zeta)}{\zeta} d\zeta \right\}, \quad T_{2,t_0} f := \left\{ \int_{t_0}^{t} f(\zeta) d\zeta \right\}$$

and the limit conditions

$$s_{1,t_0} x = s_{2,t_0} x := \{x(t_0)\}$$

correspond.

Therefore, the integrals

$$T_{t_0} f \equiv T_{1,t_0} T_{2,t_0} f = \left\{ \int_{t_0}^{t} \frac{1}{\eta} \int_{t_0}^{\eta} f(\zeta) d\zeta d\eta \right\}, \quad f \in L^0$$

and the limit conditions

$$\begin{aligned} s_{t_0} x = x - T_{t_0} S x &= x - T_{1,t_0}(T_{2,t_0} S_2) S_1 x = x - T_{1,t_0}(I - s_{2,t_0}) S_1 x \\ &= x - T_{1,t_0} S_1 x + T_{1,t_0} s_{2,t_0} S_1 x = x - (I - s_{1,t_0}) x + T_{1,t_0} s_{2,t_0} S_1 x \\ &= s_{1,t_0} x + T_{1,t_0} s_{2,t_0} S_1 x, \end{aligned}$$

that is

$$s_{t_0} x = \left\{ x(t_0) + t_0\, x'(t_0) \ln\left(\frac{t}{t_0}\right) \right\}, \quad x \in L^1,$$

where $t_0 \equiv q \in Q := [1, +\infty)$, correspond to the Bessel derivative.

The function

$$z(t) = \frac{I_0(2\sqrt{\lambda t})}{I_0(2\sqrt{\lambda t_0})}\, c_0 \tag{20}$$

is one of the solutions of the equation

$$t\, z''(t) + z'(t) = \lambda\, z(t).$$

In (20) c_0 is a fixed real number and

$$I_\nu(t) := \sum_{n=0}^{\infty} \frac{1}{n!(n+\nu)!} \left(\frac{t}{2}\right)^{2n+\nu}$$

is the so-called *ν-th order modified Bessel function of the first kind* ($\nu \in \mathbb{N}_0$) [18].

In this case

$$z(t_0) = c_0, \quad z'(t_0) = \frac{\sqrt{\lambda}\, I_1(2\sqrt{\lambda t_0})}{\sqrt{t_0}\, I_0(2\sqrt{\lambda t_0})}\, c_0 =: c_1.$$

Therefore, (20) satisfies (17) and the limit condition (18) of the form

$$s_{t_0} z = c_0 + t_0\, c_1 \ln\left(\frac{t}{t_0}\right) =: c \in \operatorname{Ker}\left(\frac{d}{dt}\left(t\frac{d}{dt}\right)\right),$$

i.e. it is the exponential element in the considered operational calculus model (cf. [9]).

If we present (20) in an algebraic form

$$z(t) = x(t) + \mathrm{i}\, y(t),$$

then, particularly for $t_0 = 1$, we have

$$x(t) = \frac{\frac{\Phi(1,\lambda t)}{\Phi(1,\lambda)} + \frac{\Phi^*(1,\lambda t)}{\Phi^*(1,\lambda)}}{2}\, c_0,$$
$$y(t) = \frac{\frac{\Phi(1,\lambda t)}{\Phi(1,\lambda)} - \frac{\Phi^*(1,\lambda t)}{\Phi^*(1,\lambda)}}{2\,\mathrm{i}}\, c_0,$$

where

$$\Phi(1,t) \equiv {}_0\widetilde{F}_1(;1;t) := \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2}$$

is the so-called *confluent regularized hypergeometric function* [19], while '$*$' means the complex conjugate operation.

The hodograph of the vector function

$$\boldsymbol{r}_3(t) = [x(t), y(t)]$$

will be called the *Bessel spiral*.

From the figure 5 graphs it follows that the Bessel spiral is opening out independently of the parameter a sign. Moreover, it is not an equiangular spiral. From computer calculations for a very large range of values of t it follows that the measure of α is becoming fixed with an increase of t, which is illustrated in figure 6 graphs.

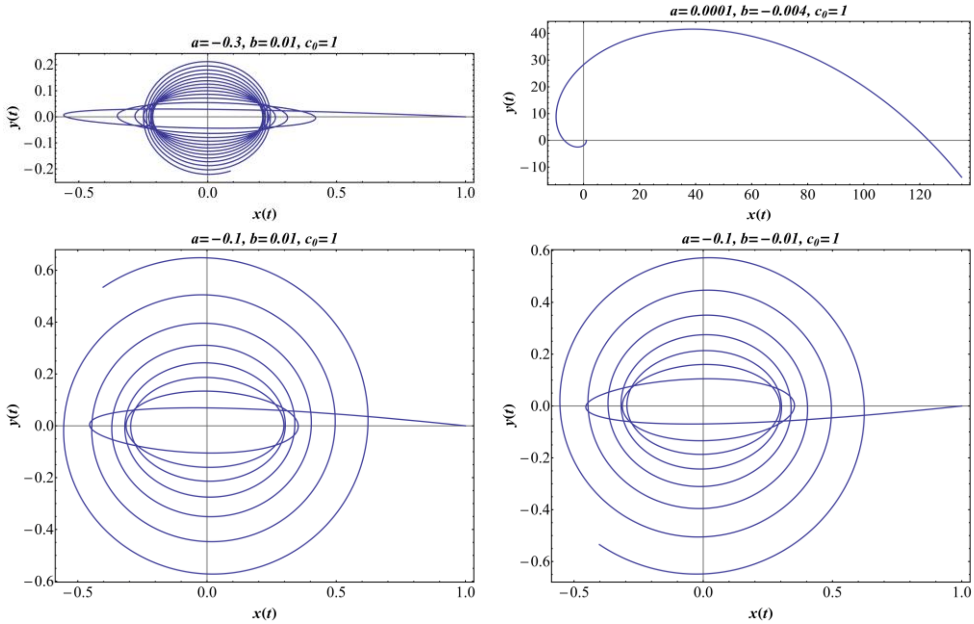

Fig. 5. The Bessel spiral

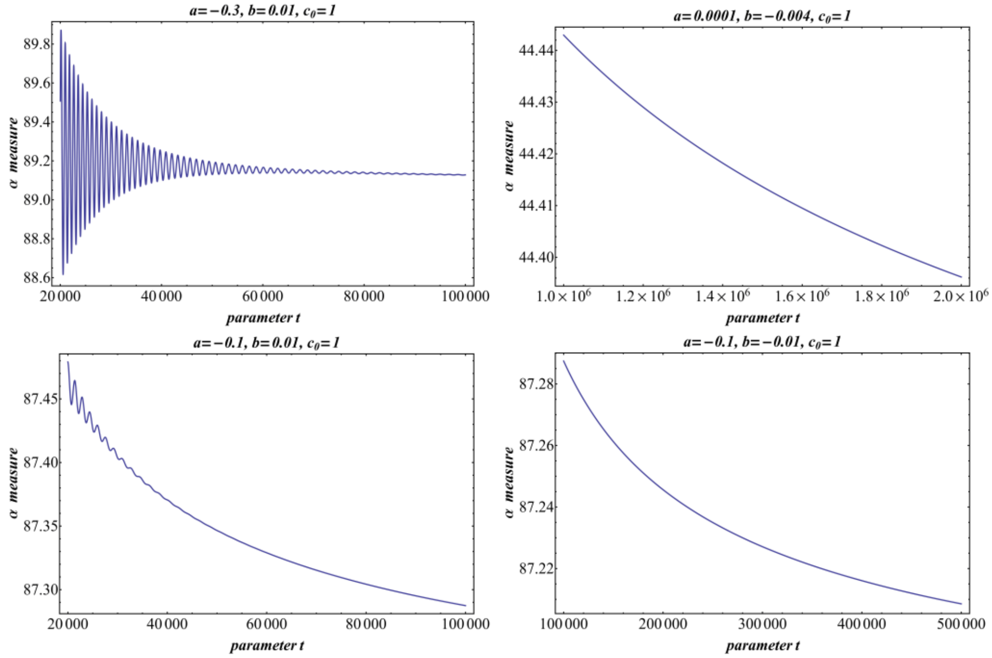

Fig. 6. Stabilization of the measure of α for the Bessel spiral

The model with the mixed partial derivative of the second order

Let $\Omega := [0, +\infty) \times [0, +\infty)$ and $L^0 := C^0(\Omega, \mathbb{R})$ be the space of all real functions $f = \{f(t, \tau)\}$ (with common addition and multiplication by real scalars) determined and continuous on Ω. Moreover, let $L^1 := C^2(\Omega, \mathbb{R})$ and $q \equiv (t_0, \tau_0) \in Q := \Omega$.

The integrals

$$T_q f := \left\{ \int_{t_0}^{t} \int_{\tau_0}^{\tau} f(\zeta, \eta) d\eta \, d\zeta \right\}, \quad f \in L^0$$

and the limit conditions

$$s_q x := \{x(t, \tau_0) + x(t_0, \tau) - x(t_0, \tau_0)\}, \quad x \in L^1$$

correspond to the *mixed partial derivative* [4]

$$S\,x := \left\{ \frac{\partial^2 x}{\partial t \, \partial \tau} \right\}, \quad x \in L^1.$$

In this case, the exponential element for $(t_0, \tau_0) = (0, 0)$ is the complex function (cf. [25])

$$z(t, \tau) = I_0(2\sqrt{\lambda t \tau})\, c, \tag{21}$$

where $c \equiv \{c\} \in \operatorname{Ker} S$.

For a fixed $t \in \mathbb{R}_{>0}$ (or $\tau \in \mathbb{R}_{>0}$), (21) is the Bessel spiral (fig. 7).

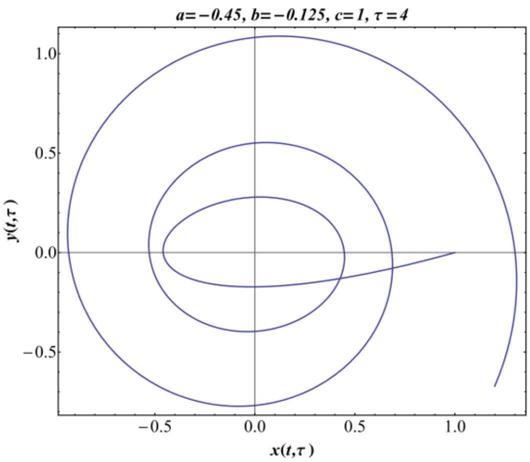

Fig. 7. The Bessel spiral in the model with the mixed partial derivative

The hodograph of the vector function of two variables

$$\boldsymbol{r}_4(t,\tau) = [x(t,\tau), y(t,\tau), \tau],$$

where

$$x(t,\tau) = \mathrm{Re}[z(t,\tau)] = \frac{\Phi(1,\lambda t\tau) + \Phi^*(1,\lambda t\tau)}{2}\,c$$
$$y(t,\tau) = \mathrm{Im}[z(t,\tau)] = \frac{\Phi(1,\lambda t\tau) - \Phi^*(1,\lambda t\tau)}{2\,\mathrm{i}}\,c,$$

will be called the *Bessel band.*

Figure 8 presents this surface for $c = 1, a = -0.45, b = -0.125,$ when $t \in [0, 120]$ and $\tau \in [0, 3]$.

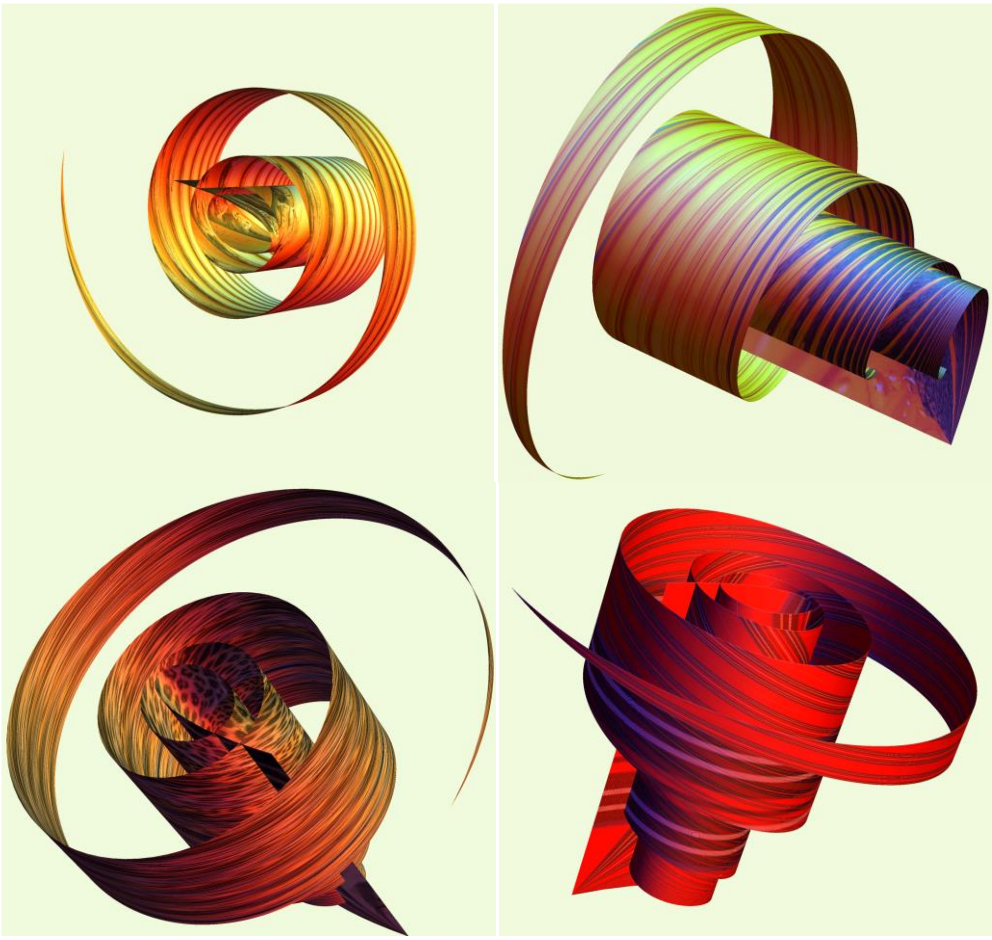

Fig. 8. The Bessel band

SPIRALS IN DISCRETE MODELS

The model with the shift operation

In a discrete model with a derivative understood as the *shift operation* (to the left) [3], [4]

$$S\{x(k)\} := \{x(k+1)\}, \quad \{x(k)\} \in L^1,$$

where $L^0 = L^1 := C(\mathbb{N}_0, \mathbb{R})$ is the real space of infinite real sequences $x = \{x(k)\}$, $k \in \mathbb{N}_0$ with common addition of sequences and their multiplication by real numbers, when $q = 0 \in Q := \{0\}$, the limit condition

$$s_0\{x(k)\} := \{x(0)\delta_0(k)\},$$

where $\delta_0(k)$ is the Kronecker symbol, i.e.

$$\delta_0(k) := \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k > 0 \end{cases},$$

corresponds to the integral

$$T_0\{x(k)\} := \begin{cases} 0 & \text{for} \quad k = 0 \\ x(k-1) & \text{for} \quad k > 0 \end{cases}.$$

In this model the exponential element is the sequence

$$z(k) = \lambda^k c,$$

where $c \equiv \{c\delta_0(k)\} \in \operatorname{Ker} S$.

Therefore, the hodograph of the vector function

$$\boldsymbol{r}_5(k) = [\, c\,|\lambda|^k \cos(k\operatorname{Arg}(\lambda)), c\,|\lambda|^k \sin(k\operatorname{Arg}(\lambda)) \,], \tag{22}$$

where $\operatorname{Arg}(\lambda)$ means the main argument of $\lambda = a + b\,\mathrm{i}$, is a spiral in the considered model. It is a *discrete spiral* if $k \in \mathbb{N}_0$, and a *continuous one* if $k \in \mathbb{R}_{\geqslant 0}$ (fig. 9).

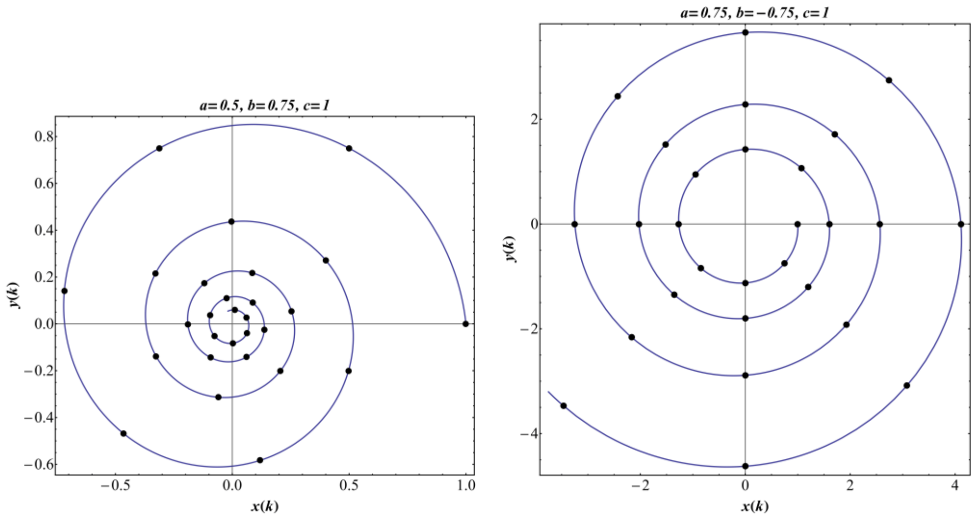

Fig. 9. Discrete and continuous spirals in the model with the shift operation

In the case of a continuous spiral, when $a^2 + b^2 > 0$ and $b \neq 0$, the points of the spiral grow away from its origin if $|\lambda| > 1$ and converge to it when $|\lambda| < 1$. If $|\lambda| = 1$, then (22) is a circle. The movement of the points on the spiral is anti-clockwise if $b > 0$ and clockwise if $b < 0$. The spiral is equiangular as

$$\alpha(k) \equiv \alpha = \arccos \sqrt{\frac{\ln|\lambda|}{\operatorname{Arg}(\lambda^2) + \ln|\lambda|}}$$

does not depend on the parameter k.

If $b = 0$ and $a = 1$, then (22) reduces itself to the point $B = (c, 0)$, while for $a = -1$ it is a circle. If $b = 0$ and $0 < a < 1$, then (22) is a segment, while for $a > 1$ it is a half-line on the abscissas axis. If $b = 0$ and $-1 < a < 0$, then the spiral (22) coils itself anticlockwise to the origin O, while for $a < -1$ it opens out anticlockwise to the pole B.

The model with the forward difference

In [1], [3], [4] there was also considered a discrete model with the derivative as a *forward difference*, i.e.

$$S\,x \equiv \Delta x := \{x(k+1) - x(k)\},$$

to which the integral

$$T_0x := \begin{cases} 0 & \text{for} \quad k = 0 \\ \sum_{j=0}^{k-1} x(j) & \text{for} \quad k > 0 \end{cases}, \quad k \in \mathbb{N}_0 \text{[11]}$$

and the limit condition

$$s_0x := \{x(0)\},$$

where $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{R})$, correspond.

It is not difficult to verify that this time the sequence

$$z(k) = (1 + \ell)^k c,$$

where $c \equiv \{c\} \in \operatorname{Ker} S$, is the solution of (17), (18) with the exponent $\ell = a + b\,\mathrm{i}$.

Therefore, the hodograph of the function

$$\boldsymbol{r}_6(k) = [\, c\,|1 + \ell|^k \cos(k\operatorname{Arg}(1 + \ell)), c\,|1 + \ell|^k \sin(k\operatorname{Arg}(1 + \ell))\,]$$

is the spiral (22) if

$$(\,1 + \ell = \lambda\,) \Longleftrightarrow (\,a = a - 1, b = b\,).$$

The model with the backward difference

In [31] the author proved that the system in which

$$S\,x := \{x(k) - x(k-1)\} \tag{23}$$

and

$$T_{k_0}x := \begin{cases} -\sum_{j=k+1}^{k_0} x(j) & \text{for} \quad k < k_0 \\ 0 & \text{for} \quad k = k_0 \\ \sum_{j=k_0+1}^{k} x(j) & \text{for} \quad k > k_0 \end{cases}, \quad k \in \mathbb{Z}, \text{[12]}$$

$$s_{k_0}x := \{x(k_0)\},$$

[11] It is assumed that $\sum_{j=0}^{-1} x(j) := 0$.

[12] It is assumed that $\sum_{j=k_0+1}^{k_0} x(j) := 0$.

where $L^0 = L^1 := C(\mathbb{Z}, \mathbb{R})$ means the space of two-sided real sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$ and $k_0 \equiv q \in Q := \mathbb{Z}$[13], forms a discrete representation of the Bittner operational calculus with the derivative as the *backward difference* (23).

In this case the exponential element (with the exponent ℓ) is a two-sided sequence [32]

$$z(k) = (1 - \ell)^{k_0 - k} c,$$

where $c \equiv \{c\} \in \operatorname{Ker} S$.Hence, for $k_0 = 0$, we get

$$\boldsymbol{r}_7(k) = [\, c\,|1 - \ell|^{-k} \cos(k\operatorname{Arg}(1 - \ell)), -c\,|1 - \ell|^{-k} \sin(k\operatorname{Arg}(1 - \ell))\,]. \tag{24}$$

Considering the fact that $k \in \mathbb{Z}$, we say that (24) is the hodograph of a *two-sided spiral.*

For this type of spiral, there follows the coiling of spiral from the pole B to the origin O if $k \to -\infty$ ($k \to +\infty$), while when $k \to +\infty$ ($k \to -\infty$) the spiral opens out from the pole B and thereby moves away from the origin O (fig. 10).

Notice also that the hodograph (24) is the spiral (22) for $k \in \mathbb{N}_0$, if

$$(\, 1 - \ell = \lambda^{-1}\,) \iff (\, \alpha = \frac{a}{a^2 + b^2}, \beta = -\frac{b}{a^2 + b^2}\,).$$

Similarly it happens when $k \in \mathbb{Z} \setminus \mathbb{N}$. Then, the curve (24) is the spiral (22), if

$$(\, 1 - \ell = \lambda\,) \iff (\, \alpha = 1 - a, \beta = -b\,).$$

The model with the central difference

First, we shall introduce the model of the operational calculus with the *central difference*

$$S\,x := \{x(k+1) - x(k-1)\} \tag{25}$$

by proving the following theorem:

[13] $\mathbb{Z}$ means the set of integers.

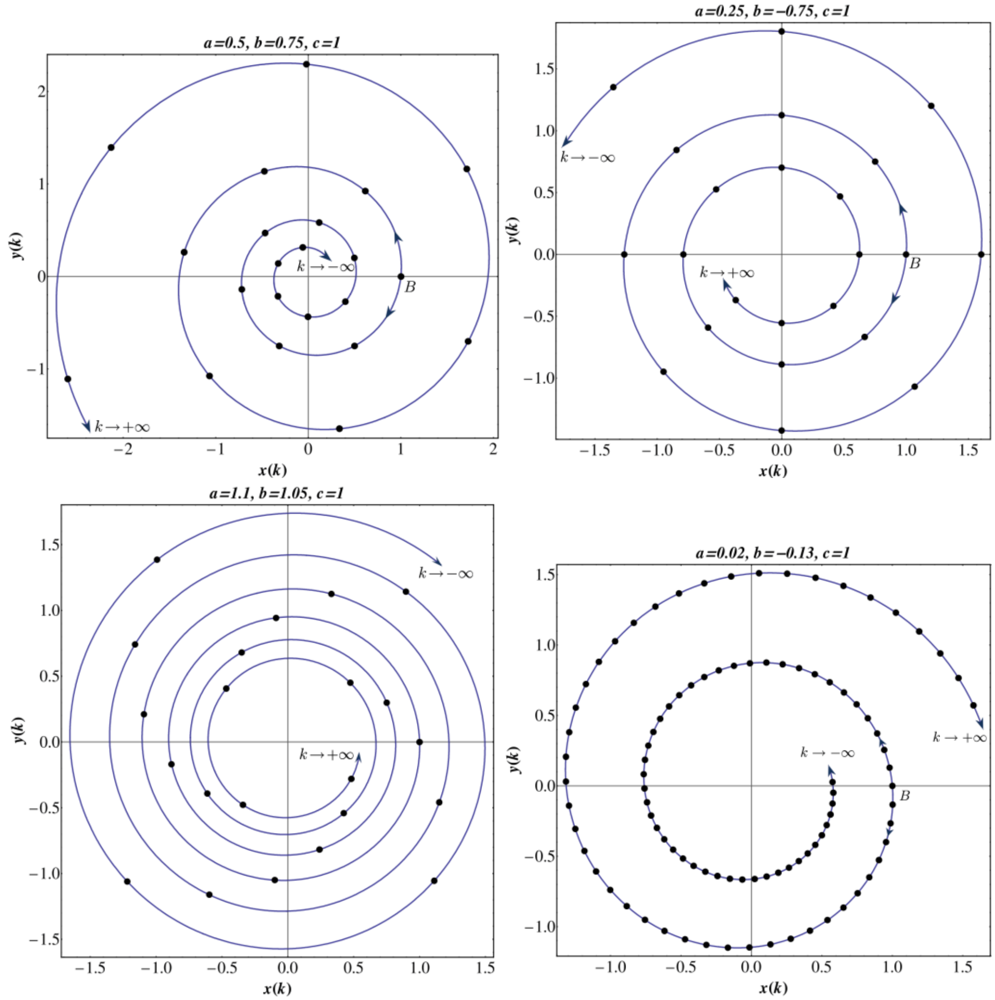

Fig. 10. Two-sided discrete and continuous spirals in the model
with the backward difference

Theorem. *The system (16), where* $x=\{x(k)\}\in L^0=L^1:=C(\mathbb{Z},\mathbb{R})$, $k_0\equiv q\in Q:=\mathbb{Z}$, *the operation* S *is the central difference (25) and*

$$T_{k_0}x := \begin{cases} -\sum\limits_{j=k+1}^{k_0} \frac{1}{2}[1-(-1)^{k-j}]x(j) & \text{for} \quad k<k_0 \\ 0 & \text{for} \quad k=k_0 \\ \sum\limits_{j=k_0+1}^{k} \frac{1}{2}[1-(-1)^{k-j}]x(j) & \text{for} \quad k>k_0 \end{cases}, \quad k\in\mathbb{Z}, \tag{26}$$

$$s_{k_0}x := \{\frac{1}{2}[x(k_0) + x(k_0 + 1)] + \frac{1}{2}(-1)^{k-k_0}[x(k_0) - x(k_0 + 1)]\} \tag{27}$$

forms the discrete model of the Bittner operational calculus.

Proof. It is easy to notice that the operations (25) – (27) are linear. Let $\{y(k)\} := T_{k_0}\{x(k)\}$. Then for $k = k_0$ we obtain

$$S\{y(k)\}|_{k=k_0} = \{y(k_0 + 1) - y(k_0 - 1)\}$$
$$= \{\sum_{j=k_0+1}^{k_0+1} \frac{1}{2}[1 - (-1)^{k_0+1-j}]x(j) + \sum_{j=k_0}^{k_0} \frac{1}{2}[1 - (-1)^{k_0-1-j}]x(j)\}$$
$$= \{0 + x(k_0)\} = \{x(k)\}|_{k=k_0}.$$

Next, for $k < k_0$ and $k + 1 < k_0$ we have

$$S\{y(k)\} = \{y(k + 1) - y(k - 1)\}$$
$$= \{-\sum_{j=k+2}^{k_0} \frac{1}{2}[1 + (-1)^{k-j}]x(j) + \sum_{j=k}^{k_0} \frac{1}{2}[1 + (-1)^{k-j}]x(j)\} = \{x(k)\}.$$

Whereas if $k < k_0$ and $k + 1 = k_0$, then

$$S\{y(k)\} = \{y(k + 1) - y(k - 1)\} = \{y(k_0) - y(k_0 - 2)\}$$
$$= \{\sum_{j=k_0-1}^{k_0} \frac{1}{2}[1 - (-1)^{k_0-j}]x(j)\} = \{x(k_0 - 1)\} = \{x(k)\}.$$

For $k > k_0$ and $k - 1 > k_0$ we get

$$S\{y(k)\} = \{y(k + 1) - y(k - 1)\}$$
$$= \{\sum_{j=k_0+1}^{k+1} \frac{1}{2}[1 + (-1)^{k-j}]x(j) - \sum_{j=k_0+1}^{k-1} \frac{1}{2}[1 + (-1)^{k-j}]x(j)\} = \{x(k)\}.$$

However, if $k > k_0$ and $k - 1 = k_0$, then

$$S\{y(k)\} = \{y(k + 1) - y(k - 1)\} = \{y(k_0 + 2) - y(k_0)\}$$
$$= \{\sum_{j=k_0+1}^{k_0+2} \frac{1}{2}[1 - (-1)^{k_0-j}]x(j)\} = \{x(k_0 + 1)\} = \{x(k)\}.$$

Finally, we are able to ascertain that the axiom $ST_{k_0}\{x(k)\} = \{x(k)\}$ is satisfied.

Let $\{f(k)\} := S\{x(k)\} = \{x(k+1) - x(k-1)\}$. Then, for $k < k_0$ we obtain

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \{-\sum_{j=k+1}^{k_0} \frac{1}{2}[1-(-1)^{k-j}]f(j)\}$$

$$= \{-\sum_{j=k+1}^{k_0} \frac{1}{2}[1-(-1)^{k-j}]x(j+1) + \sum_{j=k+1}^{k_0} \frac{1}{2}[1-(-1)^{k-j}]x(j-1)\}$$

$$= \{-\sum_{\ell=k+2}^{k_0+1} \frac{1}{2}[1+(-1)^{k-\ell}]x(\ell) + \sum_{\ell=k}^{k_0-1} \frac{1}{2}[1+(-1)^{k-\ell}]x(\ell)\}$$

$$= \{x(k)\} - \{\frac{1}{2}[x(k_0) + x(k_0+1)] + \frac{1}{2}(-1)^{k-k_0}[x(k_0) - x(k_0+1)]\}$$

$$= \{x(k)\} - s_{k_0}\{x(k)\}.$$

Similarly, if $k > k_0$, then

$$T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\} = \{\sum_{j=k_0+1}^{k} \frac{1}{2}[1-(-1)^{k-j}]f(j)\}$$

$$= \{\sum_{j=k_0+1}^{k} \frac{1}{2}[1-(-1)^{k-j}]x(j+1) - \sum_{j=k_0+1}^{k} \frac{1}{2}[1-(-1)^{k-j}]x(j-1)\}$$

$$= \{\sum_{\ell=k_0+2}^{k+1} \frac{1}{2}[1+(-1)^{k-\ell}]x(\ell) - \sum_{\ell=k_0}^{k-1} \frac{1}{2}[1+(-1)^{k-\ell}]x(\ell)\}$$

$$= \{x(k)\} - \{\frac{1}{2}[x(k_0) + x(k_0+1)] + \frac{1}{2}(-1)^{k-k_0}[x(k_0) - x(k_0+1)]\}$$

$$= \{x(k)\} - s_{k_0}\{x(k)\}.$$

In the end, when $k = k_0$, then from the definition of integrals we have $T_{k_0}S\{x(k)\}|_{k=k_0} = 0$. Moreover, $s_{k_0}\{x(k)\}|_{k=k_0} = \{x(k_0)\}$. In that case

$$T_{k_0}S\{x(k)\}|_{k=k_0} = 0 = [\{x(k)\} - s_{k_0}\{x(k)\}]_{k=k_0}.$$

Therefore, the axiom $T_{k_0}S\{x(k)\} = \{x(k)\} - s_{k_0}\{x(k)\}$ is also satisfied.

Now, consider the bilateral difference equation

$$z(k+1) - z(k-1) = \lambda\, z(k), \quad \lambda \in \mathbb{C} \setminus \{\pm 2\,\mathrm{i}\,, 0\} \tag{28}$$

with the initial conditions

$$z(0) = 0, \quad z(1) = 1. \tag{29}$$

The solution of this problem is called a *λ-Fibonacci sequence.* It is expressed by a *λ-Binet formula*

$$z(k) = \frac{\Phi_\lambda^k - \varphi_\lambda^k}{\Phi_\lambda - \varphi_\lambda}, \quad k \in \mathbb{Z}, \tag{30}$$

where

$$\Phi_\lambda := \frac{\lambda + \sqrt{4+\lambda^2}}{2}, \quad \varphi_\lambda := \frac{\lambda - \sqrt{4+\lambda^2}}{2}. \tag{31}$$

The real λ-Fibonacci sequences were considered previously by Gazale [14] and Stakhov [28] for $\lambda \in \mathbb{R}_{>0}$ and $k \in \mathbb{Z}$, while for $\lambda \in \mathbb{N}$ and $k \in \mathbb{N}_0$ — by Falcón and Plaza [12].

The number Φ_λ is called a *λ-proportion* or a *λ-mean.*

Vera de Spinadel [26], [27] called $\{\Phi_\lambda\}_{\lambda\in\mathbb{N}}$ the *family of metalic means* (*proportions*). For $\lambda = 1, 2, 3$ the proportions are called *golden, silver* and *bronze*, respectively.

For $\lambda = 1$, (30) is a classical two-sided Fibonacci sequence (cf. [8], [13], [20], [32]):

k	…	-7	-6	-5	-4	-3	-2	-1	**0**	**1**	2	3	4	5	6	7	…
$z(k)$	…	13	-8	5	-3	2	-1	1	**0**	**1**	1	2	3	5	8	13	…

Here, it is also worth to mention a considered in literature [7], [11], [17], [30], random λ-Fibonacci sequence

$$z(k+1) = \begin{cases} z(k-1) + \lambda\, z(k) & \text{with pr. } 0.5 \\ z(k-1) - \lambda\, z(k) & \text{with pr. } 0.5 \end{cases}, \; z(0) = 0, z(1) = 1, \; k \in \mathbb{N} \tag{32}$$

related with tossing a symmetric coin. A successive term of the sequence (32) (called a *Vibonacci sequence* [17]) is formed by adding (subtracting) to (from) the term $z(k-1)$ a component $\lambda z(k), \lambda \in \mathbb{C}$, when in the $(k+1)$-th coin toss heads (tails) is facing upwards.

In the considered operational calculus model, the initial-value problem (28), (29) with $q \equiv k_0 = 0$ can be written as (17), (18), where

$$s_0 z = c = \{\frac{1}{2}[\,1-(-1)^k\,]\} \in \operatorname{Ker} S\,.$$

Thus, the λ-Binet formula (30) presents a two-sided sequence, which is an exponential element in the model with the central difference (25).

Taking into consideration that $\lambda = a + b\,\mathrm{i}$ and $z(k) = x(k) + \mathrm{i}\,y(k)$, from the λ-Binet formula (30) we obtain

$$\begin{aligned} x(k) = \frac{1}{|\Phi_\lambda - \varphi_\lambda|}\Big(&|\Phi_\lambda|^k \cos[\operatorname{Arg}(\Phi_\lambda - \varphi_\lambda) - k\operatorname{Arg}(2\Phi_\lambda)] \\ &-|\varphi_\lambda|^k \cos[\operatorname{Arg}(\Phi_\lambda - \varphi_\lambda) - k\operatorname{Arg}(2\varphi_\lambda)]\Big), \\ y(k) = \frac{1}{|\Phi_\lambda - \varphi_\lambda|}\Big(&|\varphi_\lambda|^k \sin[\operatorname{Arg}(\Phi_\lambda - \varphi_\lambda) - k\operatorname{Arg}(2\varphi_\lambda)] \\ &-|\Phi_\lambda|^k \sin[\operatorname{Arg}(\Phi_\lambda - \varphi_\lambda) - k\operatorname{Arg}(2\Phi_\lambda)]\Big). \end{aligned}$$

The hodograph of the function

$$\boldsymbol{r}_8(k) = [x(k), y(k)]$$

will be called a *two-sided λ-Fibonacci spiral.*

Its graphs for various values of a, b parameters are presented in figures 11 and 12.

In the classical case, i.e. when $\lambda = 1$, the *Kepler formula*

$$\lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \Phi$$

holds for the one-sided Fibonacci sequence (11).

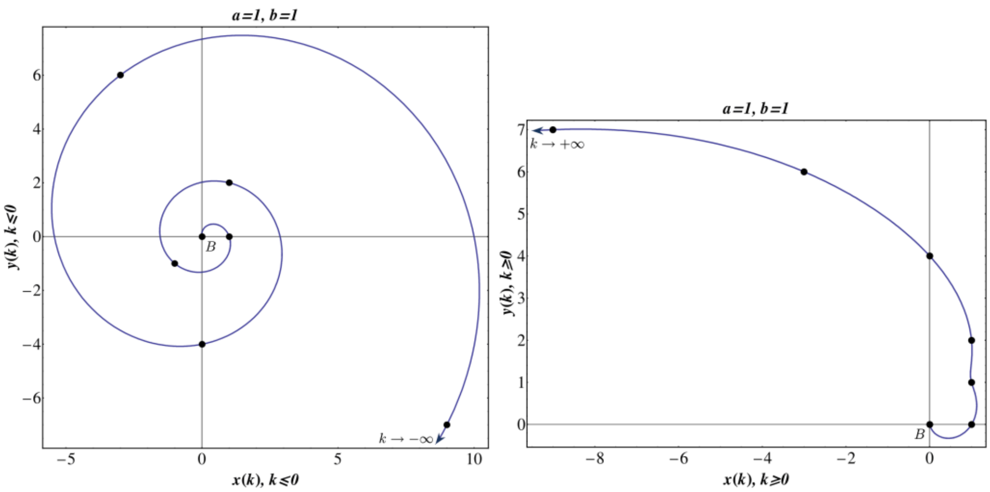

Fig. 11. Discrete and continuous $(1 + \mathrm{i})$-Fibonacci spiral

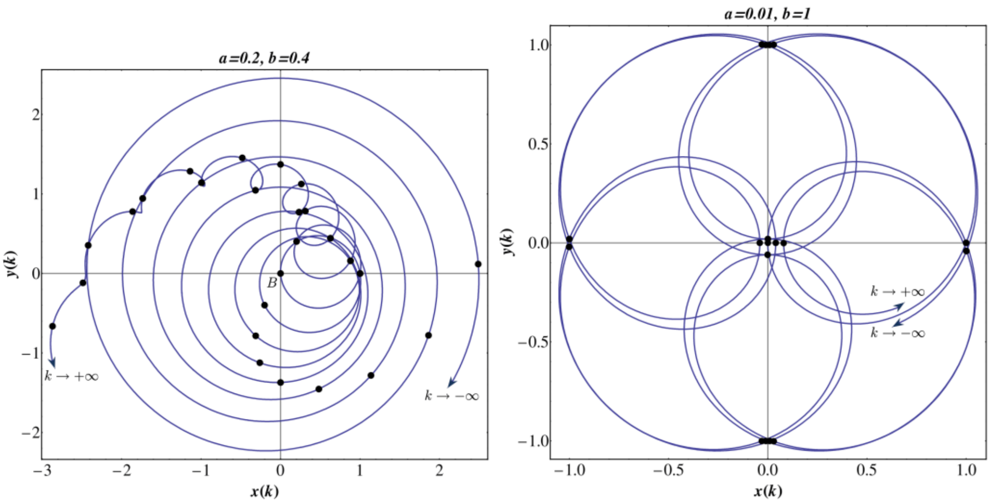

Fig. 12. Two-sided discrete and continuous spiral in the model with the central difference

Let us examine the limits

$$g_{\pm} := \lim_{k \to \pm\infty} \frac{|\boldsymbol{r}_8(k+1)|}{|\boldsymbol{r}_8(k)|} \tag{33}$$

in the general case, i.e. when, instead of the sequence (11), we will be considering the two-sided λ-Fibonacci sequence (30).

If $|\Phi_\lambda| > |\varphi_\lambda|$, which is equivalent to the condition $|\Phi_\lambda| > 1$, then on the basis of (30), we obtain

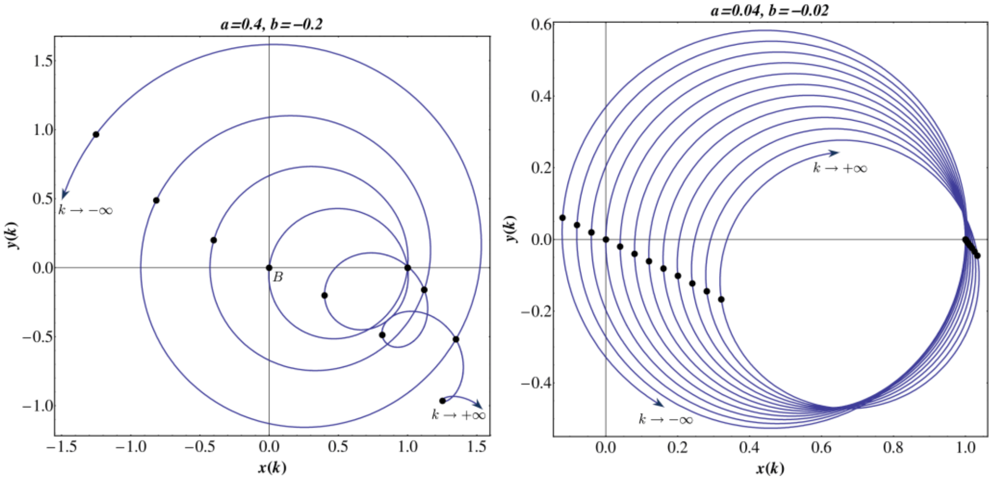

Fig. 12. continued

$$g_+ = \lim_{k\to+\infty}\left|\frac{z(k+1)}{z(k)}\right| = \lim_{k\to+\infty}\left|\frac{\Phi_\lambda^{k+1}-\varphi_\lambda^{k+1}}{\Phi_\lambda^{k}-\varphi_\lambda^{k}}\right| = \lim_{k\to+\infty}\left|\frac{\Phi_\lambda-\left(\frac{\varphi_\lambda}{\Phi_\lambda}\right)^k\varphi_\lambda}{1-\left(\frac{\varphi_\lambda}{\Phi_\lambda}\right)^k}\right| = |\Phi_\lambda|.$$

Next, if $|\Phi_\lambda| < |\varphi_\lambda|$, i.e. $|\Phi_\lambda| < 1$, then

$$g_+ = \lim_{k\to+\infty}\left|\frac{\left(\frac{\Phi_\lambda}{\varphi_\lambda}\right)^k\Phi_\lambda-\varphi_\lambda}{\left(\frac{\Phi_\lambda}{\varphi_\lambda}\right)^k-1}\right| = |\varphi_\lambda| = \frac{1}{|\Phi_\lambda|}.$$

For example, for $\lambda = \pm 1 + \mathrm{i}$ we have

$$|\Phi_{1+\mathrm{i}}| = |\varphi_{-1+\mathrm{i}}| \approx 1.70002 \text{ and } |\varphi_{1+\mathrm{i}}| = |\Phi_{-1+\mathrm{i}}| \approx 0.58823,$$

respectively. Therefore, in both cases

$$g_+ \approx 1.70002.$$

When $k \to -\infty$, notice first that $z(-k) = (-1)^{k+1}z(k)$. Because of that

$$
\begin{aligned}
g_- &= \lim_{k\to-\infty}\left|\frac{z(k+1)}{z(k)}\right| = \lim_{k\to+\infty}\left|\frac{z(-k+1)}{z(-k)}\right| \\
&= \lim_{k\to+\infty}\left|\frac{z(-(k-1))}{z(-k)}\right| = \lim_{k\to+\infty}\left|\frac{1}{\frac{z(k)}{z(k-1)}}\right| \\
&= \frac{1}{g_+} = \begin{cases} |\varphi_\lambda|, & \text{if} \quad |\Phi_\lambda| > |\varphi_\lambda| \\ |\Phi_\lambda|, & \text{if} \quad |\Phi_\lambda| < |\varphi_\lambda| \end{cases}.
\end{aligned}
$$

For $|\Phi_\lambda| = |\varphi_\lambda|$, which means that $|\Phi_\lambda| = 1$ and holds when $\lambda = b\,\mathrm{i}$, $b \in (-2, 2)$, the limits (33) are improper.

INSTEAD OF CONCLUSIONS

Nature loves logarithmic spirals. From sunflowers, seashells, and whirlpools, to hurricanes and giant spiral galaxies, it seems that nature chose this marvelous shape as its favorite 'ornament' (Mario Livio [22]).

REFERENCES

[1] Bellert S., *Prace wybrane: wydanie pośmiertne*, PWN, Warszawa — Wrocław 1980 [*Selected Papers: Posthumous Edition* — available in Polish].

[2] Bittner R., *Algebraic and analytic properties of solutions of abstract differential equations*, 'Rozprawy Matematyczne' ['Dissertationes Math.'], 41, PWN, Warszawa 1964.

[3] Bittner R., *Operational calculus in linear spaces*, 'Studia Math.', 1961, 20, pp. 1–18.

[4] Bittner R., *Rachunek operatorów w przestrzeniach liniowych*, PWN, Warszawa 1974 [*Operational Calculus in Linear Spaces* — available in Polish].

[5] Borodin A. I., Bugaj A. S., *Biografičeskij slovar' dejatelej v oblasti matematiki*, Radjans'ka Škola, Kiev 1979 [*Biographical Dictionary of Activists in the Field of* Mathematics — available in Russian].

[6] Boyer C. B., *The History of the Calculus and its Conceptual Development*, Dover Publ., Inc., New York 1949.

[7] Caruso H. A., Marotta S. M., *Sequences of complex numbers resembling the Fibonacci series*, 'Revista Ciências Exatas e Naturais', 2000, 2(1), pp. 49–59.

[8] Dimovski I. H., Kiryakova V. S., *Discrete operational calculi for two-sided sequences*, 'The Fibonacci Quarterly' (Proc. 5th Internat. Conf. on Fibonacci Numbers and Their Applications), 1993, 5, pp. 159–168.

[9] Ditkin V. A., Prudnikov A. P., *Integral'nye preobrazovaniâ i operacionnoe isčislenie*, Nauka, Moskva 1974 [*Integral Transforms and Operational Calculus* — available in Russian].

[10] Do Carmo M. P., *Differential Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey 1976.

[11] Embree M., Trefethen L. N., *Growth and decay of random Fibonacci sequences*, 'Proceedings: Mathematical, Physical and Engineering Sciences', 1987, 455 (Jul. 8, 1999), The Royal Society, pp. 2471–2485.

[12] Falcón S., Plaza A., *On the Fibonacci k-numbers*, 'Chaos, Solitons and Fractals', 2007, 32(5), pp. 1615–1624.

[13] Fishburn P. C., Odlyzko A. M., Roberts F. S., *Two-sided generalized Fibonacci sequences*, 'The Fibonacci Quarterly', 1989, 27, 352–361.

[14] Gazalé M. J., *Gnomon: From Pharaohs to Fractals*, Princeton Univ. Press, New Jersey 1999.

[15] Gdowski B., *Elementy geometrii różniczkowej z zadaniami*, PWN, Warszawa 1982 [*Elements of Differential Geometry with Exercises* — available in Polish].

[16] Hambidge J., *Dynamic Symmetry: The Greek Vase*, Yale Univ. Press, New Haven 1920.

[17] Hayes B., *The Vibonacci numbers*, 'American Scientist', 1999, 87(4), pp. 296–301.

[18] http://functions.wolfram.com/03.02.02.0001.01 [access 27.06.2015].

[19] http://functions.wolfram.com/07.18.02.0001.01 [access 27.06.2015].

[20] Knuth D. E., *NegaFibonacci numbers and the hyperbolic plane*, The 'Pi Mu Epsilon J. Sutherland Frame' memorial lecture at 'MathFest 2007' in San José, CA, 2007-08-04, http://www.pme-math.org/conferences/national/2007/2007.html [access 29.06.2015].

[21] Koshy T., *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., New York 2001.

[22] Livio M., *Golden Ratio — The Story of Phi, the World's Most Astonishing Number*, Broadway Books, New York 2002.

[23] Mikusiński J., *Operational Calculus*, Pergamon Press, New York 1959.

[24] Posmantier A. S., Lehmann I., *The Fabulous Fibonacci Numbers*, Prometheus Books, Amherst, New York 2007.

[25] Przeworska-Rolewicz D., *Algebraic Analysis*, D. Reidel & PWN, Dordrecht, Warszawa 1988.

[26] Spinadel V. W., *New Smarandache Sequences: The Family of Metallic Means*, [online], http://vixra.org/abs/1403.0507 [access 29.06.2015].

[27] Spinadel V. W., *The Family of Metallic Means*, 'VisMath — Visual Mathematics', Electronic Journal, 1999, Vol. 1(3), [online], http://www.mi.sanu.ac.rs/vismath/spinadel/index.html [access 29.06.2015].

[28] Stakhov A., *The Mathematics of Harmony: From Euclid to Contemporary Ma-thematics and Computer Science*, Series of Knots and Everything, Vol. 22, World Scientific, Singapore 2009.

[29] Thompson D. W., *On Growth and Form*, Cambridge Univ. Press, The Macmillian Comp., New York 1945.

[30] Viswanath D., *Random Fibonacci sequences and the number 1.13198824...*, 'Mathematics of Computation', 1999, 69(231), pp. 1131–1155.

[31] Wysocki H., *Model nieklasycznego rachunku operatorów Bittnera dla różnicy wstecznej*, 'Zeszyty Naukowe Akademii Marynarki Wojennej', 2010, 2(181), pp. 37–48 [*Bittner non-classical operational calculus model for the backward difference* — available in Polish].

[32] Wysocki H., *Rozwiązanie liniowego równania różnicowego w przestrzeni wyników generowanej przez ciągi dwustronne*, 'Zeszyty Naukowe Akademii Marynarki Wojennej', 2010, 3(182), pp. 85–101 [*The solution of a linear difference equation in the space of results generated by two-sided sequences* — available in Polish].

SPIRA MIRABILIS W WYBRANYCH MODELACH RACHUNKU OPERATORÓW BITTNERA

STRESZCZENIE

Korzystając z pojęcia elementu wykładniczego w nieklasycznym rachunku operatorów Bittnera oraz stosując wybrane modele tego rachunku, wyznaczono opisy parametryczne spiral będących odpowiednikami spirali logarytmicznej.

Słowa kluczowe:

spirala logarytmiczna, rachunek operatorów, pochodna, pierwotna, warunek graniczny, element wykładniczy.