OSCILLATION ON THE LEFT
AND ON THE RIGHT

Ján Ohriska

Department of Mathematics, Catholic University, Pl. Andrej Hlinka 56/1
034 01 Ružoberok, Slovak Republic
e-mail: jan.ohriska@fedu.ku.sk

Abstract

In the present paper we derive sufficient conditions for the linear differential equation $(r(t)y'(t))' + p(t)y(t) = 0$ to be either oscillatory or non-oscillatory on the left and eventually on the right. Some estimations of count of zero points for solutions to considered equation on an interval are also presented.

1. Introduction

We consider the linear differential equation

$$(r(t)y'(t))' + p(t)y(t) = 0 \tag{1}$$

on the interval (a, b), where $-\infty \le a < b \le \infty$. The following conditions are assumed to hold throughout the paper:

(i) $r \in C(a, b)$, $r(t) > 0$;

(ii) $p \in C(a, b)$, $p(t) \ge 0$, $p(t)$ is not identically zero in any right neighborhood of a and in any left neighborhood of b.

We call a function u a solution of the equation (1) on the interval (a, b) if $u \in C^1(a, b)$, $ru' \in C^1(a, b)$ and it satisfies the equation (1) for $t \in (a, b)$.

In the sequel we shall restrict our attention to non-trivial solutions of the equations considered. Such a solution u is called oscillatory on the left if there exists a decreasing sequence $\{t_n\}_{n=1}^{\infty}$ of the points $t_n \in (a, b)$ with the property $\lim_{n \to \infty} t_n = a$ and $u(t_n) = 0$ for $n = 1, 2, 3, \ldots$. By analogy,

a solution u is called oscillatory on the right if there exists an increasing sequence $\{t_n\}_{n=1}^{\infty}$ of the points $t_n \in (a, b)$ with the property $\lim_{n \to \infty} t_n = b$ and $u(t_n) = 0$ for $n = 1, 2, 3, \ldots$. An equation is said to be oscillatory on the left, eventually on the right, if all its solutions are oscillatory on the left, eventually on the right.

It is well known that second order differential equations are very important in applications. Numerous phenomena in physical, biological, and engineering sciences can be described by second order differential equations. Moreover, as we can see e.g. in [1], results on second order differential equations play important role in the study of higher order differential equations.

C'est ascavoir that in the study of the oscillatory character of solutions to differential equations we usually consider differential equation on an interval of the form $[a, \infty)$ for some real number a (see e.g. [1], [2], [3], [4] and references cited therein). In such a case, a solution u of studied equation is called oscillatory if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of the points $t_n \in [a, \infty)$ such that $\lim_{n \to \infty} t_n = \infty$ and $u(t_n) = 0$ for $n = 1, 2, 3, \ldots$. It is clear that in our terminology such a solution is oscillatory on the right, and moreover in such a special case when $b = \infty$.

The aim of this paper is to derive sufficient conditions for the linear differential equation (1) to be either oscillatory or non-oscillatory on the left and eventually on the right and give estimates of count of zero points for solutions to the differential equation (1) on a subinterval of (a, b).

2. Main results

First we pay attention to an oscillation on the left.

We put

$$R_1(t) = \int_a^t \frac{ds}{r(s)} \quad \text{for} \quad t \in (a, b). \tag{2}$$

Theorem 1. *Suppose that* $\lim_{t \to a^+} R_1(t) = 0$. *Then equation (1) is oscillatory on the left if*

$$\liminf_{t \to a^+} R_1^2(t) r(t) p(t) > 1/4 \tag{3}$$

and it is non-oscillatory on the left if

$$\limsup_{t \to a^+} R_1^2(t) r(t) p(t) < 1/4. \tag{4}$$

P r o o f. One can verified directly that the linear independent solutions of the differential equation (generalized Euler equation)

$$(r(t) u'(t))' + \frac{\alpha}{r(t) R_1^2(t)} u(t) = 0, \quad t \in (a, b) \tag{5}$$

are the functions

$$u_1(t) = [R_1(t)]^{\frac{1}{2}+\sqrt{\frac{1}{4}-\alpha}}, \quad u_2(t) = [R_1(t)]^{\frac{1}{2}-\sqrt{\frac{1}{4}-\alpha}} \quad \text{if} \quad \alpha < \frac{1}{4},$$

$$u_1(t) = [R_1(t)]^{\frac{1}{2}}, \quad u_2(t) = [R_1(t)]^{\frac{1}{2}} \ln R_1(t) \quad \text{if} \quad \alpha = \frac{1}{4}$$

and

$$u_1(t) = [R_1(t)]^{\frac{1}{2}} \cos\left\{ \sqrt{\alpha - 1/4}\ln R_1(t) \right\},$$

$$u_2(t) = [R_1(t)]^{\frac{1}{2}} \sin\left\{ \sqrt{\alpha - 1/4}\ln R_1(t) \right\} \quad \text{if} \quad \alpha > \frac{1}{4}. \tag{6}$$

Note that equation (5) was solved in [4]. However, we see that it is oscillatory on the left in the case $\alpha > 1/4$, since $\ln R_1(t) \to -\infty$ as $t \to a^+$, and it is non-oscillatory on the left in the case $\alpha \le 1/4$. Now, using the Sturm comparison theorem, it is clear that equation (1) is oscillatory on the left if for some $t_1 \in (a, b]$ and some $\alpha > 1/4$ the inequality

$$p(t) \ge \frac{\alpha}{r(t)R_1^2(t)} \quad \text{for} \quad t \in (a, t_1) \tag{7}$$

is satisfied. On the other hand, equation (1) is non-oscillatory on the left if for some $t_2 \in (a, b]$ and some $\alpha \le 1/4$ the inequality

$$p(t) \le \frac{\alpha}{r(t)R_1^2(t)} \quad \text{for} \quad t \in (a, t_2) \tag{8}$$

is satisfied.

We use the designation $\beta = \liminf_{t \to a^+} R_1^2(t)r(t)p(t)$.

Now the assumption (3) says that for every $\varepsilon > 0$, i.e. also for such ε that $\mu = \beta - \varepsilon > 1/4$, there exists $t_1 \in (a, b)$ such that for $t \in (a, t_1)$ we have

$$R_1^2(t)r(t)p(t) \ge \mu$$

which means (by (7)) that the equation (1) is oscillatory on the left.

We use the designation $\gamma = \limsup_{t \to a^+} R_1^2(t)r(t)p(t)$.

In a like manner, the assumption (4) says that for every $\varepsilon > 0$, i.e. also for such ε that $\nu = \gamma + \varepsilon < 1/4$, there exists $t_2 \in (a, b)$ such that for $t \in (a, t_2)$ we have

$$R_1^2(t)r(t)p(t) \le \nu$$

which means (by (8)) that equation (1) is non-oscillatory on the left and the proof is complete.

$\square$

Now we pay attention to the problem of oscillation on the right.
For that purpose we put

$$R_2(t) = \int_t^b \frac{ds}{r(s)} \quad \text{for} \quad t \in (a, b). \tag{9}$$

Theorem 2. *Suppose that* $\lim_{t \to b^-} R_2(t) = 0$. *Then equation (1) is oscillatory on the right if*

$$\liminf_{t \to b^-} R_2^2(t) r(t) p(t) > 1/4 \tag{10}$$

and it is non-oscillatory on the right if

$$\limsup_{t \to b^-} R_2^2(t) r(t) p(t) < 1/4. \tag{11}$$

P r o o f. The proof of this theorem is similar to the previous one with the difference that now instead of equation (5) we employ the generalized Euler equation

$$(r(t)u'(t))' + \frac{\alpha}{r(t)R_2^2(t)} u(t) = 0, \quad t \in (a, b), \tag{12}$$

the linear independent solutions of which are the functions

$$u_1(t) = [R_2(t)]^{\frac{1}{2} + \sqrt{\frac{1}{4} - \alpha}}, \quad u_2(t) = [R_2(t)]^{\frac{1}{2} - \sqrt{\frac{1}{4} - \alpha}} \quad \text{if} \quad \alpha < \frac{1}{4},$$

$$u_1(t) = [R_2(t)]^{\frac{1}{2}}, \quad u_2(t) = [R_2(t)]^{\frac{1}{2}} \ln R_2(t) \quad \text{if} \quad \alpha = \frac{1}{4}$$

and

$$u_1(t) = [R_2(t)]^{\frac{1}{2}} \cos \left\{ \sqrt{\alpha - 1/4} \ln R_2(t) \right\},$$

$$u_2(t) = [R_2(t)]^{\frac{1}{2}} \sin \left\{ \sqrt{\alpha - 1/4} \ln R_2(t) \right\} \quad \text{if} \quad \alpha > \frac{1}{4}.$$

$\square$

Note that the above Theorem 2 extends Theorem 1 of [3].

We can also introduce the sufficient conditions for the oscillation (non-oscillation) of equation (1) on the left or on the right in another form. In order to do it, first we put

$$R_3(t) = \int_t^c \frac{ds}{r(s)} \quad \text{for} \quad t \in (a, c), \tag{13}$$

where c is some number in (a, b). By using the function R_3, we obtain the following result.

Theorem 3. *Suppose that* $\lim_{t\to a^+} R_3(t) = \infty$. *Then equation (1) is oscillatory on the left if*

$$\liminf_{t\to a^+} R_3^2(t)r(t)p(t) > 1/4 \tag{14}$$

and it is non-oscillatory on the left if

$$\limsup_{t\to a^+} R_3^2(t)r(t)p(t) < 1/4. \tag{15}$$

P r o o f. To prove this theorem means to repeat the proof of the next Theorem 4 with the difference that instead of equation (20) we use for comparison the generalized Euler equation

$$(r(t)u'(t))' + \frac{\alpha}{r(t)R_3^2(t)}u(t) = 0, \quad t \in (a,c) \tag{16}$$

with the solutions

$$u_1(t) = [R_3(t)]^{\frac{1}{2}+\sqrt{\frac{1}{4}-\alpha}}, \quad u_2(t) = [R_3(t)]^{\frac{1}{2}-\sqrt{\frac{1}{4}-\alpha}} \quad \text{if} \quad \alpha < \frac{1}{4},$$

$$u_1(t) = [R_3(t)]^{\frac{1}{2}}, \quad u_2(t) = [R_3(t)]^{\frac{1}{2}} \ln R_3(t) \quad \text{if} \quad \alpha = \frac{1}{4},$$

and

$$u_1(t) = [R_3(t)]^{\frac{1}{2}} \cos\left\{\sqrt{\alpha - 1/4}\ln R_3(t)\right\},$$

$$u_2(t) = [R_3(t)]^{\frac{1}{2}} \sin\left\{\sqrt{\alpha - 1/4}\ln R_3(t)\right\} \quad \text{if} \quad \alpha > \frac{1}{4}.$$

Thus, we omit details of the proof.

$\square$

At last we put

$$R_4(t) = \int_d^t \frac{ds}{r(s)} \quad \text{for} \quad t \in (d,b), \tag{17}$$

where d is some number in (a,b), and give another result on oscillation on the right.

Theorem 4. *Suppose that* $\lim_{t\to b^-} R_4(t) = \infty$. *Then equation (1) is oscillatory on the right if*

$$\liminf_{t\to b^-} R_4^2(t)r(t)p(t) > 1/4 \tag{18}$$

and it is non-oscillatory on the right if

$$\limsup_{t\to b^-} R_4^2(t)r(t)p(t) < 1/4. \tag{19}$$

P r o o f. Now we can act as in the proof of Theorem 1. On the interval (d, b), we consider the equation (1) and the generalized Euler equation

$$(r(t)u'(t))' + \frac{\alpha}{r(t)R_4^2(t)}u(t) = 0, \quad t \in (d, b) \tag{20}$$

with the solutions

$$u_1(t) = [R_4(t)]^{\frac{1}{2}+\sqrt{\frac{1}{4}-\alpha}}, \quad u_2(t) = [R_4(t)]^{\frac{1}{2}-\sqrt{\frac{1}{4}-\alpha}} \quad \text{if} \quad \alpha < \frac{1}{4},$$

$$u_1(t) = [R_4(t)]^{\frac{1}{2}}, \quad u_2(t) = [R_4(t)]^{\frac{1}{2}} \ln R_4(t) \quad \text{if} \quad \alpha = \frac{1}{4},$$

and

$$u_1(t) = [R_4(t)]^{\frac{1}{2}} \cos\left\{\sqrt{\alpha - 1/4} \ln R_4(t)\right\},$$

$$u_2(t) = [R_4(t)]^{\frac{1}{2}} \sin\left\{\sqrt{\alpha - 1/4} \ln R_4(t)\right\} \quad \text{if} \quad \alpha > \frac{1}{4}.$$

Concerning the assumption $\lim_{t \to b^-} R_4(t) = \infty$, we see that equation (20) is oscillatory on the right if $\alpha > 1/4$ and it is non-oscillatory on the right if $\alpha \leq 1/4$.

According to the Sturm comparison theorem, we know that equation (1) is oscillatory on the right if for some $t_1 \in (d, b)$ and some $\alpha > 1/4$ the inequality

$$p(t) \geq \frac{\alpha}{r(t)R_4^2(t)} \quad \text{for} \quad t \in (t_1, b) \tag{21}$$

is satisfied. On the other hand, equation (1) is non-oscillatory on the right if for some $t_2 \in (d, b)$ and some $\alpha \leq 1/4$ the inequality

$$p(t) \leq \frac{\alpha}{r(t)R_4^2(t)} \quad \text{for} \quad t \in (t_2, b) \tag{22}$$

is satisfied.

We put $\beta = \liminf_{t \to b^-} R_4^2(t)r(t)p(t)$.

Then following the assumption (18), we know that for every $\varepsilon > 0$, i.e. also for such ε that $\mu = \beta - \varepsilon > 1/4$, there exists $t_1 \in (d, b)$ such that for $t \in (t_1, b)$ we have

$$R_4^2(t)r(t)p(t) \geq \mu$$

which means (by (21)) that equation (1) is oscillatory on the right.

Now we put $\gamma = \limsup_{t \to b^-} R_4^2(t) r(t) p(t)$.

In a like manner, the assumption (19) says that for every $\varepsilon > 0$, i.e. also for such ε that $\nu = \gamma + \varepsilon < 1/4$, there exists $t_2 \in (d, b)$ such that for $t \in (t_2, b)$ we have

$$R_4^2(t) r(t) p(t) \le \nu$$

which means (by (22)) that equation (1) is non-oscillatory on the right and the proof is complete.

$\square$

Note that the above Theorem 4 extends Theorem 2.3 of [2].

As an illustration of previous theorems, we give the following examples.

Example 1. Consider the differential equation

$$\left((t+3)^{1/4} y'(t) \right)' + \left(\left(\frac{3}{4} \right)^3 \frac{1}{(t+3)^{5/2}} + \frac{3^3}{4^4} \frac{1}{(t+3)^{7/4}} \right) y(t) = 0, \quad t \in (-3, \infty).$$

(23)

We see that $r(t) = (t+3)^{1/4}$ and $p(t) = \left(\frac{3}{4} \right)^3 \frac{1}{(t+3)^{5/2}} + \frac{3^3}{4^4} \frac{1}{(t+3)^{7/4}}$ which implies that

$$R_1(t) = \int_{-3}^t \frac{ds}{(s+3)^{1/4}} = \frac{4}{3}(t+3)^{3/4},$$

$\lim_{t \to -3^+} R_1(t) = 0$, $\liminf_{t \to -3^+} R_1^2(t) r(t) p(t) = \lim_{t \to -3^+} R_1^2(t) r(t) p(t) = \infty$. From here, according to Theorem 1, we know that equation (23) is oscillatory on the left. Putting e.g. $d = 1$, we have

$$R_4(t) = \int_1^t \frac{ds}{(s+3)^{1/4}} = \frac{4}{3} \left[(t+3)^{3/4} - 4^{3/4} \right].$$

Then $\lim_{t \to \infty} R_4(t) = \infty$, $\limsup_{t \to \infty} R_4^2(t) r(t) p(t) = \lim_{t \to \infty} R_4^2(t) r(t) p(t) = = 3/16$ and, by Theorem 4, we know that equation (23) is non-oscillatory on the right.

Note that one solution of equation (23) is $y(t) = (t+3)^{9/16} \cos(\sqrt{3}(t+3)^{-3/8})$.

$\square$

Example 2. Consider the differential equation

$$\left(t y'(t) \right)' + \frac{9}{t} y(t) = 0, \quad t \in (0, \infty).$$

(24)

We see that $r(t) = t$, $p(t) = 9/t$. Putting $c = 2$, we have

$$R_3(t) = \int_t^2 \frac{ds}{s} = \ln \frac{2}{t},$$

$\lim_{t\to 0^+} R_3(t) = \infty$, $\liminf_{t\to 0^+} R_3^2(t)r(t)p(t) = \lim_{t\to 0^+} R_3^2(t)r(t)p(t) = \infty$. According to Theorem 3, we know that equation (24) is oscillatory on the left. Putting $d = 1$, we have

$$R_4(t) = \int_1^t \frac{ds}{s} = \ln t.$$

Then $\lim_{t\to\infty} R_4(t) = \infty$, $\liminf_{t\to\infty} R_4^2(t)r(t)p(t) = \lim_{t\to\infty} R_4^2(t)r(t)p(t) = \infty$. Following Theorem 4, we know that equation (24) is oscillatory on the right.

Note that one solution of equation (24) is $y(t) = \sin\left(\ln\frac{2}{t^3}\right)$.

$\square$

Example 3. Consider the differential equation

$$\left(t^2 y'(t)\right)' + \frac{5}{4}y(t) = 0, \quad t \in (0, 100). \tag{25}$$

We see that $r(t) = t^2$, $p(t) = 5/4$. Putting $c = 10$, we have

$$R_3(t) = \int_t^{10} \frac{ds}{s^2} = \frac{1}{t} - \frac{1}{10},$$

$\lim_{t\to 0^+} R_3(t) = \infty$, $\liminf_{t\to 0^+} R_3^2(t)r(t)p(t) = \lim_{t\to 0^+} R_3^2(t)r(t)p(t) = \frac{5}{4}$. According to Theorem 3, we know that equation (25) is oscillatory on the left.

Moreover

$$R_2(t) = \int_t^{100} \frac{ds}{s^2} = \frac{1}{t} - \frac{1}{100}.$$

Then
$\lim_{t\to 100^-} R_2(t) = 0$, $\limsup_{t\to 100^-} R_2^2(t)r(t)p(t) = \lim_{t\to 100^-} R_2^2(t)r(t)p(t) = 0$.

Following Theorem 2, we know that equation (25) is non-oscillatory on the right. Note that one solution of equation (25) is $y(t) = t^{-1/2}\sin\ln t$.

$\square$

Now we give an information about count of zero points for solutions to the differential equation (1).

Note that in the rest of this paper, by $[x]$ ($x \in \mathbb{R}$) we mark the biggest integer which is less or equal to x, i.e. $x \in \mathbb{R} \Rightarrow [x] \le x < [x] + 1$.

Theorem 5. *Assume that* $0 \le \lim_{t\to a^+} R_1(t) < \infty$ *and* $t_1, t_2 \in (a, b)$ *are such that* $t_1 < t_2$. *Let for* $t \in [t_1, t_2]$ *the inequality*

$$R_1^2(t)r(t)p(t) > \frac{1}{4} \tag{26}$$

be satisfied. Denote by σ *the count of zero points for a solution to the differential equation (1) on the interval* $[t_1, t_2]$. *Then we have*

$$\left[\frac{\sqrt{\alpha - 1/4}}{\pi} \ln \frac{R_1(t_2)}{R_1(t_1)} \right] \le \sigma \le \left[\frac{\sqrt{\beta - 1/4}}{\pi} \ln \frac{R_1(t_2)}{R_1(t_1)} \right] + 1, \qquad (27)$$

where $\alpha = \min_{t \in [t_1, t_2]} R_1^2(t) r(t) p(t)$, $\beta = \max_{t \in [t_1, t_2]} R_1^2(t) r(t) p(t)$.

P r o o f. The continuity of the function $R_1^2(t) r(t) p(t)$ on the interval $[t_1, t_2]$ and inequality (26) yield the inequality $\alpha > 1/4$. We know already that for $\alpha > 1/4$ the linearly independent solutions to equation (5) are given by (6) and we see that the function $\sqrt{\alpha - 1/4} \ln R_1(t)$ maps the interval $[t_1, t_2]$ into the interval
$$\left[\sqrt{\alpha - 1/4} \ln R_1(t_1), \sqrt{\alpha - 1/4} \ln R_1(t_2) \right],$$
i.e. into the interval of the length $\sqrt{\alpha - 1/4} \ln \frac{R_1(t_2)}{R_1(t_1)}$. But then the count of zero points for a solution to equation (5) on the interval $[t_1, t_2]$ is at most
$$\left[\frac{\sqrt{\alpha - 1/4}}{\pi} \ln \frac{R_1(t_2)}{R_1(t_1)} \right] + 1.$$

The inequality $R_1^2(t) r(t) p(t) \ge \alpha$ for $t \in [t_1, t_2]$ ensures that between any two zero points of a solution to equation (5) there is at least one zero point of a solution to equation (1) and we have the left inequality of (27). One can obtain the right inequality of (27) by similar arguments.

$\square$

Similar results on count of zero points can be obtained if some of functions R_2, R_3 and R_4 defined by (9), (13) and (17) are used instead of the function R_1.

References

[1] J.H. Barrett. Oscillation theory of ordinary linear differential equations. *Adv. Math.*, **3**, 415–509, 1969.

[2] J. Ohriska. On the oscillation of a linear differential equation of second order. *Czechoslovak Math. J.*, **39** (114), 16–23, 1989.

[3] J. Ohriska. Oscillation theorems for non-canonical self-adjoint differential equations of second order. *J. Anal. Appl.*, **21**, No. 2, 515–520, 2002.

[4] J. Ohriska. Problems with one quarter. *Czechoslovak Math. J.*, **55** (130), 349–363, 2005.