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FOURTH ORDER NONLINEAR EVOLUTION EQUATION FOR INTERFACIAL GRAVITY WAVES IN THE PRESENCE OF AIR FLOWING OVER WATER AND A BASIC CURRENT SHEAR

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A fourth order nonlinear evolution equation, which is a good starting point for the study of nonlinear water waves as first pointed out by Dysthe (1979) is derived for gravity waves propagating at the interface of two superposed fluids of infinite depth in the presence of air flowing over water and a basic current shear. A stability analysis is then made for a uniform Stokes gravity wave train. Graphs are plotted for the maximum growth rate of instability and for wave number at marginal stability against wave steepness for different values of air flow velocity and basic current shears. Significant deviations are noticed from the results obtained from the third order evolution equation, which is the nonlinear Schrödinger equation.

AMS Mathematical subject classification: 47J35, 76F10, 76B15.

Key words: nonlinear evolution equation, basic current shear, Stokes gravity wave.

1. Introduction

The study of Mclean *et al*. (1981) for three-dimensional stability of finite amplitude water-waves on the surface of deep water reveals that there are two distinct types of instability of gravity waves of finite amplitude in deep water. One is predominantly two-dimensional and is related to all known results (for example Benjamin - Feir instability) for special case, and this has been designated as type-I instability. The other designated as type-II instability is predominantly three-dimensional and becomes dominant when wave steepness is sufficiently large.

Yuen (1984) made an extension of the above mentioned paper to the case of interfacial waves with current jump. The type-I instability in the particular case of long-wavelength perturbation and small wave steepness can be investigated analytically from the nonlinear evolution equation, which consists of a nonlinear Schrödinger equation coupled to an equation for the wave-induced mean flow. Such an analytical study for the stability of interfacial wave was made by Grimshaw and Pulin (1985). The corresponding numerical stability analysis for finite wave length of perturbation and finite wave steepness was carried out in a subsequent paper by Pulin and Grimshaw (1985). Pulin and Grimshaw (1986) have made an extension of the above two papers for interfacial gravity waves propagating on a basic current shear, in which both analytical and numerical results are presented. Analytical results are for long-wavelength modulational instability of small-amplitude waves. This was done starting from a third-order nonlinear evolution equation for two space dimension (i.e., one dimension in propagation space), which is a nonlinear Schrödinger

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equation. The results are presented for two superposed fluids of finite depths. The case of infinitely deep fluids on both sides of the interface was considered in detail for air-water interface.

For a small amplitude, $ka < 0.1$ the results obtained from the lowest order nonlinear Schrödinger equation, when compared with Longuet-Higgins's (1978; 1978) exact results, are fairly accurate. Here k is the wave number and a is the amplitude of the wave.

But for $ka > 0.15$ the prediction from the nonlinear Schrödinger equation do not agree with the exact results of Longuet-Higgins's (1978; 1978), Dysthe (1979), showed that a surprising improvement on these results relating to stability of a finite amplitude wave can be attained by extending the perturbation analysis one step further, i.e., adding the order ε^4 term in the nonlinear Schrödinger equation.

From this fourth-order evolution equation Janssen (1983) elaborated on the Dysthe's approach by investigating the effect of wave-induced flow on the long-time behavior of Benjamin-Feir instability and also applied this equation to the homogeneous random field of gravity waves and obtained the nonlinear energy transfer function found by Dungey and Hui (1979). Stiassnie (1984) showed that Zakharov's integral equation yields the modified or fourth-order nonlinear Schrödinger equation for the particular case of narrow spectrum. Hogan (1985) considered the stability of a train of non-linear capillary-gravity wave on the surface of an ideal fluid of infinite depth. He derived from the Zakharov equation under the assumption of a narrow band of waves, and including the full form of the interaction coefficient for capillary gravity-waves, an evolution equation for the wave envelope that is correct to fourth order in the wave steepness. Derivation of fourth order nonlinear evolution equations for deep water surface gravity waves including different context and stability analysis made from them were carried out by several authors, e.g., Dhar and Das (1999; 2001), Majumder and Dhar (2011).

The second-order corrections to the first order stability properties are shown to depend on the interaction between the mean flow and the frequency dispersion term of the wave envelope. Brinch-Nielsen and Jonsson (1986) also derived the fourth-order evolution equation for a three-dimensional Stokes wave on arbitrary water depth. In deep water the equation reduces to those of Dysthe (1979), and on finite depth the third order terms agree with these of Davey and Stewartson (1974), and Hasimoto and Ono (1972).

In the present paper we extend the paper considered by Pulin and Grimshaw (1986) for the case of wind blowing over water, starting from a fourth order nonlinear evolution equation. Therefore this paper considers the influence of wind on Benjamin-Feir instability. The present fourth order analysis shows an appreciable deviation from the third order analysis considered by Pulin and Grimshaw.

Here we derive a two dimensional fourth order nonlinear evolution equation for gravity waves propagating at the interface of two superposed fluids of infinite depth in the presence of air flowing over water and a basic current shear.

From this nonlinear evolution equation, a nonlinear dispersion relation is determined, and an expression for the maximum growth rate of instability is obtained. Graphs are plotted showing the maximum growth rate of instability against wave steepness for an air-water interface for different values of air-flow velocity v, and basic current shears Ω_1 and Ω_2. It is observed that in the fourth-order analysis, the maximum growth rate of instability first increases with the increase of wave steepness and then it decreases, while in the third-order analysis, the growth rate increases steadily with the increase of wave steepness. Again stable and unstable regions in $\zeta_0 - \lambda$ space are shown in the figures for air-water interface for different values of air-flow velocity v, and for basic current shears Ω_1 and Ω_2.

2. Basic equations

The common horizontal interface between air and water in the undisturbed state is taken as $y = 0$ plane. In the undisturbed state air flows over water with a velocity v in a direction that is taken as the *x*-axis. Each fluid has a basic current shear which has uniform vorticity Ω_1 and Ω_2, respectively corresponding to a basic horizontal current in the *x*-direction $-\Omega_1 y$ and $-\Omega_2 y$, respectively. We take $y = \zeta(x,t)$ as the equation of the common interface at any time t in the perturbed state. Let ρ_1 and ρ_2 be the density of air and water respectively. We introduce the dimensionless quantities φ^*, $\varphi^{*\prime}$, ψ^*, $\psi^{*\prime}$, ζ^*, $\left(x^*, y^*\right)$, t^*, Ω_1^*, Ω_2^*, v^* and γ^* denoting respectively perturbed velocity potentials of water, perturbed velocity potentials of air, perturbed stream function of water, perturbed stream function of air, surface elevation of air-water interface, space coordinates, time, current shear of water, current shear of air, air flow velocity and the ratio of the densities of air to water respectively. These dimensionless quantities are related to the corresponding dimensional quantities by the following relations.

$$\varphi^* = \sqrt{k^3/8g}\,\varphi, \qquad \varphi^{*\prime} = \sqrt{k^3/8g}\,\varphi',$$

$$\psi^* = \sqrt{k^3/8g}\,\psi, \qquad \psi^{*\prime} = \sqrt{k^3/8g}\,\psi',$$

$$\zeta^* = \left(k_0/2\right)\zeta, \qquad \left(x^*, y^*\right) = \left(\frac{k_0}{2}x, \frac{k_0}{2}y\right), \tag{2.1}$$

$$t^* = \left(\sqrt{k_0 g/2}\right)t, \qquad \Omega_1^* = \sqrt{2/k_0 g}\,\Omega_1,$$

$$\Omega_2^* = \sqrt{2/k_0 g}\,\Omega_2, \qquad v^* = \sqrt{\frac{k_0}{g}}\,v, \qquad \gamma^* = \frac{\rho_1}{\rho_2}$$

where k_0 is some characteristic wave number. In the future all these quantities will be written in their dimensionless form but with their asterisks deleted.

The perturbed velocity potentials φ, φ' and stream functions ψ, ψ' of the water and air respectively satisfy the two-dimensional Laplace equations

$$\nabla^2\varphi = 0, \qquad \nabla^2\psi = 0, \qquad \text{in} \qquad -\infty < y < \zeta, \tag{2.2}$$

$$\nabla^2\varphi' = 0, \qquad \nabla^2\psi' = 0, \qquad \text{in} \qquad \zeta < y < \infty. \tag{2.3}$$

The kinematic boundary conditions to be satisfied at the interface are the following

$$\frac{\partial\varphi}{\partial y} - \frac{\partial\zeta}{\partial t} = \left(\frac{\partial\varphi}{\partial x} - \Omega_2\zeta\right)\frac{\partial\zeta}{\partial x}, \qquad \text{when} \qquad y = \zeta, \tag{2.4}$$

$$\frac{\partial\varphi'}{\partial y} - \frac{\partial\zeta}{\partial t} - v\frac{\partial\zeta}{\partial x} = \left(\frac{\partial\varphi'}{\partial x} - \Omega_1\zeta\right)\frac{\partial\zeta}{\partial x}, \qquad \text{when} \qquad y = \zeta, \tag{2.5}$$

The condition of continuity of pressure at the interface is given by

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]+\Omega_2\psi-\Omega_2 y\frac{\partial\varphi}{\partial x}+\zeta=$$
$$=\gamma\left[\frac{\partial\varphi'}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\varphi'}{\partial x}\right)^2+\left(\frac{\partial\varphi'}{\partial y}\right)^2\right\}+\Omega_1\psi'-\Omega_1 y\frac{\partial\varphi'}{\partial x}+\zeta+v\frac{\partial\varphi'}{\partial x}\right] \quad \text{when} \quad y=\zeta. \tag{2.6}$$

Also φ, φ', ψ, ψ' should satisfy the following conditions at infinity

$$\varphi,\psi\to 0 \quad \text{as} \quad y\to-\infty; \qquad \varphi',\psi'\to 0 \quad \text{as} \quad y\to\infty. \tag{2.7}$$

Since a disturbance is assumed to be a progressive wave we look for solutions of the above equations in the form

$$P=P_0+\sum_{n=1}^{\infty}\left[P_n\exp in(kx-\omega t)+P_n^*\exp-in(kx-\omega t)\right] \tag{2.8}$$

where P stands for φ, φ', ψ, ψ', ζ and $*$ denotes the complex conjugate. Here it is assumed that φ_0, φ'_0, φ_n, φ'_n, ψ_0, ψ'_0, ψ_n, ψ'_n and their complex conjugates are functions of x_1, y, t_1 where $x_1=\varepsilon x$, $t_1=\varepsilon t$ while ζ_0, ζ_n, ζ_n^* are functions of x_1, t_1 only. Here ε is a slowness parameter, and ω, k satisfy the following linear dispersion relation

$$\lambda(\omega,k)\equiv(1+\gamma)\omega^2+(\gamma\Omega_1-\Omega_2)\omega-(1-\gamma)k-2\gamma v\omega k+\gamma v^2k^2-\Omega_1\gamma kv=0. \tag{2.9}$$

We now suppose that the first harmonic linear wave, whose nonlinear evolution equation we are going to study, has its wave number equal to the characteristic wave number k_0. Thus we have $k=1$ in Eq.(2.9) and the linear dispersion relation determining ω becomes

$$(1+\gamma)\omega^2+(\gamma\Omega_1-\Omega_2)\omega-(1-\gamma)-2\gamma v\omega+\gamma v^2-\Omega_1\gamma v=0. \tag{2.10}$$

3. Derivation of evolution equation

On substituting the expansion (2.8) in Eqs (2.2) and (2.3), and then equating coefficients of $\exp in(x-\omega t)$, $(n=1,2)$ we get the following equations

$$\frac{d^2\varphi_n}{dy^2}-\Delta_n^2\varphi_n=0, \qquad \frac{d^2\varphi'_n}{dy^2}-\Delta_n^2\varphi'_n=0, \tag{3.1}$$

$$\frac{d^2\psi_n}{dy^2}-\Delta_n^2\psi_n=0, \qquad \frac{d^2\psi'_n}{dy^2}-\Delta_n^2\psi'_n=0 \tag{3.2}$$

where Δ_n is an operator given by

$$\Delta_n=n-i\varepsilon\frac{\partial}{\partial x_1}, \quad (n=1,2). \tag{3.3}$$

The solution of these equations satisfy the boundary conditions (2.7) and can be put in the form

$$\varphi_n = \exp\left(y\Delta_n\right)A_n , \qquad \varphi'_n = \exp\left(-y\Delta_n\right)A'_n , \tag{3.4}$$

$$\psi_n = \exp\left(y\Delta_n\right)B_n , \qquad \psi'_n = \exp\left(-y\Delta_n\right)B'_n , \tag{3.5}$$

where A_n , B_n , A'_n , B'_n , $\left(n = 1,2\right)$ are function of x_1, t_1 .

We take the Fourier transform of Eqs (2.2) and (2.3) for $n = 0$. The solution of these transformed equations becomes

$$\overline{\varphi_0} = \overline{A_0}\exp\left(\left|\overline{k}_x\right|y\right), \qquad \overline{\varphi'_0} = \overline{A'_0}\exp\left(-\left|\overline{k}_x\right|y\right), \tag{3.6}$$

$$\overline{\psi_0} = \overline{B_0}\exp\left(\left|\overline{k}_x\right|y\right), \qquad \overline{\psi'_0} = \overline{B'_0}\exp\left(-\left|\overline{k}_x\right|y\right) \tag{3.7}$$

where $\overline{\varphi_0}$, $\overline{\varphi'_0}$, $\overline{\psi_0}$, $\overline{\psi'_0}$, are Fourier transforms of $\varphi_0, \varphi'_0, \psi_0, \psi'_0$ respectively defined by

$$\left(\overline{\varphi_0},\overline{\varphi'_0},\overline{\psi_0},\overline{\psi'_0}\right) = \int\int_{-\infty}^{\infty}\left(\varphi_0,\varphi'_0,\psi_0,\psi'_0\right)\exp i\left(\overline{k_x}x_1 - \overline{\omega}t_1\right)dx_1dt_1 \tag{3.8}$$

and $\overline{A_0}$, $\overline{A'_0}$, $\overline{B_0}$, $\overline{B'_0}$ are functions of $\overline{k}_x$, $\overline{\omega}$.

Again substituting the expansion (2.8) in the Taylor-series from Eqs (2.4)-(2.6) about $y = 0$ and then equating coefficients of $\exp in(x - \omega t)$ for $n = 0, 1, 2$ on both sides, we get three sets of equations, in each of which we substitute the solutions for $\varphi_n, \varphi'_n, \psi_n, \psi'_n$ given by Eqs (3.4)-(3.7). For the sake of convenience, we take the Fourier transform of the set of equations corresponding to $n = 0$. The set of equations corresponding to $n = 1,2,0$ has been designated respectively as the first, second and third set respectively. To solve the three sets of equations, we make the following perturbation expansion of the equations A_n , B_n , A', B'_n , ζ_n , $\left(n = 0, 1, 2\right)$.

$$E_1 = \sum_{n=1}^{\infty}\varepsilon^n E_{1n} , \qquad E_m = \sum_{n=2}^{\infty}\varepsilon^n E_{mn} , \qquad \left(m = 0,2\right) \tag{3.9}$$

where E_j stands for A_j , A'_j , B_j , B'_j , ζ_j , $\left(j = 0,1,2\right)$.

On substituting the expansion (3.9) in the above three sets of equations, and then equating various powers of ε on both sides we get a sequence of equations.

From the first-order (i.e., lowest-order) and second-order equations corresponding to the first set of equations resulting from (2.4) and (2.5), we get the solutions for A_{11}, A'_{11} and A_{12}, A'_{12} , respectively. Next, from the first-order and second-order equations corresponding to the second set of equations resulting from (2.4)-(2.6), we get the solutions for $A_{22}, A'_{22}, \zeta_{22}$ and $A_{23}, A'_{23}, \zeta_{23}$, respectively. Finally, from the first-order equations corresponding to the third set of equations resulting from (2.4)-(2.6), we get the solutions for $A_{02}, A'_{02}, \zeta_{02}$, and from the second-order equations corresponding to the third set of equations resulting from (2.6) we get the solution for ζ_{03} .

The equation corresponding to Eq.(2.6) of the first set of equations, which has not been used in getting the above perturbation solutions, can be put in the following convenient form after eliminating A_1, A_1', B_1, B_1'.

$$\lambda(\omega',k')\zeta_1 = i(\Omega_2-\omega')a_1 - i\gamma(\Omega_1+\omega')b_1 - k'c_1 + i\gamma v k' d_1 \tag{3.10}$$

where

$$\omega' = \omega + i\varepsilon\frac{\partial}{\partial t_1}, \qquad k' = k - i\varepsilon\frac{\partial}{\partial x_1}, \tag{3.11}$$

and a_1, b_1, c_1, d_1 are contributions from nonlinear terms.

We keep terms up to ε^4 in Eq.(3.10) and then substitute the solutions for various perturbed quantities appearing on its right hand side, and finally using the transformations

$$\xi = x_1 - c_g t_1, \qquad \tau = \varepsilon t_1, \tag{3.12}$$

and writing $\zeta = \zeta_{11} + \varepsilon\zeta_{12}$ we arrive at the fourth order evolution equation

$$i\frac{\partial\zeta}{\partial\tau} - \delta_1\frac{\partial^2\zeta}{\partial\xi^2} + i\delta_2\frac{\partial^3\zeta}{\partial\xi^3} = \sigma_1\zeta^2\zeta^* + i\sigma_2\zeta\zeta^*\frac{\partial\zeta}{\partial\xi} + i\sigma_3\zeta^2\frac{\partial\zeta^*}{\partial\xi} + \sigma_4\zeta H\frac{\partial(\zeta\zeta^*)}{\partial\xi} \tag{3.13}$$

where H is the Hilbert transform given by

$$\mathrm{H}(\overline{\psi}) = \frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\overline{\psi}(\xi')}{(\xi'-\xi)}d\xi' \ . \tag{3.14}$$

4. Stability of a finite amplitude wave trains

Equation (3.13) admits a Stokes wave-solution

$$\zeta = \frac{\zeta_0}{2}\exp(i\Delta\omega\tau) \tag{4.1}$$

where ζ_0 is a real constant and the nonlinear frequency shift $\Delta\omega$ is given by

$$\Delta\omega = -\frac{1}{4}\zeta_0^2\sigma_1 . \tag{4.2}$$

To study modulational instability of this uniform wave train, we introduce the following perturbation in the uniform solution

$$\zeta = \frac{\zeta_0}{2}(1+\zeta')\exp i(\theta'+\Delta\omega\tau) \tag{4.3}$$

where ζ' and θ' are small real perturbations in amplitude and phase, respectively, and are real.

Substituting (4.3) in the evolution Eq.(3.13), and linearizing with respect to ζ', θ', we get the following equations

$$P_1\zeta' + P_2\theta' - \frac{1}{4}\zeta_0^2\left(\sigma_2 + \sigma_3\right)\frac{\partial \zeta'}{\partial \xi} = 0\,, \tag{4.4}$$

$$P_1\theta' - P_2\zeta' + \frac{1}{2}\sigma_1\zeta_0^2\zeta' + \frac{1}{4}\zeta_0^2\left(\sigma_3 - \sigma_2\right)\frac{\partial \theta'}{\partial \xi} + \frac{\sigma_4\zeta_0^2}{2\pi}P\int_{-\infty}^{\infty}\frac{\partial \xi'}{\xi' - \xi}\frac{\partial \zeta'}{\partial \xi'} = 0 \tag{4.5}$$

where $$P_1 = \frac{\partial}{\partial \tau} + \delta_2\frac{\partial^3}{\partial \xi^3}\,, \qquad P_2 = -\delta_1\frac{\partial^2}{\partial \xi^2}\,. \tag{4.6}$$

Now if we suppose that τ-dependence of ζ' and θ' is of the form $\exp\left(-i\vartheta'\tau\right)$, then Eqs (4.4), (4.5) remain the same as before but P_1 now stands for

$$P_1 = -i\vartheta' + \delta_2\frac{\partial^3}{\partial \xi^3}\,. \tag{4.7}$$

Next, taking the Fourier transform of Eqs (4.4), (4.5), with respect to ξ defined by

$$\left(\overline{\theta'}, \overline{\zeta'}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\theta'(\xi), \zeta'(\xi)\right]\exp\left(-i\lambda\xi\right)d\xi\,, \tag{4.8}$$

we get two linear algebraic equations for $\overline{\theta'}$ and $\overline{\zeta'}$. The condition for the existence of a nontrivial solution of these two equations gives the dispersion relation, given by

$$\overline{P_1} = -\frac{\zeta_0^2\sigma_2\lambda}{4} \pm \left[\overline{P_2}\left\{\overline{P_2} - \frac{\zeta_0^2\sigma_1}{2}\left(1 - \frac{\sigma_4|\lambda|}{\sigma_1}\right)\right\}\right]^{1/2} \tag{4.9}$$

where $$\overline{P_1} = \vartheta - c_g\lambda - \delta_2\lambda^3\,, \qquad \overline{P_2} = \delta_1\lambda^2 \qquad \text{and} \qquad \vartheta = \vartheta' + c_g\lambda\,. \tag{4.10}$$

From Eq.(4.9) it follows that for instability we must have

$$\lambda^2 < \frac{\zeta_0^2\sigma_1}{2\delta_1}\left(1 - \frac{\sigma_4|\lambda|}{\sigma_1}\right), \tag{4.11}$$

and if this condition is met, then the maximum growth rate Γ_M is given by

$$\Gamma_M = \frac{\zeta_0^2\sigma_1}{4}\left(1 - \frac{\zeta_0\sigma_4}{2\sqrt{\delta_1\sigma_1}}\right). \tag{4.12}$$

For $\gamma = 0$, $\Omega_1 = \Omega_2 = 0$ and $v = 0$, Eq.(4.12) reduces to Eq.(3.10) of Dysthe (1979). At marginal stability we have

$$\overline{P_2}\left\{\overline{P_2} - \frac{\zeta_0^2\sigma_1}{2}\left(1 - \frac{\sigma_4|\lambda|}{\sigma_1}\right)\right\} = 0, \tag{4.13}$$

and this gives the following expression for λ at marginal stability.

$$\lambda = \zeta_0\left\{1 - \frac{\zeta_0\sigma_4}{\sqrt{8\sigma_1\delta_1}}\right\}.\sqrt{\frac{\sigma_1}{2\delta_1}}\,. \tag{4.14}$$

5. Conclusion

In the case of air-water interface, the maximum growth rate of instability Γ_M given by Eq.(4.12) has been plotted in Figs 1a and 1b against wave-steepness ζ_0, for some different values of Ω_1 with $\Omega_2 = 0$ and for air flow velocity $v = 4$ and 7, respectively. The same growth rate Γ_M for some different values of Ω_2 with $\Omega_1 = 0$ and for $v = 4$ and 7, respectively, has been plotted in Figs 2a and 2b. From these figures it is seen that Γ_M first increases with the increase of ζ_0 and then its value decreases, while the growth rate obtained from the third order evolution equation increases steadily with the increase of ζ_0.

In the case of air-water interface, the wave number λ at marginal stability given by Eq. (4.14) has been plotted in Figs 3a and 3b against wave-steepness ζ_0, for some different values of Ω_1 with $\Omega_2 = 0$ and for $v = 4$ and 7, respectively, and hence this gives the stable - unstable region in $\lambda - \zeta_0$ plane. The same wave number λ for some different values of Ω_2 with $\Omega_1 = 0$ and for $v = 4$ and 7, respectively, has been plotted in Figs 4a and 4b.

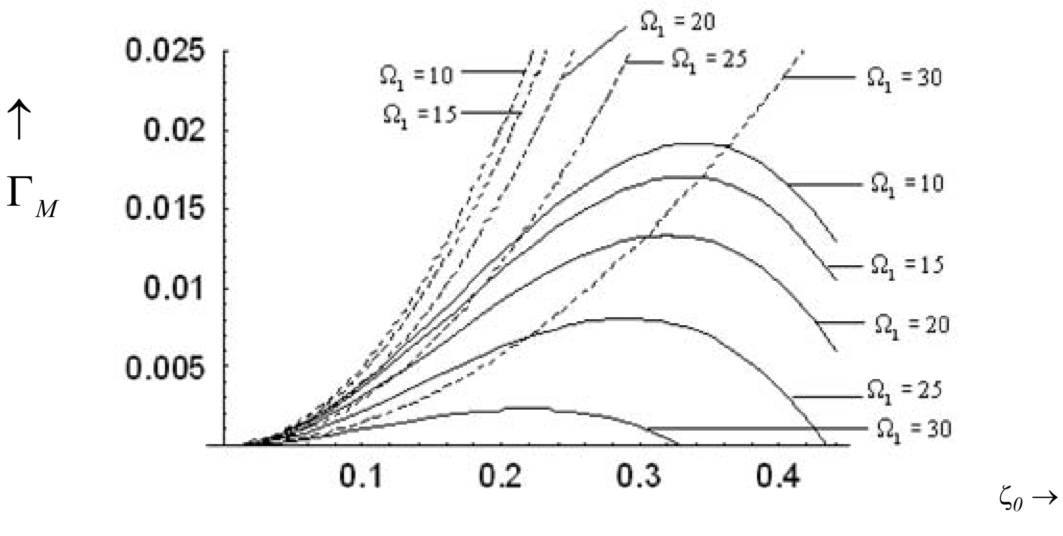

Fig.1a. One-dimensional modulational instability. Maximum growth rate Γ_M against wave steepness ζ_0, $\Omega_2 = 0$, $\gamma = 0.00129$, $v = 4$, ----------------third-order & __________________ fourth -order results.

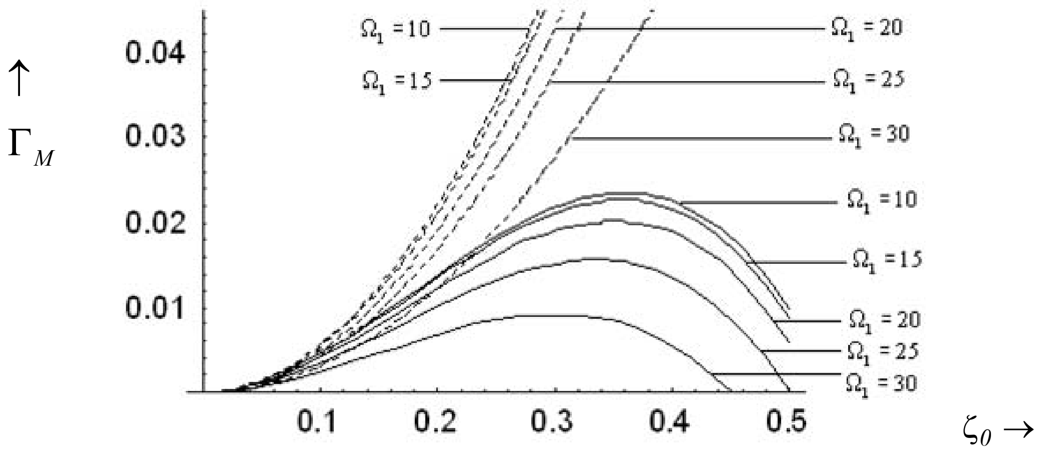

Fig.1b. One-dimensional modulational instability. Maximum growth rate Γ_M against wave steepness ζ_0, $\Omega_2 = 0$, $\gamma = 0.00129$, $\nu = 7$, -------------------third-order & ________________ fourth -order results.

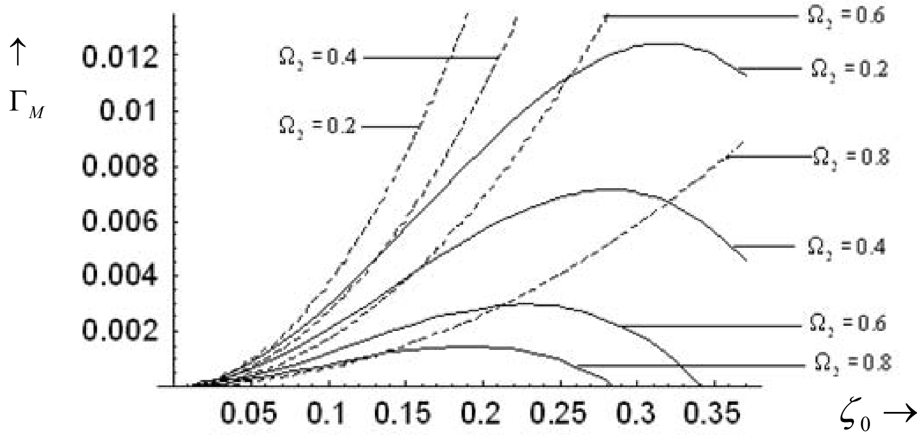

Fig.2a. One-dimensional modulational instability. Maximum growth rate Γ_M against wave steepness ζ_0, $\Omega_1 = 0$, $\gamma = 0.00129$, $\nu = 4$, -------------third-order &___________ fourth -order results.

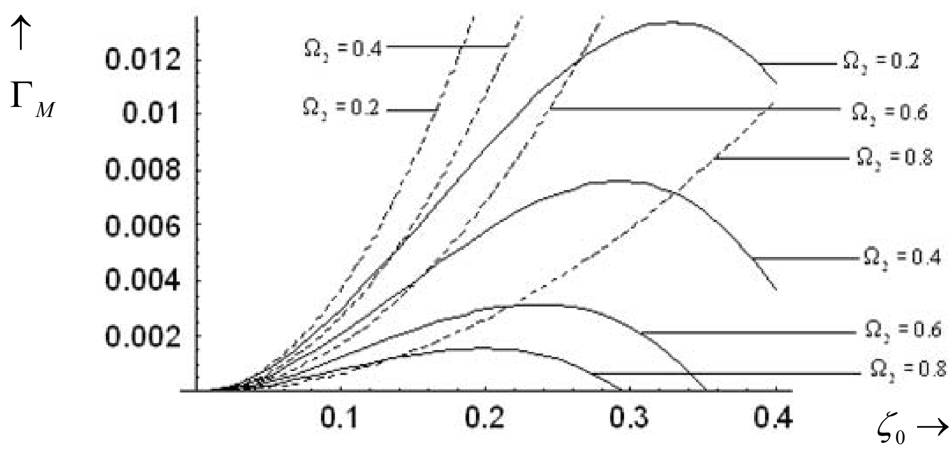

Fig.2b. One-dimensional modulational instability. Maximum growth rate Γ_M against wave steepness ζ_0, $\Omega_1 = 0$, $\gamma = 0.00129$, $\nu = 7$, -------------third-order &___________ fourth -order results.

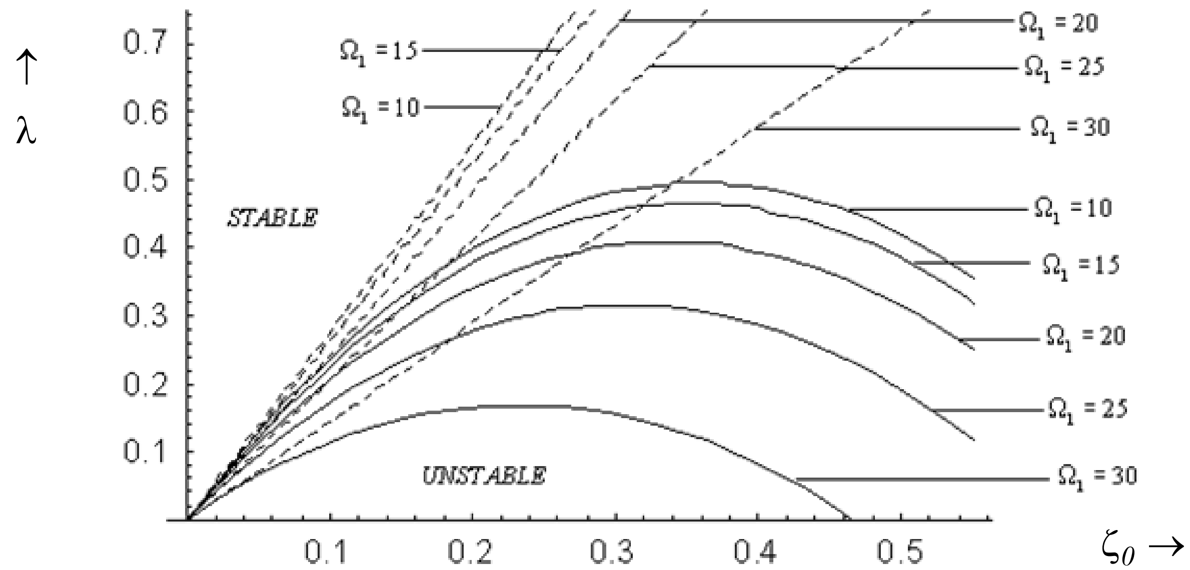

Fig.3a. One-dimensional modulational instability. Perturbed wave number λ at marginal stability against wave steepness ζ_0, $\Omega_2 = 0$, $\gamma = 0.00129$, $\nu = 4$ -------------third-order &___________ fourth-order results.

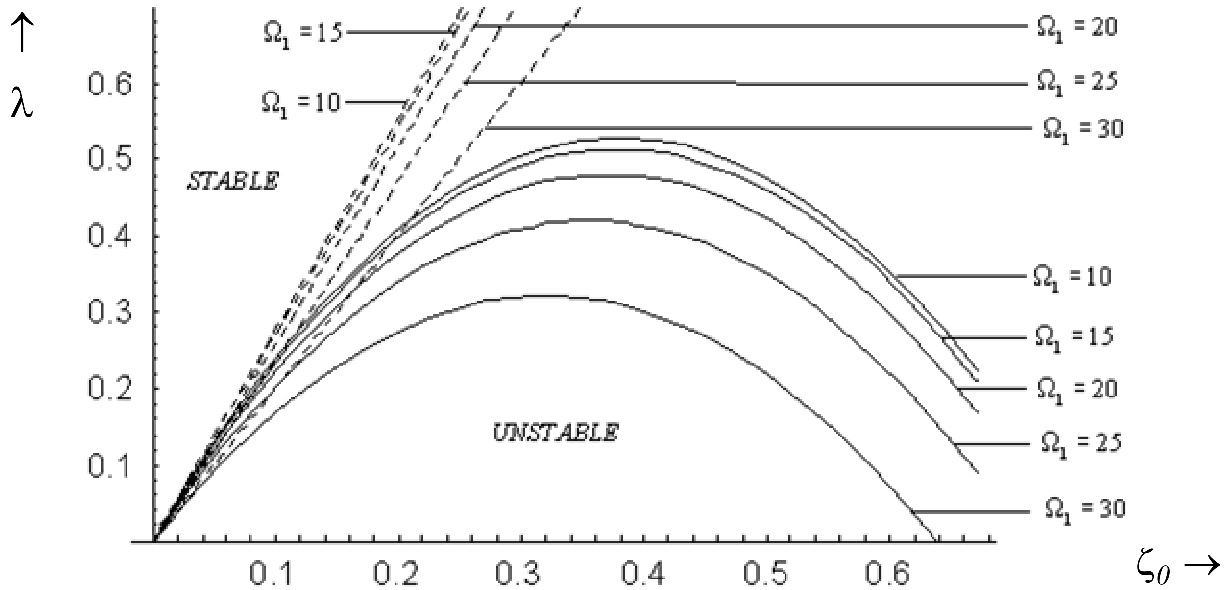

Fig.3b. One-dimensional modulational instability. Perturbed wave number λ at marginal stability against wave steepness ζ_0, $\Omega_2 = 0$, $\gamma = 0.00129$, $\nu = 7$, -------------third-order & ___________ fourth - order results.

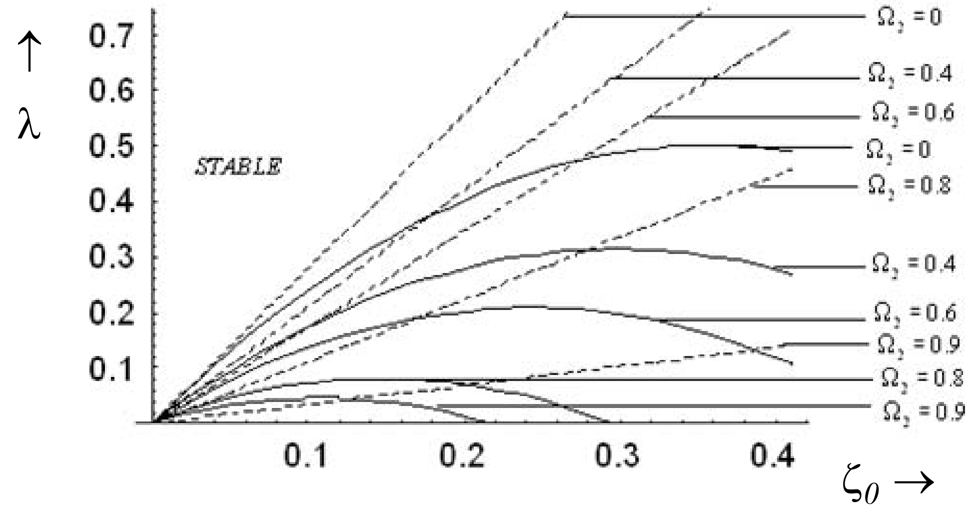

Fig.4a. One-dimensional modulational instability. Perturbed wave number λ at marginal stability against wave steepness ζ_0, $\Omega_1 = 0$, $\gamma = 0.00129$, $\nu = 4$, ---------- third-order & _______ fourth -order results.

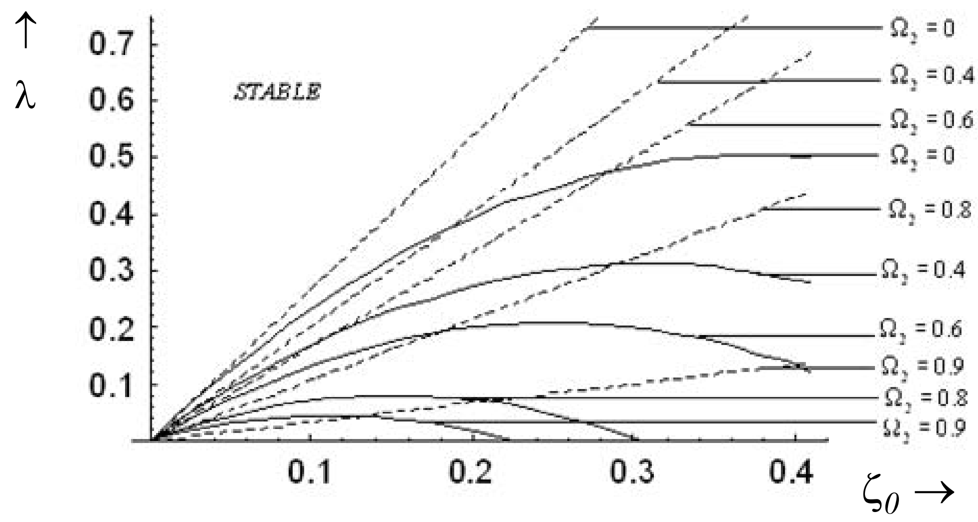

Fig.4b. One-dimensional modulational instability. Perturbed wave number λ at marginal stability against wave steepness ζ_0, $\Omega_1 = 0$, $\gamma = 0.00129$, $\nu = 7$, ---------- third-order & ________ fourth -order results.

Appendix

$$c_g = \frac{d\omega}{dk}, \qquad \delta_1 = -\frac{1}{2}\frac{dc_g}{dk}, \qquad \delta_2 = \frac{1}{2} c_g \lambda_{\omega\omega} \lambda_\omega^{-1} \frac{dc_g}{dk},$$

$$\sigma_1 = \frac{\beta_1}{\lambda_\omega}, \qquad \sigma_2 = \frac{\left(\beta_4 - c_g \beta_2 + 2\beta_1 c_g \lambda_\omega^{-1} \lambda_{\omega\omega}\right)}{\lambda_\omega},$$

$$\sigma_3 = \frac{\left(\beta_5 - c_g \beta_3 + \beta_1 c_g \lambda_\omega^{-1} \lambda_{\omega\omega}\right)}{\lambda_\omega}, \qquad \sigma_4 = \frac{\beta_6}{\lambda_\omega},$$

where β_i 's are given by

$$\beta_1=\frac{\left[2\omega^2+2\gamma\omega(\omega-v)-\gamma\Omega_1^2-\Omega_2^2-\left\{2\gamma(\omega-v)\Omega_1+\gamma\Omega_1^2+(2\omega-\Omega_2)\Omega_2\right\}^2+ +\left\{2\omega(v\gamma+\omega-\gamma\omega)+4\gamma(\omega-v)-\gamma\Omega_1^2-4\omega\Omega_2+\Omega_2^2\right\}^2\left\{2\omega(-v\gamma+\omega+\gamma\omega)\right\}^{-1}\right]}{\left\{1-\gamma-c_g\left(\gamma\Omega_1-\Omega_2\right)\right\}},$$

$$\beta_2=-\frac{\left(-v\gamma-\omega+\gamma\omega+\gamma\Omega_1+\Omega_2\right)\left\{2\gamma(\omega-v)\Omega_1+\gamma\Omega_1^2+(2\omega-\Omega_2)\Omega_2\right\}}{\left\{-1+\gamma+c_g\left(\gamma\Omega_1-\Omega_2\right)\right\}}-$$
$$\frac{2\left\{2(\omega-v\gamma+\gamma\omega)+\gamma\Omega_1+\Omega_2\right\}\left\{2\gamma(\omega-v)\Omega_1+\gamma\Omega_1^2+(2\omega-\Omega_2)\Omega_2\right\}}{\left\{-1+\gamma+c_g\left(\gamma\Omega_1-\Omega_2\right)\right\}}$$
$$\frac{\left[2\left(-\omega-v\gamma+\gamma\omega+\gamma\Omega_1+\Omega_2\right)\left\{2\omega(\omega+v\gamma-\gamma\omega)+4\gamma(\omega-v)\Omega_1+4\omega\Omega_2-\Omega_2^2\right\}\right]}{\left\{\gamma v-(1+\gamma)\omega\right\}^2}-$$
$$\frac{2\left(-\omega-v\gamma+\gamma\omega+\gamma\Omega_1+\Omega_2\right)\left\{2\omega^2+4\gamma(\omega-v)\Omega_1-\gamma\Omega_1^2-4\omega\Omega_2+\Omega_2^2\right\}}{\omega(-\gamma v+\omega+\gamma\omega)}+$$
$$+\frac{1}{2\omega^2\left(-\gamma v+\omega+\gamma\omega\right)^2}\Big[\left(4v\gamma-4\omega-4\gamma\omega-\gamma\Omega_1+\Omega_2\right).$$
$$\left\{2\omega(\gamma v+\omega-\gamma\omega)+4\gamma(\omega-v)\Omega_1-\gamma\Omega_1^2-4\omega\Omega_2+\Omega_2^2\right\}^2\Big],$$

$$\beta_3=\frac{\left(-\omega-v\gamma+\gamma\omega+\gamma\Omega_1+\Omega_2\right)\left(2\gamma(\omega-v)\Omega_1+\gamma\Omega_1^2+(2\omega-\Omega_2)\Omega_2\right)}{\left\{-1+\gamma+c_g\left(\gamma\Omega_1-\Omega_2\right)\right\}},$$

$$\beta_4=-4v\gamma\omega-4\omega^2-4\gamma\omega^2+\gamma(\omega-v)\Omega_1-\omega\Omega_2+3\left(\gamma\Omega_1^2+\Omega_2^2\right)-$$
$$\frac{\left\{2\omega(\omega+v\gamma-\gamma\omega)-4\gamma(\omega-v)\Omega_1-\gamma\Omega_1^2-4\omega\Omega_2+\Omega_2^2\right\}^2}{\omega(-\gamma v+\omega+\gamma\omega)}$$
$$\frac{\left\{2\omega(\omega+v\gamma-\gamma\omega)-4\gamma(\omega-v)\Omega_1-\gamma\Omega_1^2-4\omega\Omega_2+\Omega_2^2\right\}^2}{2\omega(+\gamma v+\omega-\gamma\omega)\left\{\gamma v-(1+\gamma)\omega\right\}^2}-\Re_1,$$

$$\beta_5=-v\gamma\Omega_1+\gamma\omega\Omega_1+\gamma\Omega_1^2-\omega\Omega_2+\Omega_2^2-$$
$$\frac{\left\{2\omega(\omega+v\gamma-\gamma\omega)-4\gamma(\omega-v)\Omega_1-\gamma\Omega_1^2-4\omega\Omega_2+\Omega_2^2\right\}^2}{2\omega(-\gamma v+\omega+\gamma\omega)}-\Re_1,$$

$$\beta_6 = \frac{1}{\left\{1-\gamma+c_g\left(\Omega_2-\gamma\Omega_1\right)\right\}}\Big[\gamma\left(2\omega+\Omega_1-2v\right)\left(2\gamma\left(v-\omega\right)+2\omega+\Omega_1-\gamma\Omega_1+\right.$$
$$+\left(2\omega-\Omega_2\right)\left(2\gamma\left(v-\omega\right)2\omega-\Omega_1\gamma c_g\Omega_1\left(4\omega+\Omega_1-\Omega_2\right)+\gamma\Omega_2\right)+$$
$$\left.+c_g\left(4\omega+\Omega_1-\Omega_2\right)\Omega_2\right)\Big]$$

where

$$-\Re_1 = \frac{1}{\left\{1-\gamma-c_g\left(\gamma\Omega_1-\Omega_2\right)\right\}^2}\Big[\left\{1-\gamma-c_g\left(\gamma\Omega_1-\Omega_2\right)\right\}\left\{2\gamma\Omega_1\left(\omega-v\right)+\gamma\Omega_1^2+\left(2\omega-\Omega_2\right)\Omega_2\right\}^2+$$
$$+\omega\left\{1-\gamma-c_g\left(\gamma\Omega_1-\Omega_2\right)\right\}\left\{2\left(1+\gamma\right)\left(\gamma v+\omega-\gamma\omega\right)-\gamma c_g\Omega_1^2-\gamma\Omega_1\left(1-\gamma-c_g\left(4\omega-\Omega_2\right)\right)+\right.$$
$$\left.-\left(1-\gamma-4\gamma\omega c_g\right)\Omega_2-\gamma c_g\Omega_2^2\right\}+c_g\left\{\gamma\left(\omega-v\right)\Omega_1+\omega\Omega_2\right\}\left\{2\left(1-\gamma\right)\left(\gamma v+\omega-\gamma\omega\right)+\right.$$
$$\left.-c_g\Omega_1^2+\left(\left(-1+\gamma-4\gamma\omega c_g\right)\Omega_2+\gamma c_g\Omega_2^2+\Omega_1\left(-1+\gamma\right)\gamma+c_g\left(\Omega_2-4\omega-\gamma\Omega_2\right)\right)\right\}\Big].$$

Nomenclature

g – acceleration due to gravity
H – Hilbert's transform operator
k – wave number
P – general solution to Eqs (2.4) - (2.6)
t – time
v – air flow velocity
(x,y) – space coordinates
$\delta_i\,(i=1,2)$, $\beta_i\,(i=1,2,3,4,5,6)$, $\sigma_i\,(i=1,2,3,4)$ – coefficients given in the Appendix
$\Delta\omega$ – frequency shift
γ – ratio of densities of air to water
ε – slowness parameter
ζ – elevation of the air-water interface
ζ_0 – wave steepness
(ζ',θ') – small real perturbations of amplitude and phase
ξ,τ – transformed variables
ϑ – perturbed frequency at marginal stability
φ,φ' – velocity potentials of air and water respectively
ψ,ψ' – stream functions of air and water respectively
Ω_1,Ω_2 – current shears of air and water respectively
ω – frequency

Reference

Benjamin T.B. and Feir J.E. (1967): *The disintegration of wave trains on deep water, I.* – Theory. J. Fluid Mech., vol.27, pp.417-430.

Brinch-Nielsen U. and Jonsson J.G. (1986): *Fourth order evolution equations and stability analysis for Stokes waves on arbitrary water depth*. – Wave Motion, vol.8, pp.455-472.

Davey A. and Stewartson K. (1974): *On three dimensional packets of surface waves*. – Proc. R. Soc. Lond., vol.A 338, pp.101-110.

Dhar A.K. and Das K.P. (1991): *Stability of small but finite amplitude interfacial waves*. – Mechanics Research Communications, vol.18, No.6, pp.367-372.

Dhar A.K. and Das K.P. (1999): *A fourth order evolution equation for capillary gravity waves including the effects of wind input and shear in the water current*. – Applied Mechanics and Engineering, vol.4, No.1, pp.5-24.

Dhar A.K. and Das K.P. (2001): *The effect of randomness on stability of surface gravity waves from fourth order nonlinear evolution equation*. – Int. J. of Applied Mechanics and Engineering, vol.6, No.1, pp.11-34.

Dungey J.C. and Hui W.H. (1979): *Nonlinear energy transfer in a narrow gravity-wave spectrum*. – Proc. R. Soc. Lond., vol.A 368, pp.239-265.

Dysthe K.B. (1979): *Note on a modification to the nonlinear Schrödinger equation for application to deep water waves*. – Proc. R. Soc. Lond., vol.A 369, pp.105-114.

Grimshaw R.H.J. and Pullin D.I. (1985): *Stability of finite amplitude interfacial waves. Part 1. Modulational instability for small amplitude waves*. – J. Fluid Mech., vol.160, pp.297-315.

Hasimoto H. and Ono H. (1972): *Nonlinear modulation of gravity waves*. – J. Phys. Soc. Japan, vol.33, No.3, pp.805-811.

Hogan S.J. (1985): *The fourth order evolution equation for deep water gravity capillary waves*. – Proc. R. Soc. Lond., vol.A 402, pp.359-372.

Janssen P.A.E.M. (1983): *On fourth order evolution equation for deep water waves*. – J. Fluid Mech., vol.126, pp.1-11.

Longuet-Higgins M.S. (1978): *The instabilities of gravity waves of finite amplitude in deep water, I. Super Harmonics*. – Proc. R. Soc. Lond., vol.A 360, pp.471-488.

Longuet-Higgins M.S. (1978): *The instabilities of gravity waves of finite amplitude in deep water, II. Sub Harmonics*. Proc. R. Soc. Lond., vol.A, 360, pp.489-506.

Majumder D.P. and Dhar A.K. (2011): *Stability of small but finite amplitude interfacial capillary gravity waves for perturbations in two horizontal directions*. – International Journal of Applied Mechanics and Engineering, vol.16, No.2, pp.425-434.

McLean J.W., Ma Y.C., Martin D.U., Saffman P.G. and Yuen H.C. (1981): *Three dimensional instability of finite amplitude water waves*. – Phys. Rev. Lett., vol.46, pp.817-820.

Pullin D.I. and Grimshaw R.H.J. (1986): *Stability of finite amplitude interfacial waves. Part 3. The effect of basic current shear for one dimensional instabilities*. – J. Fluid Mech., vol.172, pp.277-306.

Pullin D.I. and Grimshaw R.H.J. (1985): *Stability of finite amplitude interfacial waves. Part 2. Numerical results*. – J. Fluid Mech., vol.160, pp.317-336.

Stiassnie M. (1984): *Note on the modified nonlinear Schrödinger equation for deep water waves*. – Wave Motion, vol.6, pp.431-433.

Yuen H.C. (1984): *Non linear dynamics of interfacial waves*. – Physica 12D, pp.71-82.

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