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MINIMAL UNAVOIDABLE SETS OF CYCLES IN PLANE GRAPHS

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Abstract. A set S of cycles is minimal unavoidable in a graph family $\mathcal{G}$ if each graph $G \in \mathcal{G}$ contains a cycle from S and, for each proper subset $S' \subset S$, there exists an infinite subfamily $\mathcal{G}' \subseteq \mathcal{G}$ such that no graph from $\mathcal{G}'$ contains a cycle from S'. In this paper, we study minimal unavoidable sets of cycles in plane graphs of minimum degree at least 3 and present several graph constructions which forbid many cycle sets to be unavoidable. We also show the minimality of several small sets consisting of short cycles.

Keywords: plane graph, polyhedral graph, set of cycles.

Mathematics Subject Classification: 05C10.

1. INTRODUCTION

Throughout this paper, we consider connected graphs without loops and multiple edges, which are planar (that is, they can be drawn in the plane without crossing their edges). A particular plane drawing D of a planar graph G is represented by a triple (V, E, F) where V is the vertex set, E is the edge set and F is the set of faces. Two faces are adjacent if they share a common edge. Each face $\alpha \in F$ is described by its *facial walk* which is a clockwise-oriented closed walk $v_1, e_1, v_2, e_2, \dots, e_{k-1}, v_k, e_k, v_1$ whose vertices and edges are incident with α and, for all $i \in \{1, \dots, k\}$, e_i follows e_{i-1} (indices modulo k) in the counter-clockwise order of edges around v_i in D; in the sequel, we will consider facial walks simply as clockwise-ordered lists of their vertices. The number k is called the *size* of α, and is denoted by $\deg(\alpha)$. A face of size k (at least k) is further referred as k-face ($\geq k$-face); similarly, a vertex of degree k (at least k or at most k) is a k-vertex ($\geq k$-vertex or $\leq k$-vertex, respectively). A face whose facial walk is a cycle will be called *nice face*. Two adjacent faces form a *nice pair* if their common vertices are exactly the endvertices of the common edge.

By $C_k, k \geq 3$, we denote the cycle on k vertices. For positive integers $k_1, \ldots, k_\ell \geq 3$, we set

$$S_{k_1,\ldots,k_\ell} = \{C_{k_1}, \ldots, C_{k_\ell}\};$$

in addition,

$$S_{k_1,\ldots,k_\ell,k^+} = \{C_{k_1}, \ldots, C_{k_\ell}\} \cup \{C_l : \ l \geq k\}$$

(here we also allow $\ell = 0$).

We introduce the following definition: A set S of cycles is *minimal unavoidable* in a graph family $\mathcal{G}$ if each graph $G \in \mathcal{G}$ contains a cycle from S and, for each proper subset $S' \subset S$, there exists an infinite subfamily $\mathcal{G}' \subseteq \mathcal{G}$ such that no graph from $\mathcal{G}'$ contains a cycle from S'. This notion provides a certain unified framework for the study of the cycle structure of graphs of particular families. For example, the Erdős–Gyárfás conjecture says that the set $S_{\{2^k:\ k\geq 2\}}$ is unavoidable in the family of graphs of minimum degree ≥ 3 although it is not known whether is also minimal (see the discussion in [1] and its negative results for cycle lengths being powers of $q \geq 3$). The results of [4] yield that $S_{4,8,16,32,64,128}$ is unavoidable in the family of 3-connected cubic plane graphs, with the possible minimal unavoidability of $S_{4,8,16,32}$ (see also [3]).

The aim of this paper is to explore minimal unavoidable sets of cycles in plane graphs. It is well known that each plane graph of minimum degree at least 3 contains a k-cycle (in fact, a boundary of a k-face), $3 \leq k \leq 5$; moreover, it is not hard to construct infinite families of plane graphs of minimum degree 3 which avoid any two cycles from $S_{3,4,5}$. Thus, in that family, the set $S_{3,4,5}$ is minimal unavoidable. Apart of this result, it seems that a systematic research of minimal unavoidable sets in plane graphs was not performed. We contribute to this topic by specifying the large collection of cycle sets which are not unavoidable (see Section 2) and by showing that the sets $S_{3,4,11}, S_{3,4,8,9}, S_{3,4,6,8}, S_{3,4,7,9}$ and $S_{3,5,6,7}$ are minimal unavoidable (Section 3). The paper concludes with several open problems.

2. NEGATIVE RESULTS

The main tool for showing that some sets of cycles are not unavoidable in the family $\mathcal{G}_3$ of plane graphs of minimum degree at least 3 is the following construction: take a plane graph G which does not contain cycles of lengths $\ell_1, \ldots, \ell_k$, and let x be an arbitrary vertex on the outerface of G. Now, take n copies of G and identify all vertices which are counterparts of x; since the resulting graph G_n is planar, one has just take care to choose G such that $G_n \in \mathcal{G}_3$. In this way, we obtain an infinite sequence $\{G_n\}_{n=1}^{\infty}$ of graphs of $\mathcal{G}_3$, none of which containing cycles of lengths $\ell_1, \ldots, \ell_k$ and $\ell \geq c(G) + 1$ where $c(G)$ is the circumference (the length of the longest cycle) in G.

The source graph G is often built from a smaller plane graph using several operations which replace parts of graphs (mostly vertices and edges) with another configurations. The general operations are the following ones (see Figure 1):

- *truncation*: each k-vertex is replaced by k-face, resulting in plane cubic graph,
- *rectification*: each k-vertex is replaced by k-face, resulting in plane 4-regular graph,
- *edge-truncation*: each edge is replaced by 6-face,
- *edge-pentagonalization*: after edge-truncation, each of new 6-faces is divided by a new edge into two 5-faces.

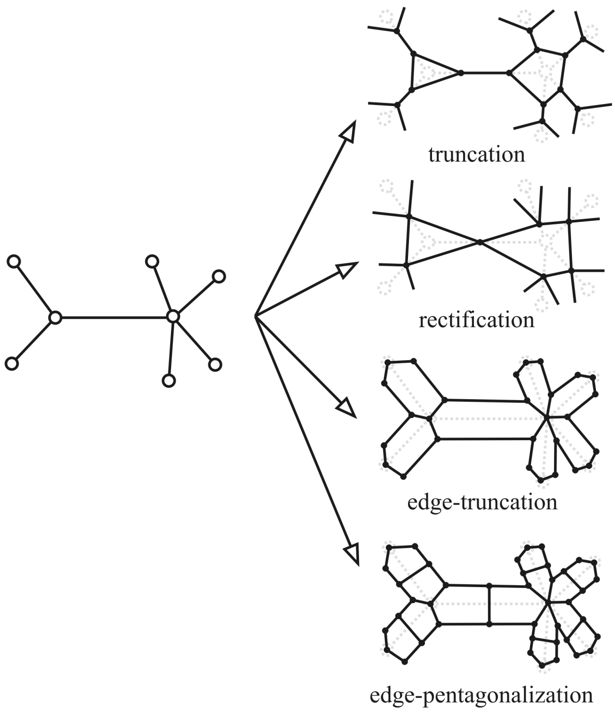

Fig. 1. Four operations on plane graphs

In addition, we use particular operations which replace 3-faces of a plane graph with suitable small graph (see Figure 2; the original vertices of 3-face correspond to 2-vertices of considered replacement graph):

- △-*substitution:* each 3-face is replaced by a 6-gon with three attached triangles,
- △-*substitution:* works similarly, just the replacement graph is a 5-gon with two attached triangles; note that, unlike △-substitution, the resulting graph depends on the way how 3-faces of the original graph were replaced,
- △-*substitution:* as above, just the replacement graph is the graph of 3-cube without a vertex.

Among small starter-graphs which are used in the subsequent constructions, there are well-known graphs of five Platonic polyhedra, and, further, soccerball graph (the truncated icosahedron graph).

For planar graphs of minimum degree at least 3, we will construct, in the above manner, a suitable sequence $\{G_n\}_{n=1}^{\infty}$ for the following sets of cycles:

(a) If $S \subset S_{5^+}$, then set $G = K_4$.
(b) If $S \subset S_{3,5,7,9^+}$, then choose G to be the graph of 3-cube.
(c) If $S \subset S_{4,\dots,9,61^+}$, then choose G to be the truncated dodecahedron graph.
(d) If $S \subset S_{4,5,13^+}$, then choose G to be the truncated tetrahedron graph.

(e) For $S \subset S_{3,4,6,7,21^+}$, choose G to be dodecahedron graph.
(f) For $S \subset S_{3,4,7,8,61^+}$, choose G to be soccerball graph.
(g) For $S \subset S_{3,4,9,85^+}$, choose G to be the graph on Figure 3.
(h) For $S \subset S_{3,4,10,51^+}$, choose G to be edge-pentagonalized graph of 3-cube.
(i) For $S \subset S_{4,5,10,\dots,19,180^+}$, choose G to be △-substituted truncated dodecahedron graph.
(j) For $S \subset S_{4,8,\dots,14,141^+}$, choose G to be △-substituted truncated dodecahedron graph.
(k) For $S \subset S_{3,5,7,\dots,14,141^+}$, choose G to be △-substituted truncated dodecahedron graph.
(l) For $S \subset S_{3,5,6,8,10,81^+}$, choose G to be 80-vertex plane cubic graph having only 4- and 7-faces such that 4-faces are not adjacent, see Figure 4.
(m) For $S \subset S_{3,6,15+}$, choose G to be plane cubic graph on 14 vertices consisting of six 5-faces and three nonadjacent 4-faces, see Figure 5.
(n) For $S \subset S_{3,6,9,145^+}$, choose G to be plane graph constructed by circular gluing of four copies of the configuration on Figure 6.

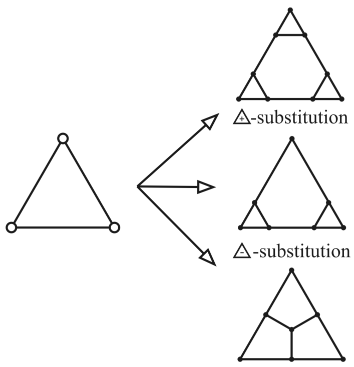

Fig. 2. The △-, △- and △-substitutions

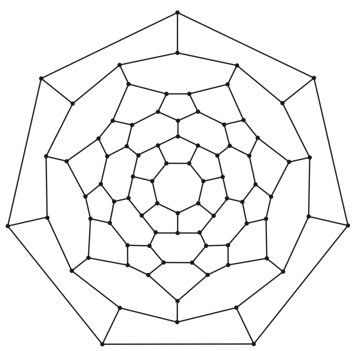

Fig. 3. The basic graph for the set $S_{3,4,9,85^+}$

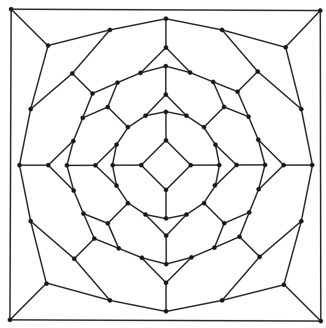

Fig. 4. The basic graph for the set $S_{3,5,6,8,10,81^+}$

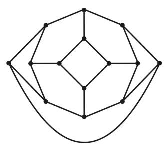

Fig. 5. The basic graph for the set $S_{3,6,15^+}$

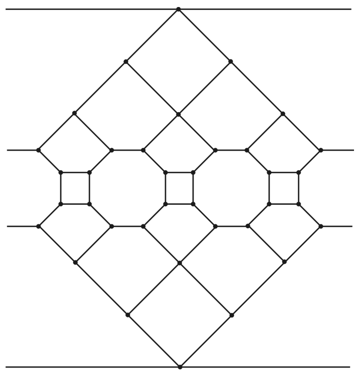

Fig. 6. The one-fourth of basic graph for the set $S_{3,6,9}$

These results imply, in particular, that $S_{k,l}$ is never unavoidable in the family $\mathcal{G}_3$.

3. MINIMAL UNAVOIDABLE SETS IN $\mathcal{G}_3$

Our main positive result is the following

Theorem 3.1. *The set $S_{3,4,11}$ is minimal unavoidable in $\mathcal{G}_3$.*

Proof. By contradiction. Consider a plane graph $G = (V, E, F)$ of minimum degree at least 3 which has neither a 3-cycle nor a 4-cycle nor else an 11-cycle.

For the purposes of this proof, we define *forbidden clusters* of small faces of G (see Figure 7; note that all vertices on the boundary of each cluster are distinct):

– three 5-faces forming a linear chain,
– two 6-faces and one 5-face incident with common 3-vertex,
– two 5-faces and one 7-face incident with common 3-vertex,
– a 7-face forming a nice pair with a 6-face,
– an 8-face forming a nice pair with a 5-face.

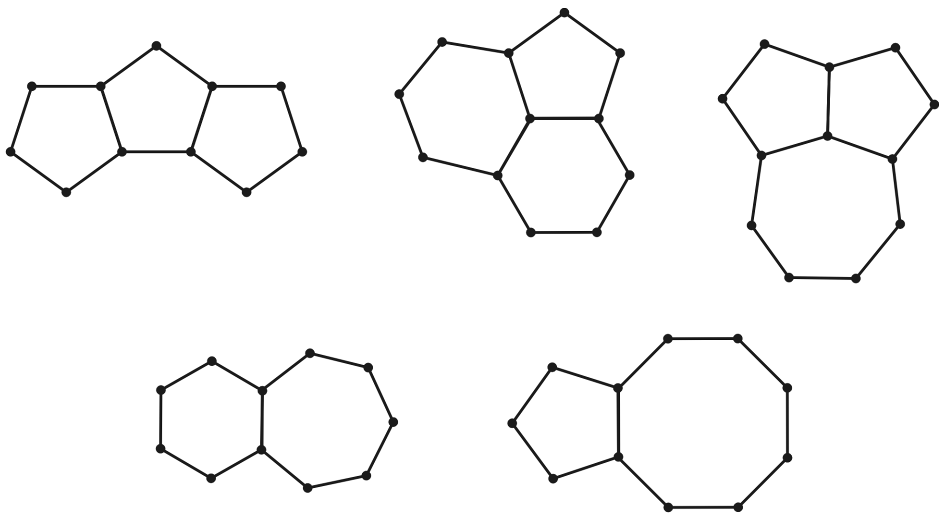

Fig. 7. Five forbidden clusters of faces

It is easy to see that each forbidden cluster contains an 11-cycle.

We proceed by Discharging Method: according to the consequence of Euler's formula on the number of vertices, edges and faces of a plane graph,

$$\sum_{v\in V}(2\deg(v)-6)+\sum_{\alpha\in F}(\deg(\alpha)-6)=-12$$

assign to vertices and faces of G initial charges $\mu : V\cup F\to\mathbb{Z}$ such that $\mu(v)=2\deg(v)-6$ for each $v\in V$ and $\mu(f)=\deg(f)-6$ for each $f\in F$. Thus $\sum\limits_{x\in V\cup F}\mu(x)=-12$.

Next, the charges of vertices and faces are redistributed locally in the way that the total sum of the charges remains the same. This redistribution is done by the following discharging rules:

Rule 1: Every k-vertex, $k\geq 4$, sends 1 to each incident 5-face.

Rule 2: Let x be a k-vertex, $k\geq 4$ which has a positive charge $\widehat{\mu}(x)$ after application of Rule 1. Denote by $\ell(x)$ the number of 5-faces with the property that each of them is incident with a 3-vertex y which is the neighbour of x. Then x sends $\dfrac{\widehat{\mu}(x)}{\ell(x)}$ to every such 5-face provided $\ell(x) > 0$; if $\ell(x) = 0$, no charge is transferred.

Rule 3: Let α be a 7-face having a common edge xy with a 5-face β which is incident only with 3-vertices. Then α sends $\frac{1}{3}$ through xy to β.

Rule 4: Let α be an ≥ 9-face having a common edge xy with a 5-face β which is incident only with 3-vertices. Then α sends $\frac{1}{2}$ through xy to β.

Before the analysis of final charges $\widetilde{\mu} : V \cup F \to \mathbb{Q}$ of vertices and faces, we prove the following auxiliary statements:

Proposition 3.2. *Three transfers of a charge from a face α through three consecutive edges on face boundary of α are not possible.*

Proof. In the opposite case, there exist three consecutive 5-faces β, γ, ρ adjacent to α. Due to the formulation of Rules 3 and 4, all vertices of these faces are distinct; but then β, γ, ρ form a forbidden cluster, a contradiction. □

Proposition 3.3. *Let x be a k-vertex, $k \geq 6$, incident with at least two ≥ 6-faces, or a 5-vertex incident with at least three ≥ 6-faces. Then the amount of charge sent from x to a 5-face by Rule 2 is at least $\frac{1}{3}$.*

Proof. In the former case, $\widehat{\mu}(x) \geq 2k - 6 - (k-2) = k - 4$ and $\ell(x) \leq k$, thus the contribution of x by Rule 2 is at least $\frac{k-4}{k} \geq \frac{1}{3}$; in the latter case, the contribution is at least $\frac{2\cdot 5-6-2\cdot 1}{5} = \frac{2}{5} > \frac{1}{3}$. □

Next, by case analysis, we show that $\widetilde{\mu}$ is nonnegative function, yielding a contradiction because $-12 = \sum_{x \in V \cup F} \mu(x) = \sum_{x \in V \cup F} \widetilde{\mu}(x) \geq 0$.

Case 1. Let v be a vertex of G.

Case 1.1. If v is a 3-vertex or a k-vertex with $k \geq 6$ then $\widetilde{\mu}(v) = \mu(v) = 2 \cdot 3 - 6 = 0$ or $\widehat{\mu}(v) \geq 2k - 6 - k \cdot 1 = k - 6 \geq 0$ which yields, due to Rule 2, $\widetilde{\mu}(x) \geq 0$.

Case 1.2. Let v be a 4-vertex. Assume that v is incident with three 5-faces $[v_1 v v_2 w_1 w_2], [v_2 v v_3 w_3 w_4], [v_3 v v_4 w_5 w_6]$. If all $w_i, i = 1, \ldots, 6$ are distinct and none of them coincides with a vertex from $\{v_1, \ldots, v_4\}$, then these 5-faces form a forbidden cluster. If there are $i \in \{1, \ldots, 6\}, j \in \{1, \ldots, 4\}$ such that $w_i = v_j$, then v and two of its neighbours form a 3-cycle. Finally, if $w_i = w_j$ for some $i, j \in \{1, \ldots, 6\}$, then w_i, v and some two neighbours of v form a 4-cycle. Therefore, we conclude that v is incident with at most two 5-faces, yielding $\widetilde{\mu}(v) \geq 2 \cdot 4 - 6 - 2 \cdot 1 = 0$.

Case 1.3. Let v be a 5-vertex. By similar argument as in Case 1.2, we obtain that v is incident with at most three 5-faces, thus $\widehat{\mu}(v) \geq 2 \cdot 5 - 6 - 3 \cdot 1 > 0$ which also yields $\widetilde{\mu}(x) \geq 0$.

Case 2. Let $\alpha = [x_1 x_2 x_3 x_4 x_5]$ be a 5-face.

Case 2.1. If α is incident with an ≥ 4-vertex, then, by Rule 1, $\widetilde{\mu}(\alpha) \geq -1 + 1 = 0$.

Case 2.2. Assume that α is incident only with 3-vertices (denote by $y_i, i = 1, \ldots, 5$ the neighbour of x_i distinct from x_{i-1}, x_{i+1}; observe that all x_i, y_i are distinct) and let $\beta, \eta, \gamma, \rho$ and ω be faces around α which contain the edge $x_1x_2, x_2x_3, x_3x_4, x_4x_5$ and x_5x_1. Note that α cannot be adjacent with an 8-face (such an 8-face must be nice and, due to absence of 3-and 4-cycles, it forms a forbidden cluster with α). If α is adjacent with at least two $\geq$ 9-faces or with at least three 7-faces or else with one $\geq$ 9-face and two 7-faces, then, by Rule 4 or 3, $\widetilde{\mu}(\alpha) \geq -1 + 2 \cdot \frac{1}{2} = 0$ or $\widetilde{\mu}(\alpha) \geq -1 + 3 \cdot \frac{1}{3} = 0$ or else $\widetilde{\mu}(\alpha) \geq -1 + \frac{1}{2} + 2 \cdot \frac{1}{3} > 0$. Hence, in subsequent analysis, we suppose that these possibilities do not occur. Then α is adjacent with (at least) three $\leq$ 6-faces. Taking into account the symmetries, we discuss several possibilities for degrees of α and neighbouring faces.

Case 2.2.1. Around α, there are two nonadjacent 5-faces, say $\beta = [x_1x_2y_2uy_1]$ and $\gamma = [x_3x_4y_4vy_3]$. To prevent α, β, γ to form a forbidden cluster, some of their vertices must coincide, which yields, up to symmetry, the following cases to analyze:

If $u = v$, then the neighbours of y_1 and y_5 incident with ω are distinct from all x_i, y_i and u. Taking into account the paths $y_5x_5x_4y_4uy_3x_3x_2x_1y_1$, $y_5x_5x_1x_2x_3x_4y_4uy_1$, ω is neither a 5-face nor a 6-face nor else a 7-face (note that, in this case, ω is nice and none of its vertices coincides with u, hence, it forms a forbidden cluster with α, β). The same argument holds for the face ρ. Hence, both ω and ρ are $\geq$ 9-faces, a contradiction.

If $u = y_4$, then again ω cannot be 5-, 6- or 7-face (as an 11-cycle always appears then), hence it is an $\geq$ 9-face. Further, observe that η cannot be a 5-face $[x_2x_3y_3zy_2]$, as it forms an 11-cycle $x_1x_5x_4x_3x_2y_2zy_3vy_4y_1x_1$. If η is a 6-face $[x_2y_2zwy_3x_3]$, then ρ is neither a 5-face nor a 6-face nor else a 7-face (because the paths $x_5x_1x_2y_2zwy_3vy_4$, $x_5x_4x_3y_3wzy_2y_4$ and $x_5x_1x_2x_3y_3vy_4$ form, together with a part of the boundary of ρ, an 11-cycle); hence, it must be an $\geq$ 9-face. Finally, η cannot be a 7-face, as the part of its boundary forms, together with path $x_2x_1x_5x_4vy_3$, an 11-cycle. Thus we obtain that α is always adjacent with at least two $\geq$ 9-faces, a contradiction.

Case 2.2.2. Around α, there are two adjacent 5-faces, say $\beta = [x_1x_2y_2uy_1]$ and $\eta = [x_2x_3y_3vy_2]$ (note that, due to Case 2.2.1, γ, ρ and ω are $\geq$ 6-faces). Then β, η form a nice pair. Consider the face γ and discuss first the case when $\gamma = [x_3x_4y_4wzy_3]$ is a 6-face and y_4, w, z are distinct from v, u, y_1, y_2. Then $x_1y_1uy_2vy_3zwy_4x_4x_5x_1$ is an 11-cycle, thus, some of the mentioned vertices must coincide. Taking into account the absence of 3- and 4-cycles, we get that either $z = y_1$ or $u = w$ or else $w = y_1$.

Let $z = y_1$. If ρ is a 6-face, then it is either incident with y_1 (which yields a 3- or 4-cycle) or it forms a forbidden cluster with α and γ. Similarly, if ρ is a 7-face, then it either forms a forbidden cluster with γ or is incident with y_1 or w (this, however, yields a 3- or 4-cycle). Hence, ρ is an $\geq$ 9-face. Next, if ω is a 6-face, then it forms, with α, β and η, a face cluster containing an 11-cycle, and if it is a 7-face, then it forms forbidden cluster with α and β. Therefore ω is also an $\geq$ 9-face, a contradiction.

If $u = w$, then ρ is neither a 6-face (as it either forms a forbidden cluster with γ and α or has three common vertices with γ resulting in a 3- or 4-cycle) nor a 7-face (from similar reason with respect to γ), hence, it is an $\geq$ 9-face. Now ω being a 6- or a 7-face forms either a forbidden cluster (with α, β, η or α, β) or yields a 3- or 4-cycle in G. Thus, it is also an ≥ 9-face, a contradiction.

The same conclusion (following the same argumentation) for the sizes of ρ and ω is obtained also in the case when $w = y_1$.

The above considerations then exclude γ from being a 6-face, which applies also to ω due to symmetry. Both γ and ω are thus ≥ 7-faces. Note that if γ is a 7-face, then it either forms a forbidden cluster with η and α, or y_2 and y_4 are adjacent; in the latter case, however, the face ρ is not a 6-face as it would form a forbidden cluster with γ, or a 3- or 4-cycle. We can conclude that β, η are the only ≤ 6-faces around α, a contradiction.

Case 2.2.3. α is incident with exactly one 5-face $\beta = [x_1x_2y_2uy_1]$. Assume first that $\gamma = [x_3x_4y_4vzy_3]$ is a 6-face having no common vertices with β. If $\eta = [y_2x_2x_3y_3wt]$ is a 6-face, then it either forms a forbidden cluster with α, γ or, when $t = y_4$, the 11-cycle $uy_1x_1x_5x_4y_4wy_3x_3x_2y_2u$. In addition, if $\eta = [y_2x_2x_3y_3wts]$ is a 7-face, then it either forms a forbidden cluster with α, β or, when $t = y_4$ or $s = v$, the 11-cycle $uy_1x_1x_5x_4x_3y_3wtsy_2u$. Hence η is necessarily an ≥ 9-face. Consider now the faces ρ, ω. If $\omega = [y_5x_5x_1y_1pq], \rho = [y_4x_4x_5y_5rs]$ are both 6-faces, then either they form a forbidden cluster with α, or $s = y_1$ with $vzy_3x_3x_4x_5y_5qpy_1y_4v$ being an 11-cycle. If $\omega = [y_5x_5x_1y_1pq]$ is a 6-face and $\rho = [y_4x_4x_5y_5rsc]$ is a 7-face, then ρ, ω either form a forbidden cluster, or $y_1 = s$ with $cy_4vzy_3x_3x_4x_5y_5rsc$ being an 11-cycle. In addition, if ω is a 7-face and ρ is a 6-face, then ω forms a bad cluster either with α, β or (if it it incident with y_2) with ρ. Thus we have, around α, either two ≥ 9-faces of one ≥ 9-face and two ≥ 7-faces, a contradiction.

Next, assume that $\gamma = [x_3x_4y_4vzy_3]$ is a 6-face having some common vertices with β (this can be assumed, due to symmetry, also for the face ρ). There are six cases: $z \in \{y_1, u\}, v \in \{u, y_1, y_2\}$, and $u = y_4$. Note that, in each of these cases, η cannot be a 6-face, as it either forms a forbidden cluster with γ, α or has at least three common vertices with γ, but this yields a 3- or 4-cycle in G. If η is a 7-face, then it either forms a forbidden cluster with γ or a forbidden cluster with α, β or else a 3- or 4-cycle in neighbourhood of α. Thus, η is always an ≥ 9-face. Similarly, ρ is not a 6-face (due to forbidden cluster with α and γ or to appearance of short forbidden cycles), and not a 7-face except of the case when $z = u$ and u is incident with ρ. In that case, there is no 11-cycle formed of some vertices of $\alpha, \beta, \gamma, \rho$, but if, in addition, ω is a 6-face, then the pair ρ, ω is nice and subsequently forms a forbidden cluster; hence α is surrounded with ≥ 9-face and two ≥ 7-faces, a contradiction.

Hence, we may assume that both γ, ρ are ≥ 7-faces, thus $\eta = [y_3x_3x_2y_2wz]$ and ω are 6-faces. Let $\gamma = [y_4x_4x_3y_3rqp]$ be a 7-face. Then either γ forms a bad cluster with η, or they have at least three vertices in common. If $w = p$, then $uy_2pqry_3x_3x_4x_5x_1y_1u$ is an 11-cycle. Similarly, if $q = y_2$, then $qwzy_3x_3x_2x_1x_5x_4y_4pq$ is an 11-cycle, and if $y_4 = w$, then $uy_2x_2x_3y_3zwx_4x_5x_1y_1u$ is also an 11-cycle. Let $p = y_2$. Then y_2 is ≥ 5-vertex and in the case when it is a 5-vertex, the face that shares the edge uy_2 with β is not a 5-face (since y_4 would be a 2-vertex). Thus, Rule 2 is applied with the contribution from y_2 to α being at least $\frac{1}{3}$, and we have, with two contributions by Rule 3 (from γ) and Rule 3 / Rule 4 (from ρ), $\widetilde{\mu}(\alpha) \geq -1 + 3 \cdot \frac{1}{3} = 0$. The same consideration can be used, due to symmetry, also when ρ is a 7-face. We conclude then that both γ, ρ are $\geq$ 9-faces, a contradiction.

Case 2.2.4. Let α be surrounded only with $\geq$ 6-faces. Note that at least three of them are exactly 6-faces, hence two of them have to be adjacent, say $\beta = [x_1x_2y_2uwy_1]$ and $\eta = [x_2x_3y_3vzy_2]$. If β and η form a nice pair, then the cluster of α, β and η is forbidden; hence, some of their vertices coincide. Taking into account that all $y_i, i = 1, \ldots, 5$ are distinct, we have $v = y_1$ or $w = y_3$ which yields symmetric cases. Thus, without loss of generality, let $v = y_1$. If ω is a 6-face or a 7-face, then it forms a forbidden cluster with α, β or with β; hence, it is an $\geq$ 9-face. Similarly, γ being a 6-face forms either a forbidden cluster with α, η), or a 3- or 4-cycle (if y_1 is incident with γ); when γ is a 7-face, it forms either a forbidden cluster with η , or again a 3- or 4-cycle (if it contains y_1). Thus γ is also an $\geq$ 9-face, a contradiction.

Case 3. Let α be a 7-face. Note that, due to the absence of 3- and 4-cycles in G, α is a nice face. Suppose that there are two transfers of a charge from α by Rule 3 to 5-faces β, γ through consecutive edges on face boundary of α. Then the only possibility to avoid 3- or 4-cycles or having $\geq$ 4-vertices on β or γ is that the faces α, β and γ form a forbidden cluster. Therefore, we conclude that, from α, at most three transfers by Rule 3 are possible, and so $\widetilde{\mu}(\alpha) \geq 7 - 6 - 3 \cdot \frac{1}{3} = 0$.

Case 4. Let α be a k-face, $k \geq 9$. According to the above proposition, at most $\lceil \frac{2}{3}k \rceil$ transfers of a charge by Rule 4 from α are possible, thus $\widetilde{\mu}(\alpha) \geq k - 6 - \frac{1}{2} \cdot \lceil \frac{2}{3}k \rceil \geq 0$.

This proves the unavoidability of $S_{3,4,11}$; its minimality follows from the constructions (5), (9) and (11) of Section 2. □

Theorem 3.4. *The sets $S_{3,4,6,8}$ and $S_{3,4,8,9}$ are minimal unavoidable in $\mathcal{G}_3$.*

Proof. Let $G \in \mathcal{G}_3$ be a graph without 3-cycle or 4-cycle. Using a non-polyhedral variant of theorem of Wernicke [8] (the details are left to reader), we obtain that G contains a 5-face adjacent to an $\leq$ 6-face. Observe that these two faces necessarily form a nice pair (otherwise a 3- or 4-cycle is found in G); since two adjacent 5-faces (a 5-face with a 6-face) then form an 8-cycle (a 9-cycle), the unavoidability of the above mentioned sets follows. The minimality of $S_{3,4,6,8}$ follows from the constructions (1), (3), (4) and (10) of Section 2, the minimality of $S_{3,4,8,9}$ similarly follows from the constructions (4), (5), (8) and (9). □

Theorem 3.5. *The set $S_{3,4,7,9}$ is minimal unavoidable in $\mathcal{G}_3$.*

Proof. Let $G \in \mathcal{G}_3$ be a graph of girth 5. By dual variant of result of Borodin [2], G contains a 3-vertex surrounded by three faces such that the sum of their sizes is at most 17. Now,

- if one of these faces is a 7-face, then it is necessarily nice, hence it bounds a 7-cycle,
- if one of these faces is a 6-face and another one is a 5-face, then they form a nice pair and, consequently, a 9-cycle,
- if all three faces are 5-faces, then, due to absence of 3- and 4-cycles in G, any two of them form a nice pair, thus together, they form a 9-cycle.

This proves the unavoidability of $S_{3,4,7,9}$; its minimality follows from the constructions (5), (7), (11) and (3) of Section 2. □

Theorem 3.6. *The set $S_{3,5,6,7}$ is minimal unavoidable in $\mathcal{G}_3$.*

Proof. Let $G \in \mathcal{G}_3$ be a graph without 3-cycle. By non-polyhedral variant of Kotzig theorem [6] (in dual form), G contains a 4-face α adjacent to an ≤ 7-face β, or a 5-face α' adjacent to an ≤ 6-face β'. Now,

– if β is a 4-face, then it forms a nice pair with α (otherwise a 3-cycle is found), thereby forming with α a 6-cycle,
– if β is a 6-face or a 7-face, then it is nice (otherwise one finds a 3-cycle in G), hence it bounds a 6-or 7-cycle,
– if none of the above possibilities happen, then a 5-face β or α' forms a 5-cycle in G.

This proves the unavoidability of $S_{3,5,6,7}$; its minimality follows from the constructions (3), (5), (11) and (12) of Section 2. □

4. CONCLUDING REMARKS

Although the negative results of Section 2 filter out many sets of cycles, the characterizations of minimal unavoidable sets is still far from being complete. The first open cases are the sets $S_{3,4,k}$ for $12 \leq k \leq 20$. The minimal unavoidability of $S_{3,4,14}$ may be also indirectly indicated by the fact from [5] that the only cycle whose heaviness could not be proved (using specialized constructions), for the family of 3-connected planar graphs without triangular and quadrangular faces, is exactly C_{14} (we note that the existence of a planar graph of girth 5 with 2-valent vertices being far apart and no 14-cycle would lead to the constructions of graphs whose 14-cycles necessarily pass through vertices of high degree). Since proving the lightness of C_{14} may be difficult, a little easier task would be to show, at least, the presence of C_{14} in these plane graphs. Note also that although a particular cycle is heavy in a graph family, still it can be unavoidable; for example, C_{11} and C_{12} might exist in each plane graph of minimum degree 5 whereas they are known to be heavy in that graph family.

The constructions of Section 2 to exclude certain sets of cycles from being unavoidable are mostly based on graphs which have cut vertices. Under the restriction of 3-connectedness on plane graphs, one has to develop other types of constructions; also, it may happen that a certain set of cycles which is not unavoidable in planar graphs of minimum degree 3 is indeed minimal unavoidable in polyhedral graphs. This is the subject of further research in the paper [7].

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