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Global rational stabilization of a class of nonlinear time-delay systems

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The present paper is mainly aimed at introducing a novel notion of stability of nonlinear time-delay systems called Rational Stability. According to the Lyapunov-type, various sufficient conditions for rational stability are reached. Under delay dependent conditions, we suggest a nonlinear time-delay observer to estimate the system states, a state feedback controller and the observer-based controller rational stability is provided. Moreover, global rational stability using output feedback is given. Finally, the study presents simulation findings to show the feasibility of the suggested strategy.

Key words: rational stability, delay system, nonlinear observer, Lyapunov functional

1. Introduction

Time-Delay Systems (TDSs) is also known as call systems with aftereffect or dead-time, hereditary systems, equations with deviating argument, or differential-difference equations. They are part of the class of functional differential equations which are infinite-dimensional, as opposed to ordinary differential equations (ODEs). Time-delay has a number of characteristics. It appears in several control systems, including aircraft, chemical [19], biological systems [17], engineering, electrical [1], economic model [5, 11], or process control systems, and communication networks, either in the state, the control input, or the measurements [2, 20]. There, we can find transported, communication, or measurement delays. It is noticeable that time delay can cause different problems, such as instability, divergence behavior, and oscillation of dynamic systems. A considerable amount of studies have analyzed the stability of dynamic systems with a delay. Therefore, the stability of systems with time delay has been investigated extensively over the past decades. It is a well known fact that stability of nonlinear time-delay systems in Lyapunov sense plays a major role in control theory, and becomes a challenging problem both in theory and applications. The stability analysis of time delay

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systems has been recently studied in many areas. There are two crucial kinds of stability of dynamical systems. These include asymptotic stability and exponential stability. In the case of asymptotic stability, for more details, the reader is referred to [3, 6, 9, 10, 12, 26, 28] and references therein. [24] addressed the problem of asymptotic stability for Markovian jump that generalized neural networks with interval time-varying delay systems. Based on the Lyapunov method, it is suggested that asymptotic stability can be used to solve linear matrix inequality with triple integral terms a delay. For exponential stability, it is requires that all solutions starting near an equilibrium point not only stay nearby, but tend to the equilibrium point very fast with exponential decay rate; see [4, 22, 23, 27].

A new notion of stability known as rational stability for systems without time delays is introduced in [14]. The study demonstrates the characteristics of rational stability. It can be characterized by means of Lyapunov functions. This notion did not know any intense progress like other tools of the stability theory. For free-delay system, [13] studied the issue of rational stability of continuous autonomous systems, followed by several examples of control systems. Under a Hamilton-Jacobi-Belleman approach, some sufficient conditions are developed by [29] to achieve the rational stability of optimal control for every dynamical control systems.

The questions which are worth being raised here are can we speak about rational stability for time-delay systems? What is the advantage of this stability? The aim of the current study is to present a new term of stability for nonlinear time-delay systems. This term is called rational stability. Sometimes the decay of the energy or the Lyapunov function is not exponential, but it can be polynomial. For rational stability and especially when the Jacobean matrix is no longer Hurwitz and the transcendental characteristic polynomial, the solutions do not decrease exponentially. However, in some cases, the solutions decrease like t^{-r}, $r > 0$ with r is called the rate decay of the solution. The real r measures the velocity of convergence of the solution which is crucial in several practical engineering such as satellite systems, unicycle systems, underwater, transport equation, string networks, etc.

The current paper introduces a novel notion of stability of nonlinear time-delay systems called rational stability. Motivated by [13] and [29], Lyapunov-Krasovskii functional is used for the purpose of obtaining to establish globally rational stability of the closed loop systems. It also investigates the problem of output feedback stabilization of a class of nonlinear time delay system written in triangular form, with constant delay. We impose a generalized condition on the nonlinearity to cover the time-delay systems is considered by [12]. We design a nonlinear observer to estimate the system states. Then, it is used to obtain a new state and input delay-dependent criterion that ensures the rational stability of the closed-loop system with a state feedback controller. The global rational stability using output feedback is also presented. As an application, we can stabilize rationally a special class of network-based systems, mechanical systems and the chained form systems, when they are subjected to input time delays. Many mechanical systems including mobile robots can be reduced to such a form by an appropriate change of variables. In this regard, these type of systems are usually linked via delay induced wireless communication channels, which may compromise the performance and the stability of the controlled system.

The rest of this paper is organized as follows. The next section presents the definition of rational stability and an auxiliary result concerning a functional should satisfy for guaranteeing the rational stability. In section 3, it also shows the system description. The main results are stated in section 4, it is concluded that parameter dependent linear state and output feedback controllers are synthesized to ensure global rational stability of the nonlinear time delay system. In section 5, we establish the problem of global rational stability using output feedback. Finally, an illustrative example, of network-based control systems (NBCSs), is discussed to demonstrate the effectiveness of the obtained results.

2. Definitions and auxiliary results

Consider time delay system of the form:

$$\begin{cases} \dot{x}(t) = f(x(t), x(t-\tau)), \\ x(\theta) = \varphi(\theta), \end{cases} \tag{1}$$

where $\tau > 0$ denotes the time delay. The knowledge of x at time $t = 0$ does not allow to deduce x at time t. Thus, the initial condition is specified as a continuous function $\varphi \in C$, where C denotes the Banach space of continuous functions mapping the interval $[-\tau, 0] \to \mathbb{R}^n$ equipped with the supremum-norm:

$$\|\varphi\|_\infty = \max_{\theta\in[-\tau,0]} \|\varphi(\theta)\|,$$

$\|\ \|$ being the Euclidean-norm. The map $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function, and satisfies $f(0,0) = 0$.

The function segment x_t is defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$. For $\varphi \in C$, we denote by $x(t,\varphi)$ or shortly $x(t)$ the solution of (1) that satisfies $x_0 = \varphi$. The segment of this solution is denoted by $x_t(\varphi)$ or shortly x_t.

Inspired from [13] and [14], we introduce some definition of rational stability for the time-delay systems.

Definition 1 *The zero solution of* (1) *is called*

- *Stable, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|\varphi\|_\infty < \delta \quad \Rightarrow \quad \|x(t)\| < \varepsilon, \ \ \forall t \geqslant 0.$$

- *Rationally stable, if it is stable and there exist positive numbers M, k, σ, $e \leqslant 1$ such that if*

$$\|\varphi\|_\infty < \sigma \quad \Rightarrow \quad \|x(t)\| \leqslant \frac{M\|\varphi\|_\infty^e}{(1 + \|\varphi\|_\infty^k t)^{\frac{1}{k}}}, \quad \forall t \geqslant 0. \tag{2}$$

- *Globally rationally stable, if it is stable and δ can be chosen arbitrarily large for sufficiently large ε, and (2) is satisfied for all $\sigma > 0$.*

Sufficient conditions for rotational stability of a functional differential equation are provided by [13], a generalization of time delay system given by following theorem. For a locally Lipschitz functional $V : \mathcal{C} \to \mathbb{R}_+$, the derivative of V along the solutions of (1) is defined as

$$\dot{V} = \lim_{h \to 0} \frac{1}{h}(V(x_{t+h}) - V(x_t)).$$

Remark 1 *It is easy to see that rational stability is satisfied then asymptotic stability is satisfied, but the converse is not true.*

Theorem 1 *Assume that there exist positive numbers λ_1, λ_2, λ_3, r_1, r_2, k and a continuous differentiable functional $V : \mathcal{C} \to \mathbb{R}_+$ such that:*

$$\lambda_1\|x(t)\|^{r_1} \leqslant V(x_t) \leqslant \lambda_2\|x_t\|_\infty^{r_2}, \tag{3}$$

$$\dot{V}(x_t) + \lambda_3 V^{1+k}(x_t) \leqslant 0, \tag{4}$$

then, the zero solution of (1) is globally rationally stable.

Proof. Using (4), we have for $V \neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} V^{-k}(x_\theta) \geqslant k\lambda_3 \,.$$

Integrating between 0 and t, one obtains

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}\theta} V^{-k}(x_\theta)\mathrm{d}\theta \geqslant \int_0^t k\lambda_3 \mathrm{d}\theta$$

equivalently, for all $t \geqslant 0$

$$V^k(x_t) \leqslant \frac{1}{k\lambda_3 t + V^{-k}(\varphi)} \,.$$

Now, it follows Theorem 1, condition (3) that

$$\|x(t)\| \leqslant \left(\frac{1}{\lambda_1}\right)^{\frac{1}{r_1}} \frac{1}{\left(\lambda_2^{-k}\|\varphi\|_\infty^{-r_2 k} + \lambda_3 k t\right)^{\frac{1}{k r_1}}} .$$

Corollary 1 *Assume that there exist positive numbers* λ_1, λ_2, λ_3, r_1, r_2, r_3, $r_2 < r_3$ *and a continuous differentiable functional* $V : C \to \mathbb{R}_+$ *such that:*

$$\begin{aligned} \lambda_1\|x(t)\|^{r_1} \leqslant V(x_t) &\leqslant \lambda_2\|x_t\|_\infty^{r_2}, \\ \dot{V}(x_t) &\leqslant -\lambda_3\|x_t\|_\infty^{r_3}, \end{aligned} \tag{5}$$

then, the zero solution of (1) *is globally rationally stable.*

Proof. The conditions (3) and (5) imply that zero solution of (1) is stable.

By combining the assertions (3) and (5), we obtain

$$\dot{V}(x_t) \leqslant -\frac{\lambda_3}{\lambda_2^{\frac{r_3}{r_2}}} V^{\frac{r_3}{r_2}}(x_t) \tag{6}$$

equivalent to

$$\dot{V}(x_t) \leqslant -\frac{\lambda_3}{\lambda_2^{\frac{r_3}{r_2}}} V^{1+k}(x_t),$$

where $k = \dfrac{r_3 - r_2}{r_2}$.

Hence, from Theorem 1, the zero solution of (1) is globally rationally stable. □

Let us recall here that a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, increasing and $\alpha(0) = 0$, of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded. The following theorem provides sufficient Lyapunov-Krasovskii conditions for global rationally stability of the zero solution of system (1).

Theorem 2 *Assume that there exist positive numbers* λ_1, λ_2, r_1, r_2, k, α *a function of class* $\mathcal{K}$ *and a continuous differentiable functional* $V : C \to \mathbb{R}_+$ *such that:*

(i) $\lambda_1\|x(t)\|^{r_1} \leqslant V(x_t) \leqslant \lambda_2\|x_t\|_\infty^{r_2}$,

(ii) $\dot{V}(x_t) \leqslant -\alpha(V(x_t))$,

(iii) $\lim\limits_{t\to 0} \dfrac{\alpha(t)}{t^{k+1}} = l \in]0, +\infty]$.

Then the zero solution of (1) *is globally rationally stable.*

Proof. First case: $0 < l < +\infty$. The conditions (i) and (ii) imply that zero solution of system (1) is stable and attractive, then asymptotically stable and $\lim_{t\to+\infty} V(x_t) = 0$.

Therefore, by using limit definition, there exists $t_0 > 0$ such that for every $0 \leqslant t \leqslant t_0$, we have $\alpha(t) \geqslant \frac{l}{2}t^{k+1}$. Since, $\lim_{t\to+\infty} V(x_t) = 0$, for this $t_0 > 0$, there exists $t_* > 0$ such that for every $t \geqslant t_*$ one gets $0 \leqslant V(x_t) \leqslant t_0$ and

$$\alpha(V(x_t)) \geqslant \frac{l}{2}V^{1+k}(x_t).$$

Using (ii), we obtain

$$0 \geqslant \dot{V}(x_t) + \alpha V(x_t) \geqslant \dot{V}(x_t) + \frac{l}{2}V(x_t)^{k+1}.$$

Thus, we have

$$\dot{V}(x_t) \leqslant \frac{l}{2}V(x_t)^{k+1}.$$

Then, using the Theorem 1, we can conclude that the zero solution of (1) is globally rationally stable.

Second case: $l = +\infty$. As in the proof of first case, there exists $t_0 > 0$ such that for every $0 \leqslant t \leqslant t_0$, we have

$$\alpha(t) \geqslant t^{k+1} \textit{ and } \dot{V}(x_t) \leqslant V(x_t)^{k+1}.$$

□

Remark 2 *The Theorem 1, Theorem 2 and Corollary 1 generalize the results given by [13] for the case of free-delay system.*

3. System description

Consider the nonlinear time-delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f(x(t), \quad x(t-\tau), u(t)), \\ y(t) = Cx(t), \end{cases} \tag{7}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the input of the system, $y \in \mathbb{R}$ is the measured output and τ is a positive known scalar that denotes the time delay affecting the state variables. The matrices A, B and C are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and the perturbed term is

$$f(x(t), x(t-\tau), u(t)) = \begin{bmatrix} f_1(x_1(t), x_1(t-\tau), u(t)) \\ f_2(x_1(t), x_2(t), x_1(t-\tau), x_2(t-\tau), u(t)) \\ \vdots \\ f_n(x(t), x(t-\tau), u(t)) \end{bmatrix}.$$

The mappings $f_i : \mathbb{R}^i \times \mathbb{R}^i \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$, are smooth and satisfy the following assumption:

We suppose that f satisfies the following assumption:

Assumption 1 *The nonlinearity $f(y, z, u)$ is smooth, globally Lipschitz with respect to y and z, uniformly with respect to u and well-defined for all $y, z \in \mathbb{R}^n$ with $f(0, 0, u) = 0$.*

We suppose also that,

Assumption 2 *For all $t \geqslant 0$, the delay τ is known and constant.*

Notation 1 *Throughout the paper, the time argument is omitted and the delayed state vector $x(t-\tau)$ is noted by x^τ. A^T means the transpose of A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximal and minimal eigenvalue of a matrix A respectively. I is an appropriately dimensioned identity matrix, diag$[\cdots]$ denotes a block-diagonal matrix.*

Remark 3 *This paper focuses on state observer design for a class of system given by (7). It specifically shows that the general high-gain observer design framework established in [8] and [15] for free delay systems can be properly extended to this class of time-delay systems.*

4. Separation principle

4.1. Observer design

The observer synthesis for triangular nonlinear system design problems along with time-delay systems have become the focal focus of various studies [3, 7] and [11] and references therein. Under the global Lipschitz condition, an observer for a class of time-delay nonlinear systems in the strictly lower triangular form was proposed by [11]. In [3] a nonlinear observer is used to investigate the output feedback controller problem for a class of nonlinear delay systems for the purpose of calculating the system states. Based on time-varying delays known and bounded, [7] propose a nonlinear observer for a class of time-delay nonlinear systems. In this section we devote to the design of the observer-based controller.

$$\dot{\hat{x}}(t) = A\hat{x} + Bu(t) + f(\hat{x}, \hat{x}^{\tau}, u) + L(\theta)(C\hat{x} - y), \tag{8}$$

where $L(\theta) = [l_1\theta, \ldots, l_n\theta^n]^T$ with $\theta > 0$ and where $L = [l_1, \ldots, l_n]^T$ is selected such that $A_L := A + LC$ is Hurwitz, $\hat{x}(s) = \hat{\phi}(s)$, $-\tau \leqslant s \leqslant 0$ with $\hat{\phi} : [-\tau, 0] \to \mathbb{R}^n$ being any known continuous function. Let P be the symmetric positive definite solution of the Lyapunov equation

$$A_L^T P + PA_L = -I. \tag{9}$$

Theorem 3 *Suppose that Assumption 1, 2 are satisfied and there exists positive constant θ such that*

$$\begin{cases} \dfrac{\theta}{2} - \|P\|\dfrac{\ln\theta}{2\tau} - 3k\|P\| > 0, \\ \dfrac{\sqrt{\theta}}{2} - k\|P\| > 0. \end{cases} \tag{10}$$

Then, (8) *is globally rationally observer for system* (7).

Proof. Denote $e = \hat{x} - x$ the observation error. We have

$$\dot{e} = (A + L(\theta)C)e + f(\hat{x}, \hat{x}^{\tau}, u) - f(x, x^{\tau}, u). \tag{11}$$

For $\theta > 0$, let $\Delta_\theta = diag\left[1, \dfrac{1}{\theta}, \ldots, \dfrac{1}{\theta^{n-1}}\right]$. One can easily check the following identities: $\Delta_\theta A \Delta_\theta^{-1} = \theta A$, $C\Delta_\theta^{-1} = C$. Let us now introduce $\eta = \Delta_\theta e$, then we get

$$\dot{\eta} = \theta A_L \eta + \Delta_\theta \left(f(\hat{x}, \hat{x}^{\tau}, u) - f(x, x^{\tau}, u)\right). \tag{12}$$

Let us choose a Lyapunov-Krasovskii functional candidate as follows

$$V(\eta_t) = V_1(\eta_t) + V_2(\eta_t) \tag{13}$$

with

$$V_1(\eta_t) = \eta^T P\eta$$

and

$$V_2(\eta_t) = \frac{\theta}{2}\theta^{\frac{-t}{2\tau}} \int_{t-\tau}^{t} \theta^{\frac{s}{2\tau}} \|\eta(s)\|^2 \mathrm{d}s.$$

Since P is symmetric positive definite then for all $\eta \in \mathbb{R}^n$,

$$\lambda_{\min}(P)\|\eta\|^2 \leqslant \eta^T P\eta \leqslant \lambda_{\max}(P)\|\eta\|^2. \tag{14}$$

This implies that on the one hand,

$$V(\eta_t) \geqslant \lambda_{\min}(P)\|\eta(t)\|^2,$$

and on the other hand,

$$\begin{aligned} V(\eta_t) &= \eta^T P\eta + \frac{\theta}{2}\int_{-\tau}^{0} \theta^{\frac{\mu}{2\tau}} \|\eta(\mu+t)\|^2 \mathrm{d}\mu \\ &= \eta^T P\eta + \frac{\theta}{2}\int_{-\tau}^{0} \theta^{\frac{\mu}{2\tau}} \|\eta_t(\mu)\|^2 \mathrm{d}\mu \\ &\leqslant \lambda_{\max}(P)\|\eta\|^2 + \frac{\theta}{2}\int_{-\tau}^{0} \theta^{\frac{\mu}{2\tau}} \|\eta_t\|_\infty^2 \mathrm{d}\mu \\ &\leqslant \left(\lambda_{\max}(P) + \frac{\theta\tau}{2}\right)\|\eta_t\|_\infty^2. \end{aligned}$$

Thus condition (i) of Theorem 2 is satisfied with

$$\lambda_1 = \lambda_{\min}(P), \qquad \lambda_2 = \lambda_{\max}(P) + \frac{\theta\tau}{2}, \qquad r_1 = r_2 = 2.$$

The time derivative of $V_1(\eta_t)$ along the trajectories of system (11) is

$$\dot{V}_1(\eta_t) = \eta^T\left(A_L^T P + PA_L\right)\eta + 2\eta^T P\left(f(\hat{x},\hat{x}^\tau,u) - f(x,x^\tau u)\right). \tag{15}$$

The time derivative of $V_2(\eta_t)$ along the trajectories of system (11) is

$$\dot{V}_2(\eta_t) = \frac{\theta}{2}\|\eta\|^2 - \frac{\sqrt{\theta}}{2}\|\eta^\tau\|^2 - \frac{\ln\theta}{2\tau}V_2(\eta_t). \tag{16}$$

Next, the time derivative of (13) along the trajectories of system (11) and making use of (9),(14), (15) and (16), we have

$$\begin{aligned} \dot{V}(\eta_t) \leqslant\; & -\frac{\theta}{2}\|\eta\| + 2\|\eta\|\|P\|\|\Delta_\theta(f(\hat{x},\hat{x}^\tau,u) - f(x,x^\tau u))\| \\ & -\frac{\sqrt{\theta}}{2}\|\eta^\tau\|^2 - \frac{\ln\theta}{2\tau}V_2(\eta_t). \end{aligned} \tag{17}$$

Using (14) we obtain

$$\dot{V}(\eta_t) + \frac{\ln\theta}{2\tau}V(\eta_t) \leqslant -\left\{\frac{\theta}{2} - \|P\|\frac{\ln\theta}{2\tau}\right\}\|\eta\|^2 + 2\|\eta\|\|P\|\|\Delta_\theta(f(\hat{x},\hat{x}^\tau,u) - f(x,x^\tau u))\| - \frac{\sqrt{\theta}}{2}\|\eta^\tau\|^2.$$

The following inequality hold globally thanks to Assumption 1 (as in [7, 11])

$$\|\Delta_\theta(f(\hat{x},\hat{x}^\tau,u) - f(x,x^\tau u))\| \leqslant k_1\|\Delta_\theta(\hat{x}-x)\| + k_2\|\Delta_\theta(\hat{x}^\tau - x)\| \tag{18}$$

$$\leqslant k\|\eta\| + k\|\eta^\tau\|, \tag{19}$$

where k_1, k_2 is a Lipschitz constant in (18) and $k = \max(k_1, k_2)$.

So, we get

$$\dot{V}(\eta_t) + \frac{\ln\theta}{2\tau}V(\eta_t) \leqslant -\left\{\frac{\theta}{2} - \|P\|\frac{\ln\theta}{2\tau} + 2k\|P\|\right\}\|\eta\|^2 + 2k\|P\|\|\eta\|\|\eta^\tau\|^2 - \frac{\sqrt{\theta}}{2}\|\eta^\tau\|^2.$$

Using the fact that

$$2\|\eta\|\|\eta^\tau\| \leqslant \|\eta\|^2 + \|\eta^\tau\|^2,$$

we deduce that

$$\dot{V}(\eta_t) + \frac{\ln\theta}{2\tau}V(\eta_t) \leqslant -\left\{\frac{\theta}{2} - \|P\|\frac{\ln\theta}{2\tau} - 3k\|P\|\right\}\|\eta\|^2 - \left\{\frac{\sqrt{\theta}}{2} - k\|P\|\right\}\|\eta^\tau\|^2. \tag{20}$$

Let

$$a(\theta) = \frac{\theta}{2} - \|P\|\frac{\ln\theta}{2\tau} - 3k\|P\|, \qquad b(\theta) = \frac{\sqrt{\theta}}{2} - k\|P\|.$$

Using (10) we have $a(\theta) > 0$ and $b(\theta) > 0$.

Now, the objective is to prove the rational convergence of the observer (11). Inequality (20) becomes

$$\dot{V}(\eta_t) \leqslant -\frac{\ln\theta}{2\tau}V(\eta_t).$$

Finally, using the stability Theorem 2, we can conclude that the error dynamics (11) is globally rationally stable if (10) hold. □

4.2. Global rational stabilization by state feedback

In this subsection, we establish a delay-dependent condition for the rational state feedback stabilization of the nonlinear system (7). The state feedback controller is given by

$$u = K(\theta)x, \tag{21}$$

where $K(\theta) = [k_1\theta^n, \ldots, k_n\theta]$ and $K = [k_1, \ldots, k_n]$ is selected such that $A_K := A + BK$ is Hurwitz. Let S be the symmetric positive definite solution of the Lyapunov equation

$$A_K^T S + SA_K = -I. \tag{22}$$

Theorem 4 *Suppose that Assumption 1, 2 are satisfied and there exists positive constant θ such that*

$$\begin{cases} \dfrac{\theta}{2} - \|S\|\dfrac{\ln\theta}{2\tau} - 3k\|S\| > 0, \\ \dfrac{\sqrt{\theta}}{2} - k\|S\| > 0, \end{cases} \tag{23}$$

then, the closed loop time-delay system (7)–(21) *is globally rationally stable.*

Proof. The closed loop system is given by

$$\dot{x} = (A + BK(\theta))x + f(x, x^\tau, u)\,. \tag{24}$$

Let $\chi = \Delta_\theta x$. Using the fact that $\Delta_\theta BK(\theta) = \theta BK\Delta_\theta$ we get

$$\dot{\chi} = \theta A_K\chi + \Delta_\theta f(x, x^\tau, u). \tag{25}$$

Let us choose a Lyapunov-Krasovskii functional candidate as follows

$$W(\chi_t) = W_1(\chi_t) + W_2(\chi_t), \tag{26}$$

with

$$W_1(\chi_t) = \chi^T S\chi$$

and

$$W_2(\chi_t) = \frac{\theta}{2}\theta^{\frac{-t}{2\tau}} \int_{t-\tau}^{t} \theta^{\frac{s}{2\tau}} \|\chi(s)\|^2 \mathrm{d}s.$$

As in the proof of Theorem 3, we have

$$\lambda_{\min}(S)\|\chi(t)\|^2 \leqslant W(\chi_t) \leqslant \left(\lambda_{\max}(S) + \frac{\theta\tau}{2}\right)\|\chi_t\|_\infty^2.$$

Thus condition (i) of Theorem 2 is satisfied with

$$\lambda_1 = \lambda_{\min}(S), \qquad \lambda_2 = \lambda_{\max}(S) + \frac{\theta\tau}{2}, \qquad r_1 = r_2 = 2.$$

The time derivative of (26) along the trajectories of system (25) is given by

$$\begin{aligned}\dot{W}(\chi_t) &= 2\chi^T S\dot{\chi} + \frac{\theta}{2}\|\chi\|^2 - \frac{\sqrt{\theta}}{2}\|\chi^\tau\|^2 - \frac{\ln\theta}{2\tau}W_2(\chi_t)\\ &= 2\theta\chi^T SA_K\chi + 2\chi^T S\Delta_\theta f(x,x^\tau,u) + \frac{\theta}{2}\|\chi\|^2 - \frac{\sqrt{\theta}}{2}\|\chi^\tau\|^2 - \frac{\ln\theta}{2\tau}W_2(\chi_t)\\ &\leqslant -\frac{\theta}{2}\|\chi\| + 2\|\chi\|\|S\|\|\Delta_\theta f(x,x^\tau,u)\| - \frac{\sqrt{\theta}}{2}\|\chi^\tau\|^2 - \frac{\ln\theta}{2\tau}W_2(\chi_t).\end{aligned}$$

Since $f(0,0,u) = 0$, (19) implies that

$$\|\Delta_\theta f(x,x^\tau,u)\| \leqslant k\|\chi\| + k\|\chi^\tau\|. \tag{27}$$

So

$$\begin{aligned}\dot{W}(\chi_t) + \frac{\ln\theta}{2\tau}W(\chi_t) &\leqslant -\left\{\frac{\theta}{2} - \|S\|\frac{\ln\theta}{2\tau} - 2k\|S\|\right\}\|\chi\|^2\\ &\quad + 2k\|S\|\|\chi\|\|\chi^\tau\| - \frac{\sqrt{\theta}}{2}\|\chi^\tau\|^2.\end{aligned} \tag{28}$$

Using the fact that

$$2\|\chi\|\|\chi^\tau\| \leqslant \|\chi\|^2 + \|\chi^\tau\|^2.$$

We deduce that

$$\begin{aligned}\dot{W}(\chi_t) + \frac{\ln\theta}{2\tau}W(\chi_t) &\leqslant -\left\{\frac{\theta}{2} - \|S\|\frac{\ln\theta}{2\tau} - 3k\|S\|\right\}\|\chi\|^2\\ &\quad - \left\{\frac{\sqrt{\theta}}{2} - k\|S\|\right\}\|\chi^\tau\|^2.\end{aligned} \tag{29}$$

Let

$$\begin{aligned}c(\theta) &= \frac{\theta}{2} - \|S\|\frac{\ln\theta}{2\tau} - 3k\|S\|,\\ d(\theta) &= \frac{\sqrt{\theta}}{2} - k\|S\|.\end{aligned}$$

Using (23), we have $c(\theta) > 0$ and $d(\theta) > 0$ which implies that

$$\dot{W}(\chi_t) \leqslant -\frac{\ln\theta}{2\tau}W(\chi_t).$$

By Theorem 2, we conclude that the origin of the closed loop system (24) is globally rationally stable. □

4.3. Observer-based control stabilization

In this subsection, we implement the control law with estimate states. The observer-based controller is given by:

$$u = K(\theta)\hat{x}, \tag{30}$$

where $\hat{x}$ is provided by the observer (8).

Theorem 5 *Suppose that Assumptions 1, 2 are satisfied, such that conditions* (10) *and* (23) *hold. Then the origin of the closed loop time-delay system* (7)–(30) *is globally rationally stable.*

Proof. The closed loop system in the (χ, η) coordinates can be written as follows:

$$\begin{aligned} \dot{\chi} &= \theta A_K \chi + \theta B K \eta + \Delta_\theta f(x, x^\tau, u), \\ \dot{\eta} &= A_L \eta + \Delta_\theta (f(\hat{x}, \hat{x}^\tau, u) - f(x, x^\tau, u)). \end{aligned} \tag{31}$$

Let

$$U(\eta_t, \chi_t) = \alpha V(\eta_t) + W(\chi_t),$$

where V and W are given by (13) and (26) respectively. Using the above results, we get

$$\dot{U}(\eta_t, \chi_t) + \frac{\ln\theta}{2\tau} U(\eta_t, \chi_t) \leqslant -\alpha a(\theta)\|\eta\|^2 - c(\theta)\|\chi\|^2 + 2\theta\|S\|\|K\|\|\eta\|\|\chi\|.$$

Now using the fact that for all $\varepsilon > 0$,

$$2\|\chi\|\|\eta\| \leqslant \varepsilon\|\chi\|^2 + \frac{1}{\varepsilon}\|\eta\|^2,$$

and select $\varepsilon = \dfrac{c(\theta)}{2\theta\|S\|\|K\|}$, we get

$$\dot{U}(\eta_t, \chi_t) + \frac{\ln\theta}{2\tau} U(\eta_t, \chi_t) \leqslant -\alpha a(\theta)\|\eta\|^2 - \frac{c(\theta)}{2}\|\chi\|^2 + \frac{2\theta^2}{c(\theta)}\|S\|^2\|K\|^2\|\eta\|^2.$$

Finally we select α such that

$$\alpha a(\theta) - \frac{2\theta^2}{c(\theta)}\|S\|^2\|K\|^2 > 0,$$

to deduce that the origin of system (31) is globally rationally stable. □

Remark 4 *It is easy to see that,* $a(\theta)$, $b(\theta)$, $c(\theta)$ *and* $d(\theta)$ *tend to* ∞ *as* θ *tends to* ∞. *This implies that there exists* $\theta_0 > 1$ *such that for all* $\theta > \theta_0$ *conditions* (10) *and* (23) *are fulfilled.*

5. Global rational stabilization by output feedback

In this subsection, we propose the following system:

$$\dot{\widetilde{\hat{x}}}(t) = A\widetilde{\hat{x}} + Bu(t) + L(\theta)(C\widetilde{\hat{x}} - y). \tag{32}$$

The output feedback controller is given by

$$u = K(\theta)\widetilde{\hat{x}}. \tag{33}$$

Under Assumption 1, 2, we give now required conditions to ensure that the origin of system (7) is rendered globally rationally stable by the dynamic output feedback control (32), (33).

Theorem 6 *Consider the time-delay system* (7) *under Assumptions 1, 2. Suppose that there exists* $\theta > 0$ *such that condition* (23) *holds and*

$$\frac{\theta}{2} - \|S\|\frac{\ln\theta}{2\tau} - 3k\|S\| > 0 \tag{34}$$

then the closed-loop time-delay system (7)–(33) *is globally rationally stable.*

Proof. Defining $\widetilde{e} = x - \widetilde{\hat{x}}$ the observation error. We have

$$\dot{\widetilde{e}} = (A + L(\theta)C)\widetilde{e} + f(x, x^{\tau}, u). \tag{35}$$

For $\theta > 0$, let $\Delta_\theta = diag\left[1, \frac{1}{\theta}, \ldots, \frac{1}{\theta^{n-1}}\right]$. Let $\widetilde{\eta} = \Delta_\theta\widetilde{e}$, then we get

$$\dot{\widetilde{\eta}} = \theta A_L\widetilde{\eta} + \Delta_\theta f(x, x^{\tau}, u). \tag{36}$$

Let us choose a Lyapunov-Krasovskii functional candidate as follows

$$V(\widetilde{\eta}_t) = \widetilde{\eta}^T P\widetilde{\eta} + \frac{\theta}{2}\theta^{\frac{-t}{2\tau}}\int_{t-\tau}^{t}\theta^{\frac{s}{2\tau}}\|\widetilde{\eta}(s)\|^2 \mathrm{d}s. \tag{37}$$

As in the proof of Theorem 3, we have thus condition (i) of Theorem 2 is satisfied with

$$\lambda_1 = \lambda_{\min}(P), \qquad \lambda_2 = \lambda_{\max}(P) + \frac{\theta\tau}{2}, \qquad r_1 = r_2 = 2.$$

Following the proof of Theorem 2, inequality (17) becomes

$$\dot{V}(\widetilde{\eta}_t) \leqslant -\frac{\theta}{2}\|\widetilde{\eta}\| + 2\|\widetilde{\eta}\|\|P\|\|\Delta_\theta f(x, x^{\tau}, u)\|$$

$$
-\frac{\sqrt{\theta}}{2}\|\widetilde{\eta}^{\tau}\|^2-\frac{\theta}{2}\frac{\ln\theta}{2\tau}\theta^{\frac{-t}{2\tau}}\int_{t-\tau}^{t}\theta^{\frac{s}{2\tau}}\|\widetilde{\eta}(s)\|^2\mathrm{d}s.
$$

Using (14) and (27) we obtain

$$
\dot{V}(\widetilde{\eta}_t)+\frac{\ln\theta}{2\tau}V(\widetilde{\eta}_t)\leqslant-\left\{\frac{\theta}{2}-\|P\|\frac{\ln\theta}{2\tau}\right\}\|\widetilde{\eta}\|^2+2k\|P\|\|\widetilde{\eta}\|\|\chi\|
$$

$$
+2k\|P\|\|\widetilde{\eta}\|\|\chi^{\tau}\|-\frac{\sqrt{\theta}}{2}\|\widetilde{\eta}^{\tau}\|^2.
$$

Using the fact that

$$
2\|\widetilde{\eta}\|\|\chi\|\leqslant\|\widetilde{\eta}\|^2+\|\chi\|^2
$$

and

$$
2\|\widetilde{\eta}\|\|\chi^{\tau}\|\leqslant\|\widetilde{\eta}\|^2+\|\chi^{\tau}\|^2,
$$

we deduce that

$$
\dot{V}(\widetilde{\eta}_t)+\frac{\ln\theta}{2\tau}V(\widetilde{\eta}_t)\leqslant-\left\{\frac{\theta}{2}-\|P\|\frac{\ln\theta}{2\tau}-2k\|P\|\right\}\|\widetilde{\eta}\|^2+k\|P\|\|\chi\|^2
$$

$$
+k\|P\|\|\chi^{\tau}\|^2-\frac{\sqrt{\theta}}{2}\|\widetilde{\eta}^{\tau}\|^2. \tag{38}
$$

Let

$$
U(\widetilde{\eta}_t,\chi_t)=\alpha V(\widetilde{\eta}_t)+W(\chi_t),
$$

where W is given by (26). Using (28) and (38), we get

$$
\dot{U}(\widetilde{\eta}_t,\chi_t)+\frac{\ln\theta}{2\tau}U(\widetilde{\eta}_t,\chi_t)\leqslant-\alpha\left\{\frac{\theta}{2}-\|P\|\frac{\ln\theta}{2\tau}-2k\|P\|\right\}\|\widetilde{\eta}\|^2
$$

$$
-\{c(\theta)-\alpha k\|P\|\}\|\chi\|^2-\{d(\theta)-\alpha k\|P\|\}\|\chi^{\tau}\|^2.
$$

Finally, we select α such that

$$
\alpha<\min\left(\frac{c(\theta)}{\alpha k\|P\|},\frac{d(\theta)}{\alpha k\|P\|}\right)
$$

to get

$$
\dot{U}(\widetilde{\eta}_t,\chi_t)+\frac{\ln\theta}{2\tau}U(\widetilde{\eta}_t,\chi_t)\leqslant 0.
$$

Therefore, the closed-loop system is globally rationally stable. □

Remark 5 *The given controller in Theorem 3 depends on the nonlinearity f and the time delay τ, but the controller* (32), (33) *is independent of f.*

Remark 6 *It is worth mentioning that exponential stability including both passivity and dissipativity of generalized neural networks with mixed time-varying delays are developed in [25] by using the Lyapunov Krasovskii approach in combination with linear matrix inequalities. These conditions rely on the bounds of the activation functions. In this paper, we utilize parameter-dependent control laws. We assume that there exists a linear feedback that asserts global rational stability of the linear part. Hence, we select the θ-parameter in order to establish global rational stability of the nonlinear system under the same controller.*

Remark 7 *[16, 18] show the sufficient conditions which guarantee that the calculation error converges asymptotically towards zero in terms of a linear matrix inequality. As compared to [16, 18], our results are less conservative and more convenient to use and it seems natural and attractive to improve feedbacks and to get solutions decreasing to zero faster.*

Remark 8 *It is worth noting that the obtained findings can be used in multiple time-delays nonlinear systems in the upper-triangular form.*

6. Simulation results

This section presents experimental results, in the case of constant delay as an example of practical application of the time-delay method in actual network-based control systems. The dynamics of the network-based system are represented by:

$$\begin{aligned} \dot{x}_1 &= x_2(t) + x_1 \cos x_1 + x_1(t-\tau)\cos u, \\ \dot{x}_2 &= u, \end{aligned} \tag{39}$$

where $x(t)$ is the augmented state vector containing the plant state vector and τ indicates the sensor-to-controller delay in the continuous-time case, is supposed to be constant. The difference between $\hat{x}(t)$ and $x(t)$ is formulated as an error of the network-based system. The system nonlinearities are globally Lipschitz. Following the notation used throughout the paper, let $f_1(x, x^\tau, u) = x_1 \cos x_1 + x_1(t-\tau)\cos u$, $f_2(x, x^\tau, u) = 0$. Now, select $L = [-14\,-28]^T$ and $K = [-30\,-30]$, A_L and A_K are Hurwitz. Using Matlab, the solutions of the Lyapunov equations (9) and (22) are given respectively by

$$P = \begin{bmatrix} 0.0377 & 0.0278 \\ 0.0278 & 1.0675 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.5172 & -0.5000 \\ -0.5000 & 0.5167 \end{bmatrix}.$$

So, $||P|| = 1.0682$ and $||S|| = 1.0169$. For our numerical simulation, we choose constant delay $\tau = 1$, and the initial conditions for the system are $x(0) = [-20 \ \ -10]^T$, for the observer $\hat{x}(0) = [10 \ \ 10]^T$ and $\theta = 8$. Corresponding numerical simulation results are shown in Figures 1, 2.

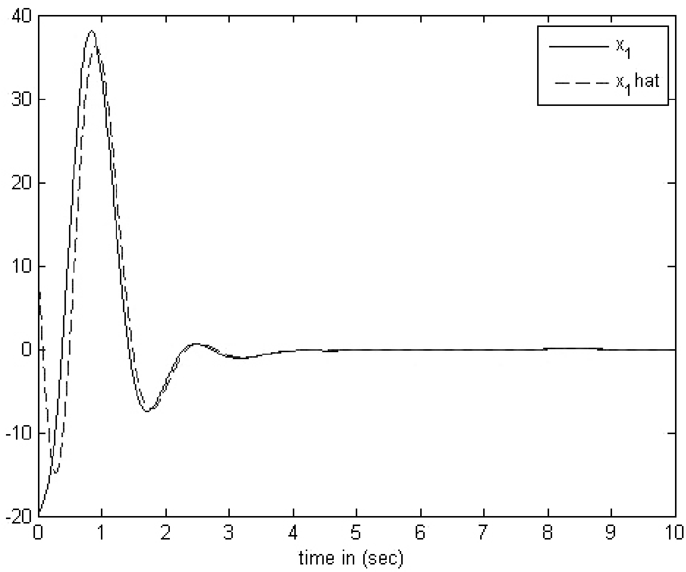

Figure 1: Trajectories of x_1 and its estimate $\hat{x}_1$

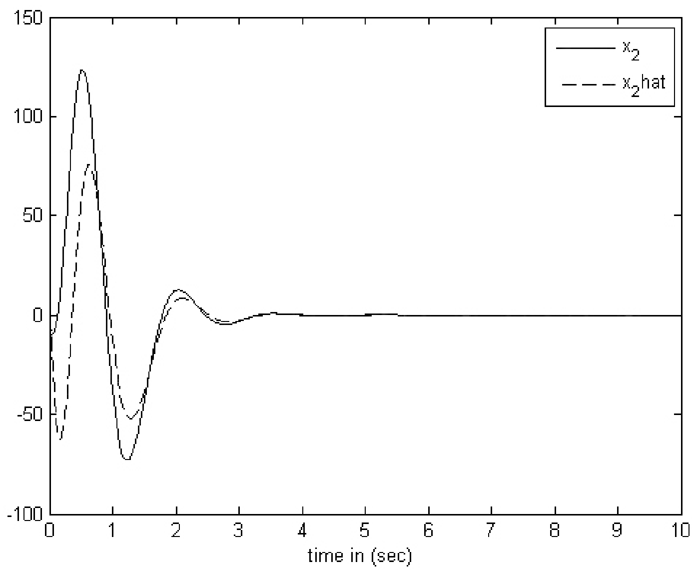

Figure 2: Trajectories of x_2 and its estimate $\hat{x}_2$

7. Conclusion

In this paper, rational stability and stabilization are investigated for nonlinear time-delay systems. We gave a rigorous construction of the rational stability of nonlinear time delay systems, followed by example of control systems. Based on this study, we reached a novel result in global rational stability and stabilization of a class nonlinear time-delay systems. This class of systems deals with the systems that have a triangular structure. Based on the result, it was found that the Lyapunov approach was used to perform sufficient conditions for rational stability. The novel design plays a crucial role in getting a rational stability condition and rendering our approach application to a general class of systems, namely the class of nonlinear time-delay systems in a lower triangular form. The numerical result of an example is provided to show the effectiveness of the proposed approach. Moreover, simulation results show that the proposed observer-based control scheme gave good results. As a perspective, It is well known that delay-dependent conditions reveal less conservative than delay-dependent ones, it can be developed in future research by considering other Lyapunov-Krasovskii functional to derive delay dependent conditions.

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