Opuscula Math. 34, no. 1 (2014), 81–96
http://dx.doi.org/10.7494/OpMath.2014.34.1.81 Opuscula Mathematica

# PROBLEM OF DETECTING INCLUSIONS BY TOPOLOGICAL OPTIMIZATION

I. Faye, M. Ndiaye, I. Ly, and D. Seck

*Communicated by Vicentiu D. Radulescu*

**Abstract.** In this paper we propose a new method to detect inclusions. The proposed method is based on shape and topological optimization tools. In fact after presenting the problem, we use topological optimization tools to detect inclusions in the domain. Numerical results are presented.

**Keywords:** topological optimization, topological gradient, shape optimization, detection of inclusions, numerical simulations.

**Mathematics Subject Classification:** 74P05, 74P10, 74P15, 65F30, 49Q10.

## 1. INTRODUCTION

From a "physical" or "experimental" point of view, we shall qualify as an inverse problem any situation where we wish to estimate a certain physical quantity $p$ inaccessible in experiment from the measure of another quantity $d$ directly accessible to the experiment, knowing a mathematical model of the direct problem which looks explicitly $d$ from $p$ (what we note symbolically $d = G(p)$). This kind of problem is an "inverse problem". In mathematics, the resolution of such problems can lead to the use of many mathematical tools and theories such as linear algebra, ordinary differential equations, partial differential equations or the minimization of an adequate functional.

An interesting and important inverse problem is the determination of cracks by over determined data. The uniqueness (identifiability) result for a buried single crack has been proved in [15]. The authors consider the case of a crack which is a perfectly insulating curve $\sigma$, located inside a 2-dimensional medium with a known reference conductivity. They proved that the presence of a crack and its actual shape and location may be determined from exactly two boundary measurements. In the same paper, they establish a partial stability result. In 1993, other authors proved a Lipschitz stability result for linear cracks. An identifiability result has been developed by Andrieux and

 

Ben Abda in [5] in 1992. In the case of a collection of cracks Bryan and Vogeluis in [10], followed by G. Alessandrini and D. Valenzuela in [3] proved uniqueness results.

In [6], the authors deal with the 2D inverse crack problem, emerging in a known point of the external boundary. They establish identifiability and stability results.

The main purpose of this note is to locate the inclusion in a beginning domain by using topological optimization tools. The first section concerns the formulation and the localization of the problem, the second one the shape optimization problem where we establish the functional space and the minimization problem. The third section is devoted to the topological optimization problem and the main result. The last section presents the numerical results.

## 2. PROBLEM FORMULATION

Another interesting and important identification problem which arises in electrical prospecting, in geophysical prospecting and medical imaging is to discover the location of metals or fluid reservoirs inside the earth.

Let us consider, the following identification problem

$$\begin{cases} \operatorname{div}((1+(k-1)\,\chi_D)\nabla u) = 0 & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega, \quad \text{or on } \Gamma \subset \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $k > 0, k \neq 1$, is the electrical conductivity and $D$ is an unknown subdomain of $\Omega, \chi_D$ denotes the characteristic function of $D$ and $u$ denotes the electrostatic potential in $\Omega$, $f$ is the boundary voltage function, and some times it is possible to get a measure only on a part of $\partial\Omega$; and $g$ is the flux across $\partial\Omega$ or current measurements.

In (2.1), one wishes to locate $D$ in the equation by knowledge of the boundary voltage and current measurements.

It is important to emphasize that the formulation of the problem needs to be explained. Our aim is to identify $D$ under the following hypotheses:

- The conductivity $k$ is given.
- Not both the boundary condition (Dirichlet and Neumann) are considered.

The problem is to obtain some information on the location of $D$ from the measurements $f$ or $g$.

In [12], the authors supposed there are some errors in the measurements, they established a local stability theorem, but in [1], the author proved that the stability is in fact global. In [14], the authors consider an inverse problem for an electrically conductive material occupying a domain $\Omega$ in $\mathbb{R}^N$, $N \geq 2$. They consider a sub domain of $\Omega$ and suppose that the conductivity coefficient of this sub domain is different for the conductivity in $\Omega$ they wish to determine the location of $D$ by injecting a current across the boundary of $\Omega$, $\partial\Omega$ and measure the voltage on a portion of the boundary. They prove that if $D$ is known to be a convex polyhedron (although its specific shape is not known), then the shape and location of $D$ are determined by one measurement only. If $\partial\Omega$ is piecewise analytic, $D$ is uniquely determined in [18]; the case where $\partial\Omega$ is Lipschitz is studied in [17]. In [13], the authors study the case of a single measurement. In [11], Friedman considers the problem of detection: if the measure of $D$ is prescribed, can one distinguish $D$ from the empty set? His result was generalized in [9] by considering the case when the conductivities in $\Omega\backslash D$ and $D$ are non constant. In [2] in 1998, the authors give constructive estimates on the measure of the unknown inclusion $D$ in terms of the boundary data provided a $C^{1,\alpha}$ bound on $\partial\Omega$ is given.

In this work, we want to use topological optimization tools to find the location of $D$. In order to define our problem, let us assume here, for the sake of simplicity, that the conductivity in $D$ is $k \equiv 2$.

Let us consider the solution $u_0$ to (2.1) when $D$ is replaced by the empty set,

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{in } \partial\Omega, \\ \dfrac{\partial u_0}{\partial \nu} = g & \text{in } \partial\Omega. \end{cases} \tag{2.2}$$

Then the problem (2.1) takes the form

$$\begin{cases} \operatorname{div}((1+\chi_D)\nabla u) = 0 & \text{on } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

In this section we deal about a problem of inclusion detections. Let $\Omega$ be a bounded open set of $\mathbb{R}^N,\ N \geq 2$ and we assume that there exists somewhere an inclusion inside the domain $\Omega$.

There are many works about the detection of inclusions: In [2], the authors prove upper and lower bounds on the size of the unknown inclusion $D$ when one pair of current density and voltage measurements from the exterior of the domain is available. In [13], from the measurements of a pair of Dirichlet and Neumann data one wishes to identify $D$. It is proved that this problem is stable in some local sense. It is also proven in the same paper that if $\Omega$ is of boundary $\mathcal{C}^{1,\alpha},\ D \subset\subset \Omega$ and $u^e = u/_{\Omega\backslash\bar{D}},\ u^i = u/_D$, then $u^e \in \mathcal{C}^{1,\beta}(\bar{\Omega}\backslash D),\ u^i \in \mathcal{C}^{1,\beta}(\bar{D}),\ \beta \in (0,1)$. Furthermore, $u^e = u^i$ on $\partial\Omega$ and $\frac{\partial u^e}{\partial n} = \frac{\partial u^i}{\partial n}$ on $\partial D$.

The equation (2.3) can be reformulated as the Poisson problem with the following boundary value problem:

$$\begin{cases} -\Delta u_D = 0 & \text{in } \Omega\backslash D, \\ \dfrac{\partial u_D}{\partial n} = 0 & \text{on } \partial D, \\ \dfrac{\partial u_D}{\partial \nu} = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

and we want to find $D \subset \Omega$ such that the solution $u_D$ to (2.4) satisfy the condition $u_{D/\partial\Omega} = g$ for a given measuring voltage function $f$.

Let us consider the boundary value problem defined in $\Omega$ without inclusions by

$$\begin{cases} -\Delta u_0 = 0 & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu} = \phi_0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Let us remark that the above problem, admits a solution only if $\int_{\partial\Omega} \phi \, d\sigma = 0$ for (2.4) and $\int_{\partial\Omega} \phi d\,\sigma = 0$ for (2.5). For finding $D$, we will use shape and topological optimization tools. The idea is to consider the following cost functional

$$J(\Omega) = \int_{\Omega} (u_D - u_0)^2 dx, \tag{2.6}$$

for the topological optimization problem. For the shape optimization problem, we are going to consider

$$J(D) = \frac{1}{2} \int_{\Omega \setminus D} |\nabla u_D|^2 dx, \tag{2.7}$$

where $D \in \mathcal{O}$, $\mathcal{O}$ is a space of admissible domains $D$. But in functional (2.6) the solution to (2.4) is not defined everywhere in $\Omega$ (it is defined in $\Omega \setminus D$), then we can use an extension of $u_D$ everywhere in $\Omega$ and we consider the new cost functional defined by

$$J(\Omega) = \int_{\Omega} (\tilde{u}_D - u_0)^2 dx, \tag{2.8}$$

where $\tilde{u}_D$ is solution to

$$\begin{cases} -\Delta \tilde{u}_D = 0 & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}_D}{\partial \nu} = \tilde{\phi} & \text{on } \partial\Omega. \end{cases}$$

As $D$ and $\Omega$ are a regular subset of $\mathbb{R}^N$, and $u_D$ is a solution to the Poisson with Neumann boundary condition, the extension of $u_D$ every where in $\Omega$ is possible. It suffices to consider all open subsets of $\mathbb{R}^n$ with the uniform extension. We recall that the uniform extension is given as follows: for each $w \in \mathcal{S}$ there exists a linear continuous extension operator $P_w \in H^1(\Omega \setminus D)$ to $H^1(\Omega)$, and there exists a constant $M$ such that for all $w \in \mathcal{S}$, $\|P_w\| \leq M$.

Consider a harmonic function defined every where in $D$ and satisfying

$$\begin{cases} \Delta f = 0 & \text{in } D, \\ \dfrac{\partial f}{\partial \nu} = m & \text{on } \partial D, \end{cases}$$

and such that $\int_{\partial D} m \, d\sigma = 0$. One can search $\tilde{u}_D$ such that $\tilde{u}_D = u_D\chi_{\Omega\setminus D} + kf\chi_D$, then for all $x \in \Omega\setminus D$, $\Delta\tilde{u}_D = \Delta u_D = 0$, for $x \in D$, then $\Delta\tilde{u}_D = \Delta f = 0$. $\frac{\partial \tilde{u}_D}{\partial \nu} = \frac{\partial(u_D\chi_{\Omega\setminus D}+kf\chi_D)}{\partial \nu} = \tilde{\phi}$ on $\partial\Omega$. Then $\tilde{u}_D$ is defined in $\Omega$ and is a solution to

$$\begin{cases} -\Delta\tilde{u}_D = 0 & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}_D}{\partial n} = \tilde{\phi} & \text{on } \partial\Omega. \end{cases}$$

## 3. SHAPE OPTIMIZATION PROBLEM

In this section, we need some definitions and auxiliary results to study the existence of a solution for a shape optimization problem.

**Definition 3.1.** Let $\zeta$ be an unitary vector of $\mathbb{R}^N$, $\epsilon$ be a positive real number and $y$ be in $\mathbb{R}^N$. We call a cone with vertex $y$, of direction $\zeta$ and angle to the vertex and height $\epsilon$, the set defined by

$$\mathcal{C}(y,\zeta,\epsilon,\epsilon) = \{x \in \mathbb{R}^N : |x-y| \le \epsilon \text{ and } |(x-y)\zeta| \ge |x-y|\cos\epsilon\}.$$

Let $D$ be an open set of $\mathbb{R}^N$, $D$ is said to have the $\epsilon$-cone property if for all $x \in \partial D$ then there exists a direction $\zeta$ and a strictly positive real number $\epsilon$ such that

$$\mathcal{C}(y,\zeta,\epsilon,\epsilon) \subset D \quad \text{for all} \quad y \in B(x,\epsilon) \cap \bar{\Omega}.$$

Let $K_1$ and $K_2$ be two compact subsets of $\Omega$. Let

$$d(x,K_1) = \inf_{y\in K_2} d(x,y), \quad d(x,K_2) = \inf_{y \in K_1} d(x,y).$$

Note that

$$\rho(K_1,K_2) = \sup_{x\in K_2} d(x,K_1)\,, \quad \rho(K_2,K_1) = \sup_{x\in K_1} d(x,K_2)\,.$$

Let

$$d_H(K_1,K_2) = \max\{\rho(K_1,K_2), \rho(K_2,K_1)\}.$$

We call the Hausdorff distance of $K_1$ and $K_2$, the following positive number, denoted $d_H(K_1,K_2)$.

Let $(D_n)$ be a sequence of open subsets of $\Omega$ and $D$ be an open subset of $\Omega$. We say that the sequence $(D_n)$ converges on $D$ in the Hausdorff sense and we denote by $D_n \xrightarrow{H} D$ if $\lim_{n\to+\infty} d_H(\bar{\Omega}\setminus D_n, \bar{\Omega}\setminus D) = 0$.

Let $(D_n)$ be a sequence of open sets of $\mathbb{R}^N$ and $D$ be an open set of $\mathbb{R}^N$. We say that the sequence $(D_n)$ converges on $D$ in the sense of $L^p$, $1 \le p < \infty$, if $\chi_{D_n}$ converges on $\chi_D$ in $L^p_{\text{loc}}(\mathbb{R}^N)$, $\chi_D$ being the characteristic functions of $D$.

**Lemma 3.2.** *Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^N$ having the $\epsilon$-cone property, with $\bar{D}_n \subset F \subset \Omega$, $F$ a compact set and $D$ a ball, then, there exists an open set $D$, included in $F$, which satisfies the $\frac{\epsilon}{2}$-cone property and a subsequence $(D_{n_k})_{k\in\mathbb{N}}$ such that*

$$\chi_{D_{n_k}} \stackrel{L^1}{\to} \chi_D, \quad D_{n_k} \stackrel{H}{\to} D,$$
$$\partial D_{n_k} \stackrel{H}{\to} \partial D, \quad \bar{D}_{n_k} \stackrel{H}{\to} \bar{D}.$$

*Proof.* See [16]. □

Let $\mathcal{O}_\epsilon$ be the class of admissible domains defined by

$$\mathcal{O}_\epsilon = \Big\{ D \subset \Omega : \ D \text{ is an open set satisfying the } \epsilon\text{-cône property and } \int_D dx = m_0 \Big\},$$

where $m_0$ is a fixed volume in $\mathbb{R}_+^*$.

The first question is related to the existence problem expressed as follows: Find $D,\ u_D$ such that

$$\min\{J(D),\ D \in \mathcal{O}_\epsilon\}$$

admits a solution and $u_D$ is a solution to the Neumann problem

$$\begin{cases} -\Delta u_D = 0 & \text{in } \Omega\backslash D, \\ \dfrac{\partial u_D}{\partial n} = 0 & \text{in } \partial D, \\ \dfrac{\partial u_D}{\partial \nu} = 0 & \text{in } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$J(D) = \frac{1}{2} \int_{\Omega\backslash D} |\nabla u_D|^2 dx. \tag{3.2}$$

We have the following existence result which is a classical one. For more details see for instance [16, 19]. But we are going to present the proof in this work.

**Theorem 3.3.** *Let $J(D)$ be given by (3.2), where $u_D$ is solution to (3.1). Then the problem: Find $D_0 \in \mathcal{O}_\epsilon$ such that*

$$J(D_0) = \min\{J(D),\ D \in \mathcal{O}_\epsilon\}$$

*admits a solution.*

*Proof.* Let us define the extension $\tilde{u}$ of $u$ by

$$\tilde{u} = \begin{cases} u & \text{if } x \in \Omega\backslash\overline{D}, \\ 0 & \text{if } x \in \overline{D}, \end{cases}$$

$$\nabla \tilde{u} = \begin{cases} \nabla u & \text{if} \quad x \in \Omega \backslash \overline{D}, \\ 0 & \text{if} \quad x \in \overline{D}. \end{cases}$$

$J(D) = \frac{1}{2} \int_{\Omega \backslash D} |\nabla u_D|^2 dx \geq 0$. This implies that $\inf\{J(D),\ D \in \mathcal{O}_\epsilon\} \geq 0$. Let $\alpha = \inf\{J(D),\ D \in \mathcal{O}_\epsilon\}$. Then there exists a minimizing sequence $(D_n)_{n \in \mathbb{N}} \subset \mathcal{O}_\epsilon$ such that $J(D_n)$ converges on $\alpha$.

Since the sequence $(D_n)_{n \in \mathbb{N}}$ is bounded, according to Lemma 3.2 there exists a compact set $F$ such that $\overline{D}_n \subset F \subset \Omega$. There exists also a subsequence $D_{n_k}$ verifying the $\epsilon$-cone property and

$$D_{n_k} \xrightarrow{H} D_0 \quad \text{and} \quad \chi_{D_{n_k}} \xrightarrow{L^1\ p.p} \chi_{D_0}.$$

Let $u_{D_n} = u_n$, the sequence $u_n$ is bounded in $W^{1,2}(\Omega)$. If not, for all $s$ there exists a subsequence $\tilde{u}_n^s \in W^{1,2}(\Omega)$ such that $\frac{1}{2} \int\limits_\Omega |\nabla \tilde{u}_n^s|^2 dx > 0$ and

$$\int\limits_\Omega |\nabla \tilde{u}_n^s|^2 dx = \int\limits_{\Omega \backslash D_n} |\nabla \tilde{u}_n^s|^2 dx + \int\limits_{D_n} |\nabla \tilde{u}_n^s|^2 dx.$$

Thus $J(\Omega_n) \to +\infty$, and then it is a contradiction.

As $H_0^1(\Omega)$ is a reflexive space, there exist a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and a function $u^*$ such that $u_{n_k} \rightharpoonup u^*$ in $H^1(\Omega/\mathbb{R})$ and $u_{n_k} \xrightarrow{L^2 p.p} u^*$. We have

$$\int\limits_{\Omega \backslash D_{n_k}} \nabla u_{n_k} \cdot \nabla v = 0, \quad v \in H^1(\Omega)/\mathbb{R}. \tag{3.3}$$

Taking the limit as $n \to +\infty$, we get

$$\int\limits_{\Omega \backslash D} \nabla u^* \cdot \nabla v = 0, \quad v \in H^1(\Omega)/\mathbb{R}. \tag{3.4}$$

Taking also $v = u_{n_k}$ in (3.3) and $v = u^*$ in (3.4), we obtain

$$\int\limits_{\Omega \backslash D} |\nabla u^*|^2 dx = \int\limits_{\Omega \backslash D_{n_k}} |\nabla u_{n_k}|^2 dx. \tag{3.5}$$

It follows from (3.5) that $u^* = u_{n_k}$ and $J(u_{n_k}) \xrightarrow{p.p} J(\Omega \backslash D) = \inf_{w \in \mathcal{O}} J(\omega)$. □

## 4. TOPOLOGICAL OPTIMIZATION PROBLEM

Let $\Omega$ be a regular and bounded open domain of $\mathbb{R}^N (N = 2, 3)$. We assume that there exists an inclusion $D$ inside the domain $\Omega$ and we do not know where it is situated. Our aim is to locate $D$ by using topological optimization tools. We consider the following Poisson problem:

$$\begin{cases} -\Delta \tilde{u}_D = 0 & \text{in } \Omega, \\ \frac{\partial \tilde{u}_D}{\partial \nu} = \tilde{\phi} & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\tilde{u}_D$ is an extension of $u_D$ in $\Omega$. The function $u_D$ satisfies:

$$\begin{cases} -\Delta u_D = 0 & \text{in } \Omega\backslash D, \\ \frac{\partial u_D}{\partial \nu} = 0 & \text{on } \partial D, \\ \frac{\partial u_D}{\partial \nu} = \phi & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The solutions (4.1) and (4.2) are unique up to a constant.

The conditions $\int_{\partial\Omega} \tilde{\phi}\, d\sigma = 0,\ \int_{\partial\Omega} \phi\, d\sigma = 0$ are added in order to ensure uniqueness of problems (4.1) and (4.2).

We consider the functional $J(\Omega, \tilde{u}_D)$ to be minimized defined by

$$J(\Omega, \tilde{u}_D) = J(\tilde{u}_D) = \int_{\Omega} (\tilde{u}_D - u_0)^2 dx, \tag{4.3}$$

where $\tilde{u}_D$ solution to (4.1) is an extension of $u_D$ solution to (2.4) and $u_0$ is solution to (2.5).

The idea of topological asymptotic analysis is to measure the impact of perturbation of the domain on the cost function. For a small parameter $\epsilon$, we introduce the perturbed domain $\Omega_\epsilon = \Omega\backslash\bar{\omega}_\epsilon$ obtained by insertion of a small domain $\omega_\epsilon = x_0 + \epsilon\omega$, where $x_0 \in \Omega$ and $\omega \subset \Omega$.

In $\Omega_\epsilon,\ u_\epsilon$ satisfies:

$$\begin{cases} -\Delta u_\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u_\epsilon}{\partial \nu} = \tilde{\phi} & \text{on } \partial\Omega, \\ u_\epsilon = 0 & \text{or} \quad \dfrac{\partial u_\epsilon}{\partial \nu} = 0 \text{ on } \partial\omega_\epsilon, \end{cases} \tag{4.4}$$

and the corresponding functional is

$$J(\Omega_\epsilon, u_\epsilon) = J(u_\epsilon) = \int_{\Omega_\epsilon} (u_\epsilon - u_0)^2 dx. \tag{4.5}$$

In this section the aim is to determine the variation of the cost function induced by the insertion of this small hall. Our wish is to get the following asymptotic expansion of $J$ when $\epsilon$ tends to zero:

$$J(u_\epsilon) - J(\tilde{u}_D) = f(\epsilon)g(x_0) + o(f(\epsilon)), \tag{4.6}$$

where $f(\epsilon)$ is a positive function satisfying $\lim_{\epsilon\to 0} f(\epsilon) = 0$ and where $g$ is called the topological gradient or topological sensitivity. The insertion of small halls, where $g$

gets its lower bound, gives the best approximation in the sense $J(u_\epsilon) \simeq J(\tilde{u}_D)$. If this approximation is obtained, we will be able to claim that where $u_\epsilon$ is far from $\tilde{u}_D$ the inclusion is located. Before going on it is important to underline that one of the first steps to prove is $\|u_\epsilon - \tilde{u}_D\|_{H^1(\Omega_\epsilon)} = o(f(\epsilon))$.

Now let us introduce the weak formulation. Multiplying (4.4) by a test function $v$ and integrating we have

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u_\epsilon}{\partial \nu}.v \, d\sigma - \int_{\partial\omega_\epsilon} \frac{\partial u_\epsilon}{\partial \nu}.v \, d\sigma = 0.$$

As $\frac{\partial u_\epsilon}{\partial \nu} = \tilde{\phi}$ on $\partial\Omega$, we get finally the bilinear form

$$\int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v dx = \int_{\partial\Omega} \tilde{\phi}.v \, d\sigma.$$

Let

$$\mathcal{V}_\epsilon = \Big\{ v \in H^1(\Omega_\epsilon) \colon \frac{\partial v}{\partial \nu} = \tilde{\phi} \text{ on } \partial\Omega, \ v = 0 \text{ on } \partial\omega_\epsilon \Big\},$$

$$a_\epsilon(u_\epsilon, v) = \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v \, dx,$$

and

$$L_\epsilon(v) = \int_{\Omega_\epsilon} \tilde{\phi} v \, dx.$$

Following the same idea, in $\Omega$, let

$$\mathcal{V} = \Big\{ v \in H^1(\Omega) \colon \frac{\partial v}{\partial \nu} = \tilde{\phi} \text{ on } \partial\Omega, \ v = 0 \text{ on } \partial\Omega \Big\},$$

$$a_0(u, v) = \int_{\Omega} \nabla u \nabla v,$$

$$a_\epsilon(u_\epsilon, v) - a_0(u, v) = \int_{\Omega_\epsilon} \nabla(u_\epsilon - u) \nabla v dx + \int_{\tilde{\omega}_\epsilon} \nabla u \nabla v dx.$$

Let $\mathcal{V}$ be a fixed Hilbert space and $\mathcal{L}(\mathcal{V})$ (resp. $\mathcal{L}_2(\mathcal{V})$) denotes the spaces of linear (resp. bilinear) forms on $\mathcal{V}$. Let us state the following hypotheses:

(H$_1$) There exists a real function $f$ defined in $\mathbb{R}^+$, a bilinear and continuous form $a_0$ defined in $\mathcal{L}_2(\mathcal{V})$ and a linear form $\delta_a$ such that:

$$\lim_{\epsilon \to 0} f(\epsilon) = 0, \tag{4.7}$$

$$\|a_\epsilon - a_0 - f(\epsilon)\delta_a\|_{\mathcal{L}_2(\mathcal{V})} = o(f(\epsilon)), \tag{4.8}$$

$$\|l_\epsilon - l_0 - f(\epsilon)\delta_{\mathcal{L}}\|_{\mathcal{L}(\mathcal{V})} = o(f(\epsilon)). \tag{4.9}$$

($H_2$) The bilinear form $a_0$ is coercive: There exists a constant $\alpha > 0$ such that

$$a_0(u,u) \geq \alpha \|u\|^2 \quad \text{for all} \quad u \in \mathcal{V}. \tag{4.10}$$

According to (4.10), the bilinear form $a_\epsilon$ depends continuously on $\epsilon$. Hence there exist $\epsilon_0$ and $\beta > 0$ such that for all $\epsilon \in [0;\epsilon_0]$ the following uniform coercivity condition holds.

$$a_\epsilon(u,u) \geq \beta \|u\|^2 \qquad \text{for all} \quad u \in \mathcal{V}. \tag{4.11}$$

Using inequality (4.11) for $u_1 = u_\epsilon - u_0$, for all $\epsilon \geq 0$, the function $u_\epsilon$ is solution to (4.4), the equalities (4.7) and (4.8) and the continuity of $\delta_a$, we obtain the following lemma.

**Lemma 4.1.** *Let $u_\epsilon$ and $\tilde{u}$ be the solutions to* (4.4) *and* (4.1). *Under the hypotheses* ($H_1$) *and* ($H_2$), *we have*

$$\|u_\epsilon - \tilde{u}\|_{H^1(\Omega_\epsilon)} = O(f(\epsilon)).$$

*Proof.* If follows from hypothesis ($H_2$) that there exists $v \neq 0$ such that

$$\begin{aligned}
\beta\|u_\epsilon - \tilde{u}\|^2 \leq a(u_\epsilon - \tilde{u}, v) \leq |a_\epsilon(u_\epsilon, v) - l_\epsilon(v)| = \\
&= |a_\epsilon(u_\epsilon, v) - (l_\epsilon - l_0 - \delta_l f(\epsilon))(v) - l_0(v) - f(\epsilon)\delta_l(v)| = \\
&= |a_\epsilon(u_\epsilon, v) - a_0(\tilde{u}, v) - (l_\epsilon - l_0 - \delta_l f(\epsilon))(v) - f(\epsilon)\delta_l(v)| \leq \\
&\leq |a_\epsilon(u_\epsilon, v) - a_0(\tilde{u}, v) - f(\epsilon)\delta_a(\tilde{u}, v)| + \\
&\quad + |l_\epsilon(v) - l_0(v) - \delta_l(v) f(\epsilon)| + f(\epsilon)(|\delta_a(\tilde{u}, v)| + |\delta_l(v)|).
\end{aligned}$$

If hypothesis ($H_1$) is satisfied, we obtain

$$\beta\|u_\epsilon - \tilde{u}\|^2 \leq o(f(\epsilon)) + f(\epsilon)(\|\delta_a\|_{\mathcal{L}_2(\mathcal{V})}\|\tilde{u}\| + \|\delta_l\|_{\mathcal{L}(\mathcal{V})})\|v\|.$$ □

($H_3$) Consider a cost function $j(\epsilon) = J(u_\epsilon)$, where the functional $J$ is differentiable. For $u \in \mathcal{V}$, there exists a linear and continuous form $DJ(u) \subset L(\mathcal{V})$ and $\delta_J$ such that

$$J_\epsilon(v) - J_0(u) = DJ_0(u)(v-u) + f(\epsilon)\delta_J + o(f(\epsilon)).$$

The Lagrangian is defined by

$$\mathcal{L}_\epsilon(u,v) = J_\epsilon(u) - a_\epsilon(u,v) - l_\epsilon(v), \quad u, v \in \mathcal{V}.$$

**Theorem 4.2.** *If hypothesis* ($H_1$), ($H_2$) *and* ($H_3$) *are satisfied and let $u_\epsilon$ be the solution to* (4.4). *Then the functionals $J_\epsilon$ admits the following asymptotic expansion:*

$$J_\epsilon(u_\epsilon) - J_0(u) \ = \ f(\epsilon)\delta_{\mathcal{L}}(u,U) + o(f(\epsilon)),$$

*where $\delta_{\mathcal{L}}(u,v) = \delta_J(u) + \delta_a(u,v) - \delta_l$ and $U$ is the solution of the adjoint problem: to look for $U \in \mathcal{V}$ such that*

$$a_0(W,U) = -DJ_0(\Phi)W \quad \textit{for all} \quad W \in \mathcal{V}.$$

In order to get the asymptotic expansion of the cost functional, we will use the fact that variation of the Lagrangian is equal to the one of the cost functional. Then

$$J(u_\epsilon) - J(u) = \mathcal{L}_\epsilon(u_\epsilon, v) - \mathcal{L}_0(u,v) \quad \text{for all} \quad v \in \mathcal{V}.$$

4.1. VARIATION OF THE BILINEAR FORM

Let us take $w_\epsilon = u_\epsilon - u$. Then $w_\epsilon$ is solution to

$$\begin{cases} -\Delta w_\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \dfrac{\partial w_\epsilon}{\partial n} = 0 & \text{on } \partial\Omega, \\ \dfrac{\partial w_\epsilon}{\partial n} = -u & \text{on } \partial\omega_\epsilon. \end{cases}$$

Let us approximate the solution $w_\epsilon$ by $h_\epsilon$ solution to

$$\begin{cases} \Delta h_\epsilon = 0 & \text{in } \mathbb{R}^2 \backslash \bar{\omega}_\epsilon, \\ h_\epsilon = -u & \text{in } \partial\omega_\epsilon, \\ h_\epsilon = 0 & \text{at } \quad \infty. \end{cases}$$

Let us take $h_\epsilon = \epsilon H_\epsilon(\frac{x}{\epsilon})$. Then we get

$$\begin{cases} \Delta H = 0 & \text{in } \mathbb{R}^2 \backslash \bar{\omega}, \\ H = -u & \text{on } \partial\omega, \\ H = 0 & \text{at } \infty. \end{cases} \tag{4.12}$$

The function $H$ solution to (4.12) can be expressed by a single layer potential on $\partial\omega$. Let $E$ be a fundamental solution of the Laplace operator, then we have

$$E(y) = \frac{1}{2\pi} \ln r, \quad r = |y|.$$

Then the function $H$ reads

$$H(x) = \int_{\partial\omega} \lambda(y) E(x - y) ds(y), \quad x \in \mathbb{R}^2 \backslash \bar{\omega},$$

where $\lambda \in H_0^{-1/2}(\partial\omega)$ is the unique solution of the integral equation

$$\int_{\partial\omega} E(y - x) p_\omega(x) d\gamma(x) = -u(x_0), \quad y \in \partial\omega.$$

For $x$ bounded and large, $r = |y|$, we have

$$E(y - x) = E(y) + O\Big(\frac{1}{r^2}\Big).$$

The asymptotic expansion at infinity of the function $H$ is given by

$$\begin{aligned} H(y) &= P_w(y) + W_\omega(y), \\ P_\omega(y) &= A_\omega(u(x_0)) E(y), \\ A_\omega(u(x_0)) &= \int_{\partial\omega} p_\omega d\gamma(x), \\ W_\omega(y) &= 0\Big(\frac{1}{r^2}\Big). \end{aligned}$$

If $\omega = B(0,1)$, then $v_\omega(y)$, $p_\omega(y)$ and $W_\omega(y)$ can be computed explicitly (see [20]):

$$v_\omega(y) = \frac{u(x_0)}{r} = p_\omega(y),\ W_\omega(y) = 0,\ y \neq y \in \mathbb{R}^2.$$

Then

$$A_\omega(u(x_0)) = 2\pi u(x_0).$$

**Theorem 4.3.** *Let $J_\epsilon$ be the functional defined by* (4.5)*, where $u_\epsilon$ is solution to* (4.4) *with $\dfrac{\partial u_\epsilon}{\partial n} = 0$ on $\partial\omega_\epsilon$. The following asymptotic expansion holds*

$$J_\epsilon(u_\epsilon) - J(\tilde{u}) = \epsilon^3(-2\pi\nabla\tilde{u}(0)\nabla U(0) - 2\pi|\tilde{u}(0) - u_0(0)|^2) + o(\epsilon^n), \tag{4.13}$$

*where $\tilde{u}$ is a solution to* (4.1) *and $U$ is a solution to the adjoint state*

$$\begin{cases} -\Delta U = 2(\tilde{u} - u_0) & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

**Remark 4.4.** If $u_\epsilon = 0$ in $\omega_\epsilon$ and $\omega = B(0,1)$, the topological gradient associated to the functional (4.5) is given by

$$g(x_0) = -4\pi\tilde{u}\,U, \tag{4.15}$$

where $U$ is solution to (4.14), and $\tilde{u}$ is solution to (4.1).

## 5. NUMERICAL SIMULATIONS

In this section we present numerical simulations obtained for finding the position of the unknown subset $D$ inside the domain $\Omega$. We use topological optimization tools obtained in the last section. The numerical result show that the domain $D$ is located where the topological gradient $g(x_0)$ is most negative. The difference between the solution $u_0$ of problem (2.5) and $\tilde{u}_D$ solution to (4.1) is very big. That this is reflected by the fact that the conductivities inside $D$ and $\Omega$ are different. By a measurement of the Neumann data $\frac{\partial u_0}{\partial n} = \phi$, we want to locate $D$ inside the domain $\Omega$. The function $\tilde{\phi}$ obtained by the extension of $u_D$ in $\Omega$ depends on the conductivity of the domain $D$.

In the first case, we use $\phi = 2x + y$, $\tilde{\phi} = k(x + y)$, $k$ which is very small. We plot the solution $\tilde{u}_D$, the adjoint state $v_D$ and the topological derivative in the domain $\Omega = [-1,1] \times [-1,1]$. The results obtained are given in Figure 1.

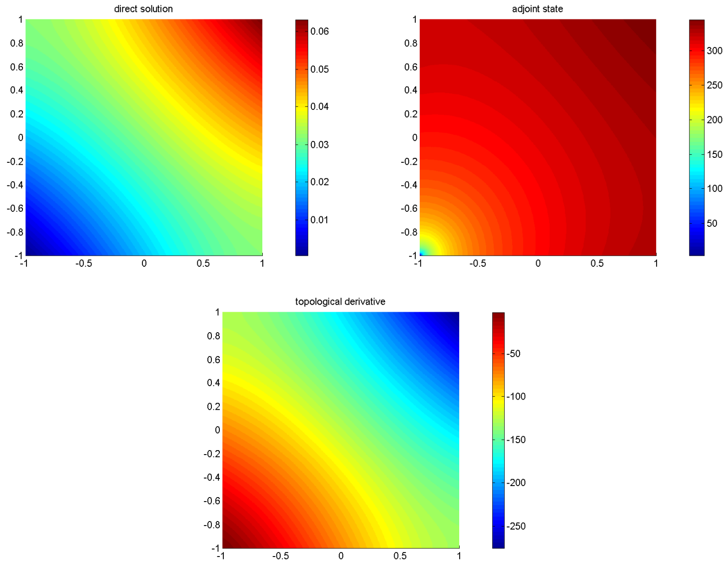


**Fig. 1.** Direct solution, adjoint state and topological derivative

In the second case, we take $\phi = (x^2 - 1)(y^2 - 1) - \frac{4}{9}$, $\tilde{\phi} = -k(x + y)$, $k = 2$. The results obtained are given in Figure 2.

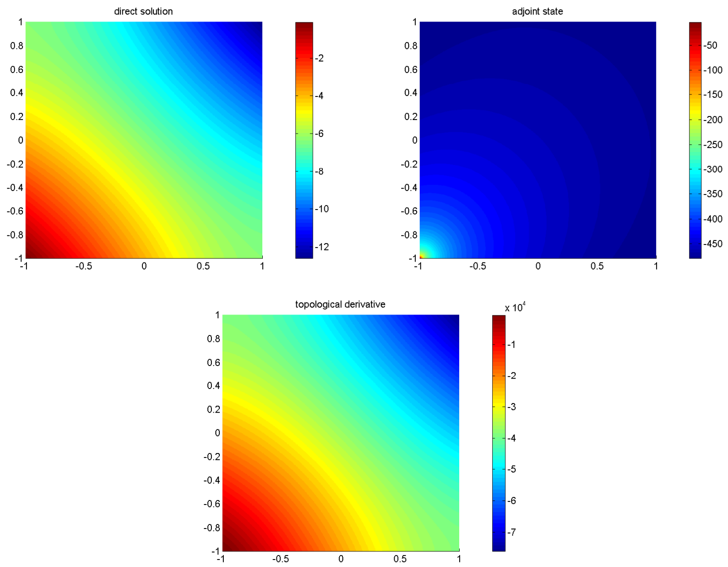


**Fig. 2.** Direct solution, adjoint state and topological derivative

In the last case, we take $\phi(x,y) = \cos\pi(x+y) + \sin\pi(x+y)$, $\tilde{\phi} = -2k\cos\pi(x+y) + \sin\pi(x+y)$, $k = 1, 2$. The results obtained are given in Figure 3.

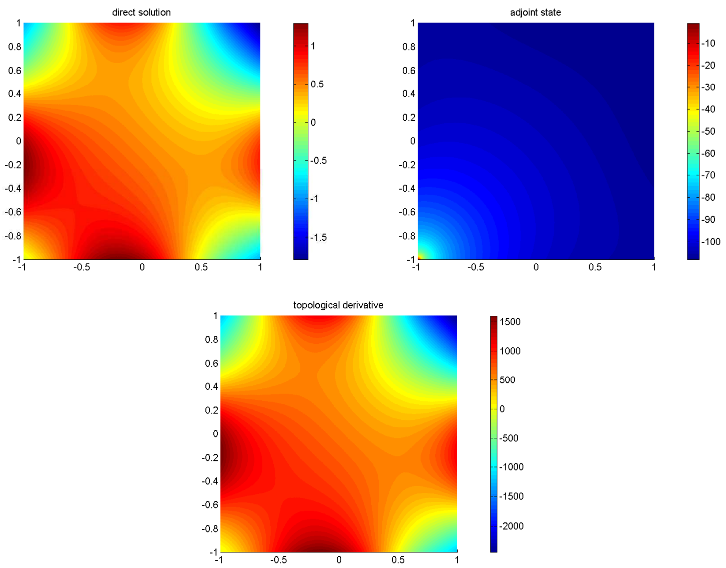


**Fig. 3.** Direct solution, adjoint state and topological derivative

**Acknowledgments**

*This work was supported by the NLAGA-project.*

REFERENCES

[1] G. Alessandrini, *Remark on a paper by Bellout and Friedman*, Boll. Un. Mat. Ital. A (7) **3** (1989), 243–249.

[2] G. Alessandrini, E. Rosset, *The inverse conductivity problem with one measurement: Bounds on the size of the unknown object*, SIAM J. Appl. Math. **58** (1998) 4, 1060–1071.

[3] G. Alessandrini, A. Diaz Valenzuela, *Unique determination of multiple cracks by two measurements*, SIAM J. Control Optim. **34** (3) (1996), 913–921.

[4] H. Ammari, H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Math., Berlin 2004.

[5] S. Andrieux, A.B. Abda, *Identification de fissures planes par une donnée au bord unique; un procédé direct de localisation et d'identification*. Comptes Rendus à l'Académie des Sciences, Série 1, 315, 1992.

[6] S. Andrieux, A.B. Abda, M. Jaoua, *On the inverse emerging plane crack problem*, Raport de recherche 3012, Inria, Octobre 1996.

[7] D. Auroux, M. Masmoudi, *A one-shot inpainting algorithm based on the topological asymptotic analysis*, Comp. Appl. Math. **25** (2006) 2–3, 1–17.

[8] H. Bellout, A. Friedmann, V. Isakov, *Stability for an inverse problem in potential theory*, Trans. Amer. Math. Soc. **332** (1992), 271–296.

[9] K. Bryan, *Single measurement detection of a discontinuous conductivity*, Comm. Partial Differential Equations **15** (1990), 503–514.

[10] K. Bryan, M. Vogelius, *A uniqueness result concerning the identification of a collection of cracks from finitely many electrostatic boundary measurements*, J. Math. Anal. **23** (1992), 950–958.

[11] A. Friedman, *Detection of mines by electric measurements*, SIAM J. Appl. Math. **47** (1987), 201–212.

[12] A. Friedman, H. Bellout, *Identification problems in potential theory*, Archive Rat. Mech. Anal. **101** (1988), 143–160.

[13] A. Friedmann, B. Gustafson, *Identification of the conductivity coefficient in an elliptic equation*, Siam J. Math. Anal. **18** (1987) 3, 777–787.

[14] A. Friedmann, V. Isakov, *On the uniqueness in the inverse conductivity problem with one measurement*, Indiana Univ. Math. J. **38** (1989), 563–570.

[15] A. Friedman, M. Vogelius, *Determining Cracks by Boundary Measurements*, Indiana Univ. Math. J. **38** (1989), 527–556.

[16] A. Henrot, M. Pierre, *Variation et Optimisation de formes : Une analyse géométrique,* Collection Mathématiques et Applications, vol. 48, Springer-Verlag, 2005.

[17] V. Isakov, *On uniqueness of recovery of a discontinuous conductivity coefficient*, Comm. Pure Appl. Math. **41** (1988), 865–878.

[18] R. Kohn, M. Vogelius, *Identification of an unknown conductivity by means of measurements at the boundary*, [in:] Inverse Problems (Ed. D.W. McLaughlin), SIAM-AMS Proceedings **14** (1984), 113–123.

[19] I. Ly, D. Seck, *Optimisation de forme et problème à fontière libre: cas du p-laplacien*, Ann. Fac. Sci. Toulouse Math. (6) **12** (2003) 1, 103–126.

[20] B. Samet, *L'analyse asymptotique topologique pour les équations de Maxwell et applications*, Thèse de Doctorat, Univ P. Sabatier Toulouse III 2004.

[21] B. Samet, S. Amstutz, M. Masmoudi, *The topological asymptotic for the Helmholtz equation*, SIAM J. Control Optim. **42** (2003) 5, 1523–1544.

Ibrahima Faye
ibrahima.faye@uadb.edu.sn

Université de Bambey
Bp 30 Sénégal

Ecole Doctorale de Mathématiques et Informatique
Laboratoire de Mathématique de la Décision
et d'Analyse Numérique (L.M.D.A.N) F.A.S.E.G
Bp 16 889 Dakar-Fann, Sénégal

M. Ndiaye
mndiak@yahoo.fr

Université Gaston Berger de Saint Louis

Ecole Doctorale de Mathématiques et Informatique
Laboratoire de Mathématique de la Décision
et d'Analyse Numérique (L.M.D.A.N) F.A.S.E.G
Bp 16 889 Dakar-Fann, Sénégal

Idrissa Ly
idrissa.ly@ucad.edu.sn

Ecole Doctorale de Mathématiques et Informatique
Laboratoire de Mathématique de la Décision
et d'Analyse Numérique (L.M.D.A.N) F.A.S.E.G
Bp 16 889 Dakar-Fann, Sénégal

Diaraf Seck
diaraf.seck@ucad.edu.sn

Ecole Doctorale de Mathématiques et Informatique
Laboratoire de Mathématique de la Décision
et d'Analyse Numérique (L.M.D.A.N) F.A.S.E.G
Bp 16 889 Dakar-Fann, Sénégal

*Received: December 29, 2012.*
*Revised: July 29, 2013.*
*Accepted: September 28, 2013.*