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PARABOLIC TURBULENCE k-EPSILON MODEL WITH APPLICATIONS IN FLUID FLOWS THROUGH PERMEABLE MEDIA

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Abstract. In this work, we study a one-equation turbulence k-epsilon model that governs fluid flows through permeable media. The model problem under consideration here is derived from the incompressible Navier–Stokes equations by the application of a time-averaging operator used in the k-epsilon modeling and a volume-averaging operator that is characteristic of modeling unsteady porous media flows. For the associated initial- and boundary-value problem, we prove the existence of suitable weak solutions (average velocity field and turbulent kinetic energy) in the space dimensions of physics interest.

Keywords: turbulence, k-epsilon modelling, permeable media, existence.

Mathematics Subject Classification: 76F60, 76S05, 35Q35, 35K55, 35A01, 76D03.

1. INTRODUCTION

In this work, we study a particular case of porous media flows, which will be called turbulent flows through permeable media. The expression permeable media is used in this paper to lay emphasis on the fact that we are considering porous media where the size of the void space is large enough so that the flow can be considered in turbulent regime.

The study of turbulence in permeable media has gained extensive attention lately, in particular due to its applications in oil and gas extraction, materials science and natural sciences. For instance, in the oil or gas extraction industry the fluid flow is accelerated when approaching a well, becoming turbulent. Therefore an appropriate mathematical characterization of the flow is necessary to reduce the uncertainties in the performance of the useful life of the well. Other example, is solidification and fusion of certain alloys that are characterized by the presence of a fluid domain, a mushy zone and a solid structure. When the flow in the fluid domain is turbulent, the precise prediction of the metal alloy depends on the proper characterization of the turbulence transport process within the mushy zone. Turbulence in permeable media is also being applied in natural sciences for predicting bio-diversity and mitigation of forest fires propagation, where the vegetation is seen as the permeable structure. In this case, an accurate simulation of turbulent air flow that goes through the vegetation is extremely important to study the spreading of seeds and to help fight forest fires. In the same way, transport and dispersion of air pollution through heavily built cities can also benefit from accurate modeling of turbulent flow through permeable media [18, 32, 33].

The most widely used approach to study turbulence in permeable media is based in the k-epsilon modelling. By this approach, macroscopic turbulence models for incompressible single-phase flow in rigid and fully saturated permeable media are derived by using two distinct average concepts. The momentum balance equation is first developed at the microscale, by time-averaging the incompressible Navier–Stokes equations. By this procedure, we obtain the so-called Reynolds-averaged Navier–Stokes (RANS) equations. Then, by volume-averaging (over a representative elementary volume) the RANS equations, we obtain a macroscale equation for the evolution of the turbulence. The differences on the order of the application of these two average concepts may led to distinct equations for the transport of the turbulence [1, 21]. This can be avoided if the total drag effect due to the permeable matrix is modeled only after the two average concepts have been applied, regardless of the order in which they are applied [28]. Motivated by this approach, we consider in this work the following general one-equation turbulence model,

$$\partial_t \boldsymbol{u} + \mathbf{div}\,(\boldsymbol{u}\otimes\boldsymbol{u}) - \mathbf{div}\,(\nu_T(k)\mathbf{D}(\boldsymbol{u})) + \nabla p = \boldsymbol{g} - \boldsymbol{f}(\boldsymbol{u}) \quad \text{in } Q_T, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q_T, \tag{1.2}$$

$$\partial_t k + \boldsymbol{u}\cdot\nabla k - \operatorname{div}(\nu_D(k)\nabla k) = \nu_T(k)|\mathbf{D}(\boldsymbol{u})|^2 + P(\boldsymbol{u},k) - \varepsilon(k) \quad \text{in } Q_T, \tag{1.3}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{and} \quad k = k_0 \quad \text{in } \Omega\times\{0\}, \tag{1.4}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{and} \quad k = 0 \quad \text{on } \Gamma_T, \tag{1.5}$$

set in a cylinder $Q_T := \Omega\times(0,T)$ with lateral boundary $\Gamma_T := \partial\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^d$ is a bounded domain (open and connected) with its boundary denoted by $\partial\Omega$, and T is a given positive constant. The real world problems correspond to consider $d=3$, and in certain particular cases to $d=2$. In (1.1)–(1.5), the velocity field $\boldsymbol{u}$ and the pressure p are, in fact, averages that result by the application of the two aforementioned averaging concepts [18]. The averaged tensor $\mathbf{D}(\boldsymbol{u})$ is the symmetric part of the averaged gradient $\nabla\boldsymbol{u}$. The function k is an unknown of the problem and is usually called turbulent kinetic energy (TKE). By definition, we always have

$$k \geq 0.$$

Turbulent kinetic energy can be produced by fluid shear, friction or buoyancy, or through external forcing at low-frequency eddy scales. Turbulent kinetic energy is then transferred down the turbulence energy cascade, and is dissipated by viscous forces at the Kolmogorov scale [12, 20]. The rate of dissipation of the TKE is described, in the model, by the function ε, which, accordingly, is denoted by dissipation of the TKE, or, briefly, turbulence dissipation. The scalar function ν_T is the turbulent, or eddy, viscosity, that may depend on k, whereas ν_D is the turbulent diffusion that may also depend on k.

In the scope of porous media, all the terms in the momentum equation (1.1) should come multiplied by the porosity of the medium, say ϕ, which is obtained by applying spatial averaging to the characteristic function of the fluid phase, and therefore may depend on the space variable, and ranging in the interval $(0,1)$ [32]. Similarly to what we have assumed that other properties of the fluid are constant, such as viscosity and density, in this work we assume the porosity is constant as well.

Fluid flow through permeable media is governed by various forces, including viscous forces, drag forces, forces due to pressure from surrounding fluid that act on the fluid, and body forces as gravitational force, electromagnetic forces and buoyancy forces. The symbol $\boldsymbol{g}$ on the r.h.s. of the mean flow equation (1.1) stay in this work for a general (averaged) body force. In the same equation, the feedback term $\boldsymbol{f}(\boldsymbol{u})$ accounts for the resistance made by the rigid matrix of the permeable medium to the flow. This resistance is usually characterized by the Darcy law,

$$\boldsymbol{f}(\boldsymbol{u}) = c_{Da}\boldsymbol{u},$$

where c_{Da} is the Darcy coefficient, a positive constant that is experimentally determined. However, as Reynolds number increases, small-scale drag effects, due to the flow through the permeable medium, can be captured by adding extra terms to the Darcy law, giving rise to various non-Darcy models such as the Darcy–Forchheimer,

$$\boldsymbol{f}(\boldsymbol{u}) = c_{Da}\boldsymbol{u} + c_{Fo}\,|\boldsymbol{u}|\boldsymbol{u},$$

where c_{Fo} is the Forchheimer coefficient, a distinct positive constant that is also experimentally determined. We will consider in this work the following generalized Darcy–Forchheimer law,

$$\boldsymbol{f}(\boldsymbol{u}) = c_{Da}\boldsymbol{u} + c_{Fo}|\boldsymbol{u}|^{\alpha-2}\boldsymbol{u}, \quad \alpha > 1, \tag{1.6}$$

where α is a constant that characterizes the flow. The consideration of (1.6) is only for the sake of mathematical generality, despite generalized Darcy–Forchheimer laws appeared in several works on porous media already (see [6] and the references cited therein). In particular, when $\alpha = 2$ or $\alpha = 3$ in (1.6), we recover the Darcy law or the Darcy–Forchheimer law, respectively. The additional term $P(\boldsymbol{u}, k)$ in equation (1.3) appears as an output of the averaging process, and it is a production term of turbulent kinetic energy that accounts for the solids inside the fluid. The influence of this term in the turbulence equations is modeled in this work by

$$P(\boldsymbol{u}, k) = \nu_P(k)|\boldsymbol{u}|^{\beta}, \quad \beta \geq 1, \tag{1.7}$$

where ν_P shall be denoted as the turbulence production. The expression (1.7) comprises many models governing turbulent flows through permeable media (see e.g. [18, 21, 28]). Finally, in standard models,

$$\varepsilon(k) = \frac{k\sqrt{k}}{\ell} \tag{1.8}$$

where $\ell : Q_T \longrightarrow \mathbb{R}$ is the Prandtl length scale (function) of the motion, which is usually assumed to satisfy $\ell \geq \ell_0$ a.e. in Q_T for some positive constant ℓ_0. Therefore, without loss of generality, we can assume that

$$\varepsilon(k) = k\,e(k), \tag{1.9}$$

with

$$e(k) \geq 0 \quad \forall k \in \mathbb{R}_0^+, \quad \text{a.e. in } Q_T. \tag{1.10}$$

In particular,

$$\varepsilon(k)k \geq 0 \quad \forall\, k \in \mathbb{R}_0^+, \quad \text{a.e. in } Q_T. \tag{1.11}$$

Problem (1.1)–(1.5) can be easily adapted to encompass other situations of turbulence modelling not directly related with permeable media. In fact, we can make slight modifications to the problem so it can model turbulent flows in a rotating frame, where the drag forces in the mean flow equation are replaced by the Coriolis acceleration and, in that case, the turbulence production term $P(\boldsymbol{u}, k)$ is zero. Problem (1.1)–(1.5) is also suited to study turbulent flows controlled by a given magnetic field, where the drag forces are now replaced by the Lorentz force, a term where the Navier–Stokes equations are coupled to Maxwell's equations and Ohm's law (see [25] and the references cited therein). In particular, by considering zero drag forces and no production term of turbulence, and assuming the turbulence dissipation $\varepsilon(k)$ is of the order of $k^{\frac{3}{2}}$, we recover the one-equation turbulent k-epsilon model (see e.g. [12, 20]).

The mathematical analysis of the turbulent k-epsilon model has been investigated during the last 20-30 years, although important questions, as the case of real turbulent viscosity and turbulent diffusion functions, remain open. For questions of existence, uniqueness and regularity of the solutions, related to the problem (1.1)–(1.5) with zero drag forces and no turbulence production, we address the reader to the works [10, 15–17, 19, 22]. Looking only at the equations, our model differs from the turbulence models studied in these references, because in the momentum and turbulence equations we have two extra nonlinear terms: the generalized Forchheimer drag and the production of turbulence due to the permeable medium. With respect to the mathematical analysis of the problem (1.1)–(1.5) it has started, to our best knowledge, in the work [23]. In the subsequent works [24–26], it was proved the existence of weak solutions to the steady version of the problem under different conditions on the growth of the drag forces and of the turbulence dissipation, and by considering these terms as strong nonlinearities as well. In [27], it has been established some partial regularity results to the solutions of the steady problem considered in [25]. On the other hand, the effect of the generalized Forchheimer term $|\boldsymbol{u}|^{\alpha-2}\boldsymbol{u}$ on the incompressible Navier–Stokes equations (in the laminar regime) has been studied in [2–5], in particular to obtain the confinement of the solutions, either in space [2–4] or in time [5].

Our problem has some resemblances with the Navier–Stokes–Fourier system governing clear flows (outside a permeable medium), in the laminar regime, of incompressible fluids with temperature-dependent coefficients [9]. Mathematically speaking, the main difficulty of these problems lies in the first r.h.s. term of the turbulence equation (1.3) (or energy equation for the Navier–Stokes–Fourier case), which is only in L^1, making that passing the approximate equation of the weak formulation to the limit does not preserve the identity. To overcome the low regularity of that nonlinear term, the authors in [9] considered the equation that results from adding the scalar product of the momentum equation and the velocity field with the energy equation, obtaining an extra equation for a new quantity that is expressed as the sum of the kinetic energy with the internal energy. However, in this new equation, it is not possible to get rid of the pressure, as we can in the incompressible Navier–Stokes equations. Thus, and as the applicability of de Rham's lemma to Navier–Stokes equations with variable coefficients is still unknown, the authors [9] preferred to work with Navier's slip boundary conditions for the velocity field. This together with the assumption that the boundary is, at least, $C^{1,1}$, lead to the existence of globally integrable pressure. Furthermore, the authors recovered an inequality of the type (2.14) (see below) by making use of the second law of thermodynamics. By these approach the authors [9] were able to prove the long-time and large-data existence of suitable weak solutions. The same reasoning was used in [10] to study a one-equation k-epsilon model governing turbulence in clear flows.

The notion of weak solution satisfying only (2.13)–(2.14), without requiring an extra (opposite in)equality, may be considered very week, because there can easily be many solutions in that conditions. The alternative would be to proceed as in [9, 10], considering Navier's slip boundary conditions, so that we can recover the pressure. But then we would no longer be studying the same problem. This issue has also been extensively studied in previous works of turbulence in clear flows [10, 14, 16, 17, 19, 22], but it has not yet been possible to solve it, in particular in the case of Dirichlet boundary conditions that we consider here. Not only the case of Navier's slip boundary conditions on the velocity field, but also the consideration of unbounded functions of turbulent viscosity, turbulent diffusion and turbulence production will be investigated shortly by the author.

The rest of the paper is organized as follows. In Section 2, we state the weak formulation of the problem (1.1)–(1.5) and we present there the main result of this work: Theorem 2.2. Most part of the subsequent text is dedicated to prove Theorem 2.2, whose proof is carried out from Section 3 till Section 9.

With respect to the notation used in this work, it is quite standard in this field. In any case, we address the interested reader to some of the monographs cited hereinafter [12, 15, 31]. We just want to point out that boldface letters denote tensor-valued (capital) and vector-valued (small) functions and non-boldface letters stay for scalars. The letters C and K will always denote positive constants, whose values may change from line to line, but whose dependence on other parameters or data will always be clear from the exposition.

2. WEAK FORMULATION

In the mathematical treatment of the turbulence problem (1.1)–(1.5), there is a set of usual assumptions that although do not follow from the real situation they are physically admissible,

$$\nu_T,\ \nu_D,\ \nu_P,\ \varepsilon,\ e : Q_T \times \mathbb{R} \to \mathbb{R}_0^+ \quad \text{are Carathéodory functions.} \tag{2.1}$$

On the functions of turbulent viscosity ν_T, turbulent diffusion ν_D and turbulent production ν_P, we assume the existence of couples of positive constants c_T, C_T, c_D,

C_D, and c_P and C_P such that

$$0 < c_T \leq \nu_T(k) \leq C_T \quad \forall k \in \mathbb{R}_0^+, \quad \text{a.e. in } Q_T, \tag{2.2}$$

$$0 < c_D \leq \nu_D(k) \leq C_D \quad \forall k \in \mathbb{R}_0^+, \quad \text{a.e. in } Q_T, \tag{2.3}$$

$$0 < c_P \leq \nu_P(k) \leq C_P \quad \forall k \in \mathbb{R}_0^+, \quad \text{a.e. in } Q_T. \tag{2.4}$$

Motivated by the standard models used in the applications (see (1.8)), we assume on the function of turbulence dissipation that, for certain constant $\vartheta \in \mathbb{R}_0^+$, there exist a couple of positive constants c_ε and C_ε such that

$$c_\varepsilon k^{\vartheta+1} \leq \varepsilon(k) \leq C_\varepsilon k^{\vartheta+1} \quad \forall k \in \mathbb{R}_0^+, \quad \text{a.e. in } Q_T. \tag{2.5}$$

In order to define the notion of a weak solution to the problem (1.1)–(1.5), let us introduce the following function spaces:

$$\begin{aligned} &\mathcal{V} := \{\boldsymbol{v} \in C_0^\infty(\Omega)^d : \operatorname{div}\boldsymbol{v} = 0\}, \\ &\mathbf{H} := \text{closure of } \mathcal{V} \text{ in } L^2(\Omega)^d, \\ &\mathbf{V}^s := \text{closure of } \mathcal{V} \text{ in } W^{s,2}(\Omega)^d, \end{aligned}$$

where s is a positive integer. In the particular case of $s = 1$, we denote $\mathbf{V}^s$ solely by $\mathbf{V}$. Let us also define the scalar function space

$$V^s := \text{closure of } C_0^\infty(\Omega) \text{ in } W^{s,2}(\Omega).$$

Below we define the notion of weak solutions to the problem (1.1)–(1.5) that we are interested in. We assume on the external forces field that

$$\boldsymbol{g} \in L^2(0,T; L^2(\Omega)^d). \tag{2.6}$$

By $\|\boldsymbol{g}\|_{2,L^2(\Omega)^d}$ we shall denote in the sequel the norm of $\boldsymbol{g}$ in $L^2(0,T; L^2(\Omega)^d)$.

Let

$$r_u := \max\left\{\frac{2(d+2)}{d}, \alpha\right\}, \quad \rho_k := \max\left\{\frac{2(d+2)}{d}, \vartheta + 2\right\}, \quad r_k := \max\left\{\frac{d+2}{d}, \vartheta + 1\right\}. \tag{2.7}$$

In order to properly define the weak formulation of our problem, we assume that:

$$\vartheta < \frac{2}{d} \tag{2.8}$$

to make sure that $\varepsilon(k) \in L^q(0,T; L^q(\Omega))$, for some $q > 1$;

$$\beta < r_u \tag{2.9}$$

to make sure that $\nu_P(k)|\boldsymbol{u}|^\beta \in L^q(0,T; L^q(\Omega))$, for some $q > 1$. It should be stressed that, in view of the assumption (2.8), one has $\rho_k = \frac{2(d+2)}{d}$ and $r_k = \frac{d+2}{d}$ in (2.7).

We will use the notations ρ_k and r_k mostly to simplify some expressions, especially in Sections 4, 5 and 7.

On the initial data, we assume that

$$\boldsymbol{u}_0 \in \mathbf{H}, \tag{2.10}$$

$$k_0 \in L^1(\Omega). \tag{2.11}$$

In addition, we assume the existence of a positive constant C_0 such that

$$k_0 \geq C_0 > 0 \quad \text{a.e. in } \Omega. \tag{2.12}$$

Definition 2.1. Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 2$, with its boundary $\partial\Omega$ supposed to be Lipschitz-continuous. We say that a pair $(\boldsymbol{u}, k)$ is a suitable weak solution to the problem (1.1)–(1.5) if:

(1) $\boldsymbol{u} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}) \cap L^{r_u}(0,T;L^{r_u}(\Omega)^d)$, for r_u given in (2.7);
(2) $k \in L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega)) \cap L^{r_k}(0,T;L^{r_k}(\Omega))$, for q satisfying (7.3) below;
(3) $k \geq C_0$ a.e. in Q_T;
(4) For every $\boldsymbol{\psi} \in C^\infty(Q_T)$, with $\operatorname{div}\boldsymbol{\psi} = 0$ in Q_T and $\operatorname{supp}\boldsymbol{\psi} \subset\subset \Omega \times [0,T)$,

$$\begin{aligned} &-\int_0^T\int_\Omega \boldsymbol{u}\cdot\partial\boldsymbol{\psi}\,dxdt - \int_0^T\int_\Omega \boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{\psi}\,dxdt + \int_0^T\int_\Omega \nu_T(k)\,\mathbf{D}(\boldsymbol{u}) : \nabla\boldsymbol{\psi}\,dxdt \\ &+\int_0^T\int_\Omega \left(c_{Da} + c_{Fo}|\boldsymbol{u}|^{\alpha-2}\right)\boldsymbol{u}\cdot\boldsymbol{\psi}\,dxdt = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\psi}(0)\,dx + \int_0^T\int_\Omega \boldsymbol{g}\cdot\boldsymbol{\psi}\,dxdt; \end{aligned} \tag{2.13}$$

(5) For every $\varphi \in C^\infty(Q_T)$, with $\varphi \geq 0$ in Q_T and $\operatorname{supp}\varphi \subset\subset \Omega \times [0,T)$,

$$\begin{aligned} &-\int_0^T\int_\Omega k\partial_t\varphi\,dxdt - \int_0^T\int_\Omega k\boldsymbol{u}\cdot\nabla\varphi\,dxdt \\ &+\int_0^T\int_\Omega \nu_D(k)\nabla k\cdot\nabla\varphi\,dxdt + \int_0^T\int_\Omega \varepsilon(k)\varphi\,dxdt \\ &\geq \int_\Omega k_0\varphi(0)\,dx + \int_0^T\int_\Omega \nu_T(k)|\mathbf{D}(\boldsymbol{u})|^2\varphi\,dxdt + \int_0^T\int_\Omega \nu_P(k)|\boldsymbol{u}|^\beta\varphi\,dxdt; \end{aligned} \tag{2.14}$$

(6) The initial conditions are satisfied in the following sense

$$\lim_{t\to 0^+}\left(\|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_2^2 + \|k(t) - k_0\|_1\right) = 0. \tag{2.15}$$

Observe that in (2.13) the notion of the solution is in the usual weak sense, but in (2.14) the solution is considered in a suitable weak sense once the equality, for the best of author's knowledge, is not known how to be reached. In a way, this resembles the notion of suitable weak solutions introduced in [11].

The main result of this work is the following.

Theorem 2.2. *Assume the hypotheses* (1.6)–(1.7)*,* (2.1)*,* (2.2)–(2.4)*,* (2.5)*,* (2.6)*,* (2.8)–(2.9)*,* (2.10)–(2.11) *and* (2.12) *are fulfilled. If* $2 \leq d \leq 3$*, and* $\alpha = 2$ *or*

$$\frac{8}{3} \leq \alpha < 4, \quad \textit{if } d = 3, \tag{2.16}$$

$$3 - \frac{2}{2+\delta} \leq \alpha < 5 - \frac{6}{2+\delta}, \quad \delta > 0, \quad \textit{if } d = 2, \tag{2.17}$$

then there exists, at least, a weak solution to the problem (1.1)–(1.5) *in the sense of Definition 2.1. Moreover,*

$$\boldsymbol{u} \in C_{\mathrm{w}}([0,T];\mathbf{H}), \tag{2.18}$$

$$\partial_t \boldsymbol{u} \in L^r(0,T;W^{-1,2}(\Omega)^d), \quad \textit{for } r \textit{ satisfying } (8.12), \tag{2.19}$$

$$\partial_t k \in \mathcal{M}(0,T;W^{-1,q'}(\Omega)), \quad \textit{for } q \textit{ satisfying } (8.7). \tag{2.20}$$

Observe that $C_{\mathrm{w}}([0,T];\mathbf{H})$ is the subset of $L^\infty(0,T;\mathbf{H})$ formed by functions that are weakly continuous with values in $\mathbf{H}$, and $\mathcal{M}(0,T;W^{-1,q'}(\Omega))$ denotes the space of Radon measures $\sigma : [0,T] \longrightarrow W^{-1,q'}(\Omega)$. In addition, s' and q' denote the Hölder conjugates of s and q.

It is worth mentioning that the cases $\alpha = 2$ and $\alpha = 3$, the most used in the applications [21, 28] when modelling turbulent flows through this sort of porous media, as we have seen in the introductory section, are covered by Theorem 2.2.

Despite the problem considered in this work is distinct, for the proof of Theorem 1, we have been inspired in the works [9, 19, 22, 25], in particular for obtaining suitable estimates for the approximate turbulent kinetic energy.

Proof of Theorem 2.2. For the proof, we start by considering a general space dimension $d \geq 2$ until we are forced to restrict it to $2 \leq d \leq 3$, which will happen in Section 8 (see (8.10)).

Let us firstly extend θ_0 to the whole $\mathbb{R}^d$ in such a way that, for this extension, say $\overline{k}_0$, $\overline{k}_0 = C_0$ in $\mathbb{R}^d \setminus \Omega$, and where C_0 is the positive constant from assumption (2.12). Next, we regularize $\overline{k}_0$ by considering its mollifying function

$$k_{0,n} := \eta_\delta \star \overline{k}_0, \quad \delta = n^{-1}, \quad n \in \mathbb{N}, \tag{2.21}$$

where η_δ is the Friedrichs mollifying kernel. In view of the assumption (2.12), one has $k_{0,n} \geq C_0 > 0$ a.e. in Ω. In addition, due to (2.11) and (2.21),

$$k_{n,0} \xrightarrow[n\to\infty]{} k_0 \ \text{ in } L^1(\Omega). \tag{2.22}$$

We consider a sequence $\boldsymbol{u}_{n,0} \in \mathbf{H}$ such that

$$\boldsymbol{u}_{n,0} \xrightarrow[n\to\infty]{} \boldsymbol{u}_0 \ \text{ in } L^2(\Omega)^d. \tag{2.23}$$

In order to be able to use the energy equality for the mean flow equation in the final step of the proof of Theorem 2.2 (see (7.14) later), we regularize the velocity field in the convective term. For that, let $\Phi \in C^\infty\big([0,\infty)\big)$ be a non-increasing function such that

$$\Phi(\tau) = \begin{cases} 1 & \text{if } \ 0 \le \tau \le 1, \\ 0 & \text{if } \ \tau \ge 2, \end{cases} \qquad 0 \le \Phi \le 1 \ \text{ in } [0,\infty). \tag{2.24}$$

For $n \in \mathbb{N}$, we set

$$\Phi_n(\tau) = \Phi\left(\frac{\tau}{n}\right), \qquad \tau \in [0,\infty). \tag{2.25}$$

The most technical and substantial part of the proof will consist in proving the following result.

Proposition 2.3. *Assume we are in the conditions of Theorem 2.2. For each $n \in \mathbb{N}$ there exists a couple of functions $\boldsymbol{u}_n$ and k_n satisfying (1)–(3) of Definition 2.1, $\boldsymbol{u}_n = \boldsymbol{u}_{n,0}$ and $k_n = k_{0,n}$, and such that for every $\boldsymbol{v} \in \mathbf{V} \cap L^\alpha(\Omega)^d$ and every $w \in W_0^{1,\infty}(\Omega)$,*

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \boldsymbol{u}_n(t)\cdot\boldsymbol{v}\,dx - \int_\Omega \Phi_n(|\boldsymbol{u}_n|^2)\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t) : \nabla\boldsymbol{v}\,dx \\ &+ \int_\Omega \nu_T(k_n(t))\,\mathbf{D}(\boldsymbol{u}_n(t)) : \nabla\boldsymbol{v}\,dx \\ &+ \int_\Omega \big(c_{Da} + c_{Fo}|\boldsymbol{u}_n(t)|^{\alpha-2}\big)\,\boldsymbol{u}_n(t)\cdot\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{g}(t)\cdot\boldsymbol{v}\,dx \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_\Omega k_n(t)w\,dx - \int_\Omega k_n(t)\boldsymbol{u}_n(t)\cdot\nabla w\,dx + \int_\Omega \nu_D(k_n(t))\nabla k_n(t)\cdot\nabla w\,dx \\ &+ \int_\Omega \varepsilon(k_n(t))w\,dx \\ &= \int_\Omega \nu_T(k_n(t))|\mathbf{D}(\boldsymbol{u}_n(t))|^2 w\,dx + \int_\Omega \nu_P(k_n(t))|\boldsymbol{u}_n(t)|^\beta w\,dx \end{aligned} \tag{2.27}$$

hold for all $t \in (0,T)$.

Proof. For the sake of comprehension, we shall split this proof into several parts in what follows, each of which corresponding to a distinct section. To easy writing, in the course of the proof of Proposition 2.3, we drop the subscript n.

3. EXISTENCE OF GALERKIN APPROXIMATIONS

The proof of the existence of solutions satisfying (2.26)–(2.27) will be carried out by using a two-level Galerkin approximation method.

By means of separability, we may consider orthogonal bases $\{\boldsymbol{v}_i\}_{i\in\mathbb{N}}$ of $\mathbf{V}^s$, for $s > 1 + \frac{d}{2}$, and $\{w_i\}_{i\in\mathbb{N}}$ of $W^{s,2}(\Omega)$, in this case for $s > \frac{d}{2}$, that are orthonormal in $L^2(\Omega)^d$ and in $L^2(\Omega)$, respectively. Given $j,\ l \in \mathbb{N}$, let us consider the j-dimensional space $\mathbf{X}_j := \operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_j\}$ and the l-dimensional space $X_l := \operatorname{span}\{w_1, \dots, w_l\}$.

For each $j \in \mathbb{N}$ and each $l \in \mathbb{N}$, we search for approximate solutions

$$\begin{aligned} \boldsymbol{u}^{j,l}(x,t) &= \sum_{i=1}^{j} a_i^{j,l}(t)\boldsymbol{v}_i(x), \quad \boldsymbol{v}_i \in \mathbf{X}_j, \\ k^{j,l}(x,t) &= \sum_{i=1}^{l} c_i^{j,l}(t)w_i(x), \quad w_i \in X_l, \end{aligned} \tag{3.1}$$

where the coefficients $a_1^{j,l}(t), \dots, a_j^{j,l}(t)$ and $c_1^{j,l}(t), \dots, c_l^{j,l}(t)$ solve the following system of $j + l$ ordinary differential equations

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \boldsymbol{u}^{j,l}(t)\cdot\boldsymbol{v}_i\,dx - \int_\Omega \Phi_n(|\boldsymbol{u}^{j,l}(t)|^2)\boldsymbol{u}^{j,l}(t)\otimes\boldsymbol{u}^{j,l}(t) : \nabla\boldsymbol{v}_i\,dx \\ &+ \int_\Omega \nu_T(k^{j,l}(t))\,\mathbf{D}(\boldsymbol{u}^{j,l}(t)) : \mathbf{D}(\boldsymbol{v}_i)\,dx + \int_\Omega \left(c_{Da} + c_{Fo}|\boldsymbol{u}^{j,l}(t)|^{\alpha-2}\right)\boldsymbol{u}^{j,l}(t)\cdot\boldsymbol{v}_i\,dx \\ &= \int_\Omega \boldsymbol{g}(t)\cdot\boldsymbol{v}_i\,dx, \quad i = 1, \dots, j, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\frac{d}{dt}\int_\Omega k^{j,l}(t)w_i\,dx - \int_\Omega k^{j,l}(t)\boldsymbol{u}^{j,l}(t)\cdot\nabla w_i\,dx + \int_\Omega \nu_D(k^{j,l}(t))\nabla k^{j,l}(t)\cdot\nabla w_i\,dx \\ &+ \int_\Omega \varepsilon(k^{j,l}(t))w_i\,dx \\ &= \int_\Omega \nu_T(k^{j,l}(t))|\mathbf{D}(\boldsymbol{u}^{j,l})|^2 w_i\,dx + \int_\Omega \nu_P(k^{j,l}(t))|\boldsymbol{u}^{j,l}(t)|^\beta w_i\,dx, \quad i = 1, \dots, l. \end{aligned} \tag{3.3}$$

System (3.2)–(3.3) is supplemented with the following initial conditions

$$\boldsymbol{u}^{j,l}(0) = \boldsymbol{u}_0^{j,l} \quad \text{and} \quad k^{j,l}(0) = k_0^{j,l} \quad \text{in } \Omega, \tag{3.4}$$

where $\boldsymbol{u}_0^{j,l}$ and $k_0^{j,l}$ are the orthogonal projections of $\boldsymbol{u}_0$ and k_0^j onto $\mathbf{X}_j$ and X^l, respectively. Whence

$$\boldsymbol{u}_0^{j,l} = \sum_{i=1}^{j} a_{0,i}^{j,l}\boldsymbol{v}_i, \quad \boldsymbol{v}_i \in \mathbf{X}_j, \quad k_0^{j,l} = \sum_{i=1}^{l} c_{0,i}^{l} w_i, \quad w_i \in X_l,$$

for some $\boldsymbol{a}_0 = (a_{0,1}^{j,l}, \ldots, a_{0,j}^{j,l}) \in \mathbb{R}^j$ and $\boldsymbol{c}_0 = (c_{0,1}^{j,l}, \ldots, c_{0,l}^{j,l}) \in \mathbb{R}^l$. We can assume that

$$\boldsymbol{u}_0^{j,l} \underset{l\to\infty}{\longrightarrow} \boldsymbol{u}_0^j \ \text{ in } \ L^2(\Omega)^d, \quad \boldsymbol{u}_0^j \underset{j\to\infty}{\longrightarrow} \boldsymbol{u}_0 \ \text{ in } \ L^2(\Omega)^d, \tag{3.5}$$

$$k_0^{j,l} \underset{l\to\infty}{\longrightarrow} k_0^j \ \text{ in } \ L^2(\Omega), \quad k_0^j \underset{j\to\infty}{\longrightarrow} k_0 \ \text{ in } \ L^1(\Omega). \tag{3.6}$$

By the application of the Carathéodory theorem (see e.g. [13, Chapter 2]), there are a time $T^* \in (0,T)$ and absolute continuous functions $\boldsymbol{a} : (0,T^*) \longrightarrow \mathbb{R}^j$ and $\boldsymbol{c} : (0,T^*) \longrightarrow \mathbb{R}^l$, with $\boldsymbol{a}(t) = (a_1^{j,l}(t), \ldots, a_j^{j,l}(t))$ and $\boldsymbol{c}(t) = (c_1^{j,l}(t), \ldots, c_l^{j,l}(t))$, such that $(\boldsymbol{a}, \boldsymbol{c})$ solve the initial-value problem (3.2)–(3.4).

4. ESTIMATES INDEPENDENT OF l

In all the estimates that will be established in this subsection, we will emphasize the dependence (or not) of the r.h.s. positive constants in terms of j, regardless of whether they may depend on other data and coefficients of the problem. More importantly, these constants must be independent of l.

We start by multiplying (3.2) by $a_i^{j,l}(t)$ and then we add the resulting equations from $i = 1$ up to $i = j$, so that one has for all $t \in (0, T^*)$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^{j,l}(t)\|_2^2 + \int_\Omega \nu_T(k^{j,l}(t))|\mathbf{D}(\boldsymbol{u}^{j,l}(t))|^2\,dx + \int_\Omega \left(c_{Da} + c_{Fo}|\boldsymbol{u}^{j,l}(t)|^{\alpha-2}\right)|\boldsymbol{u}^{j,l}(t)|^2\,dx \\ &= \int_\Omega \boldsymbol{g}(t)\cdot\boldsymbol{u}^{j,l}(t)\,dx.\end{aligned} \tag{4.1}$$

Observe that for the regularized convective term, one has

$$\begin{aligned}&\int_\Omega \Phi_n(|\boldsymbol{u}^{j,l}(t)|^2)\boldsymbol{u}^{j,l}(t)\otimes\boldsymbol{u}^{j,l}(t) : \nabla\boldsymbol{u}^{j,l}(t)\,dx \\ &= \int_\Omega \Phi_n(|\boldsymbol{u}^{j,l}(t)|^2)\boldsymbol{u}^{j,l}(t)\cdot\nabla\left(|\boldsymbol{u}^{j,l}(t)|^2\right)dx \\ &= \int_\Omega \nabla\left(\Psi_n(|\boldsymbol{u}^{j,l}(t)|^2)\right)\cdot\boldsymbol{u}^{j,l}(t)\,dx = 0, \\ &\Psi_n(\tau) := \int_0^\tau \Phi_n(s)\,ds,\end{aligned} \tag{4.2}$$

Next, we integrate (4.1) between 0 and $t \in (0, T^*)$, and we use the Korn, Cauchy–Schwarz and Sobolev inequalities, together with the assumption (2.2). Then, we use $(3.4)_1$ and $(3.5)_1$ in the term $\|\boldsymbol{u}^{j,l}(0)\|_{L^2(\Omega)^d}$ that results by the previous integration. After all, one has for all $t \in (0, T^*)$

$$\|\boldsymbol{u}^{j,l}(t)\|_2^2 + \int_0^t \int_\Omega \nu_T(k^{j,l})|\mathbf{D}(\boldsymbol{u}^{j,l})|^2 \, dxds + c_{Fo} \int_0^t \|\boldsymbol{u}^{j,l}(s)\|_\alpha^\alpha ds \le C, \tag{4.3}$$

for some positive constant $C = C(c_T, \|\boldsymbol{g}\|_{2,L^2(\Omega)^d}, \|\boldsymbol{u}_0\|_2)$.

We now multiply (3.3) by $c_i^{j,l}(t)$ and add up the resulting equations from $i = 1$ till $i = l$. We obtain for all $t \in (0, T^*)$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|k^{j,l}(t)\|_2^2 - \frac{1}{2}\int_\Omega \boldsymbol{u}^{j,l}(t) \cdot \nabla \left(k^{j,l}(t)\right)^2 dx + \int_\Omega \nu_D(k^{j,l}(t))|\nabla k^{j,l}(t)|^2 dx \\ &+ \int_\Omega \varepsilon(k^{j,l}(t))k^{j,l}(t)\, dx \\ &= \int_\Omega \nu_T(k^{j,l}(t))|\mathbf{D}(\boldsymbol{u}^{j,l}(t))|^2 k^{j,l}(t)\, dx + \int_\Omega \nu_P(k^{j,l}(t))|\boldsymbol{u}^{j,l}(t)|^\beta k^{j,l}(t)\, dx. \end{aligned} \tag{4.4}$$

The second term of (4.4) vanishes after integration by parts and using the facts that $\boldsymbol{u}^{j,l} = \mathbf{0}$ on $\partial\Omega$ and $\operatorname{div} \boldsymbol{u}^{j,l} = 0$ in Q_T. Integrating the resulting equality between 0 and $t \in (0, T^*)$ and using (2.2)–(2.4), (2.5), $(3.4)_2$ and $(3.6)_1$, one has for all $t \in (0, T^*)$

$$\begin{aligned} &\frac{1}{2}\|k^{j,l}(t)\|_2^2 + c_\varepsilon \int_0^t \|k^{j,l}(s)\|_{\vartheta+2}^{\vartheta+2} ds + c_D \int_0^t \|\nabla k^{j,l}(s)\|_2^2 ds \\ &\le \frac{1}{2}\|k_0\|_2^2 + C_T \int_0^t \int_\Omega |\mathbf{D}(\boldsymbol{u}^{j,l})|^2 k^{j,l}\, dxds + C_P \int_0^t \int_\Omega |\boldsymbol{u}^{j,l}|^\beta |k^{j,l}|\, dxds. \end{aligned} \tag{4.5}$$

Note that, by (1.11), the fourth l.h.s. term of (4.4) is nonnegative. Combining (2.2) with Korn and Hölder inequalities, and using (3.1) and (4.3), we get

$$\int_0^t \int_\Omega |\mathbf{D}(\boldsymbol{u}^{j,l})|^2 k^{j,l}\, dxds \le C(j) \sup_{i \in \{1,\ldots,j\}} \|\nabla \boldsymbol{v}_i\|_4^2 \left(\int_0^t \|k^{j,l}(s)\|_2^2 ds \right)^{\frac{1}{2}},$$

where $C = C(j, C_T, C_K, M, T^*)$ is a positive constant. Analogously, by combining (3.1) and (4.3) with the assumption (2.4), and with the Young and Sobolev inequalities, we also get

$$\int_0^t \int_\Omega |\boldsymbol{u}^{j,l}|^\beta |k^{j,l}|\, dxds \le \frac{c_\varepsilon}{2} \int_0^t \|k^{j,l}(s)\|_{\vartheta+2}^{\vartheta+2} ds + C(j) \sup_{i \in \{1,\ldots,j\}} \|\boldsymbol{v}_i\|_{\beta\frac{\vartheta+2}{\vartheta+1}}^\beta,$$

where $C(j) = C(j, c_\varepsilon, \beta, \vartheta, M, T^*)$ is another positive constant. Note that, by the choice of the basis $\{\boldsymbol{v}_i\}_{i\in\mathbb{N}}$, $\|\nabla \boldsymbol{v}_i\|_4$ and $\|\boldsymbol{v}_i\|_{\beta\frac{\vartheta+2}{\vartheta+1}}$ are bounded for all $i \in \{1, \dots, j\}$.

Then, plugging the last two inequalities into (4.5), and using the Cauchy inequality, we obtain

$$\|k^{j,l}(t)\|_2^2 + \frac{c_\varepsilon}{2}\int_0^t \|k^{j,l}(s)\|_{\vartheta+2}^{\vartheta+2} ds + c_D \int_0^t \|\nabla k^{j,l}(s)\|_2^2 ds \le C(j) + \int_0^t \|k^{j,l}(s)\|_2^2 ds$$

for some positive constant $C(j) = C(j, \beta, \vartheta, c_\varepsilon, C_T, C_P, C_K, \|k_0\|_2, T^*)$. By the Grönwall inequality, we get

$$\|k^{j,l}(t)\|_2^2 + c_\varepsilon \int_0^t \|k^{j,l}(s)\|_{\vartheta+2}^{\vartheta+2} ds + c_D \int_0^t \|\nabla k^{j,l}(s)\|_2^2 ds \le C(j). \tag{4.6}$$

The uniform estimates (4.3) and (4.6) enable us to use the Continuation Principle (see e.g. [13, Section 1.4]) to extend the solutions $(\boldsymbol{a}, \boldsymbol{c})$ of the initial-value problem (3.2)–(3.4) to the whole interval $(0, T)$.

From (4.3), one can prove the following a priori estimates independent of l hold,

$$\sup_{t\in[0,T]} \|\boldsymbol{u}^{j,l}(t)\|_2^2 + c_T \int_0^T \|\nabla \boldsymbol{u}^{j,l}(t)\|_2^2 dt + c_{Fo} \int_0^T \|\boldsymbol{u}^{j,l}(t)\|_\alpha^\alpha dt \le \Theta_1, \tag{4.7}$$

$$\int_0^T \|\boldsymbol{u}^{j,l}(t)\|_{r_u}^{r_u} dt \le \Theta_2, \tag{4.8}$$

for r_u given in (2.7), and where $\Theta_1,\ \Theta_2 = C(c_T, \|\boldsymbol{g}\|_{2,L^2(\Omega)^d}, \|\boldsymbol{u}_0\|_2)$ are distinct positive constants. On the other hand, from (4.6), we can prove the following a priori estimates independent of l also hold,

$$\sup_{t\in[0,T]} \|k^{j,l}(t)\|_2^2 + c_\varepsilon \int_0^T \|k^{j,l}(t)\|_{\vartheta+2}^{\vartheta+2} dt + c_D \int_0^T \|\nabla k^{j,l}(t)\|_2^2 dt \le \Xi_1(j), \tag{4.9}$$

$$\int_0^T \|k^{j,l}(t)\|_{\rho_k}^{\rho_k} dt \le \Xi_2(j), \tag{4.10}$$

where ρ_k is given in (2.7), and $\Xi_1,\ \Xi_2 = C(j, \beta, \vartheta, c_\varepsilon, C_T, C_P, C_K, \|k_0\|_2, T)$ are distinct positive constants.

In what follows we obtain estimates for the time derivatives of the Galerkin approximations $\boldsymbol{u}^{j,l}$ and $k^{j,l}$. We start by multiplying (3.2) by $\frac{da_i^{j,l}}{dt}$ and then, integrating in $(0,T)$ and adding up from $i=1$ till $i=j$ the resulting equation, we obtain

$$\begin{aligned}\frac{1}{2}\int_0^T \|\partial_t \boldsymbol{u}^{j,l}(t)\|_2^2 dt = & \int_0^T\int_\Omega \boldsymbol{g}\cdot\partial_t\boldsymbol{u}^{j,l}\,dxdt \\ & + \int_0^T\int_\Omega \Phi_n(|\boldsymbol{u}^{j,l}(t)|^2)\boldsymbol{u}^{j,l}\otimes\boldsymbol{u}^{j,l} : \nabla\partial_t\boldsymbol{u}^{j,l}\,dxdt \\ & - \int_0^T\int_\Omega \nu_T(k^{j,l})\,\mathbf{D}(\boldsymbol{u}^{j,l}) : \mathbf{D}(\partial_t\boldsymbol{u}^{j,l}(t))\,dxdt \\ & - \int_0^T\int_\Omega \left(c_{Da}+c_{Fo}|\boldsymbol{u}^{j,l}|^{\alpha-2}\right)\boldsymbol{u}^{j,l}\cdot\partial_t\boldsymbol{u}^{j,l}\,dxdt.\end{aligned}$$

By using the Hölder and Cauchy inequalities, together with assumptions (2.2) and (2.6), and with (2.24) and (4.8), one has

$$\begin{aligned}\int_0^T \|\partial_t\boldsymbol{u}^{j,l}(t)\|_2^2 dt \leq C\Bigg(1 & + \int_0^T\int_\Omega |\boldsymbol{u}^{j,l}|\,|\nabla\boldsymbol{u}^{j,l}|\,|\partial_t\boldsymbol{u}^{j,l}|\,dxdt \\ + \int_0^T\int_\Omega |\mathbf{D}(\boldsymbol{u}^{j,l})|\,|\nabla(\partial_t\boldsymbol{u}^{j,l})|\,dxdt & + \int_0^T\int_\Omega |\boldsymbol{u}^{j,l}|^{\alpha-1}|\partial_t\boldsymbol{u}^{j,l}|\,dxdt\Bigg),\end{aligned} \tag{4.11}$$

for some positive constant $C = C\left(\|\boldsymbol{g}\|_{L^2(0,T;L^2(\Omega)^d)}, c_{Da}, c_{Fo}, C_T, c_T\right)$. Using the fact that $\mathbf{X}_j$ is a finite-dimensional space and $\mathbf{X}_j\subset\mathbf{V}^s$, we can use the Hölder and Korn inequalities, together with (4.7)–(4.8), to show that

$$\int_0^T\int_\Omega |\boldsymbol{u}^{j,l}|\,|\nabla\boldsymbol{u}^{j,l}|\,|\partial_t\boldsymbol{u}^{j,l}|\,dxdt \leq C_1(j)\left(\int_0^T \|\partial_t\boldsymbol{u}^{j,l}(t)\|_2^2 dt\right)^{\frac{1}{2}},$$

$$\int_0^T\int_\Omega |\mathbf{D}(\boldsymbol{u}^{j,l})|\,|\nabla(\partial_t\boldsymbol{u}^{j,l})|\,dxdt \leq C_2(j)\left(\int_0^T \|\partial_t\boldsymbol{u}^{j,l}(t)\|_2^2 dt\right)^{\frac{1}{2}},$$

$$\int_0^T\int_\Omega |\boldsymbol{u}^{j,l}|^{\alpha-1}|\partial_t\boldsymbol{u}^{j,l}|\,dxdt \leq C_3(j)\left(\int_0^T \|\partial_t\boldsymbol{u}^{j,l}(t)\|_2^2 dt\right)^{\frac{1}{2}},$$

where the positive constants C_1, C_2 and C_3 depend also on T, with C_3 depending, in addition, on α.

Plugging all this information in (4.11), we obtain

$$\int_0^T \|\partial_t \boldsymbol{u}^{j,l}(t)\|_2^2 dt \leq C(j), \tag{4.12}$$

for some positive constant $C = C(j)$. Next, using the fact that the elements of the base $\mathbf{X}_j$ are orthonormal in $L^2(\Omega)^d$, we get from (3.1) and (4.12)

$$\int_0^T \left|\frac{d\boldsymbol{a}^{j,l}(t)}{dt}\right|_2^2 dt \leq C(j), \tag{4.13}$$

for the same positive constant.

On the other hand, from (3.3), we can infer that for all $t \in (0,T)$ one has

$$\begin{aligned}\partial_t k^{j,l}(t) = &- \operatorname{div}\left(k^{j,l}(t)\boldsymbol{u}^{j,l}(t)\right) + \operatorname{div}\left(\nu_D(k^{j,l}(t))\nabla k^{j,l}(t)\right)\\ &+ \nu_T(k^{j,l}(t))|\mathbf{D}(\boldsymbol{u}^{j,l})|^2 + \nu_P(k^{j,l}(t))|\boldsymbol{u}^{j,l}(t)|^\beta - \varepsilon(k^{j,l}(t)),\end{aligned} \tag{4.14}$$

in the distribution sense on X_l', where X_l' denotes the dual space of X_l. Consider $\phi \in W_0^{s,2}(\Omega)$ and let P^l be the orthogonal projection from $W_0^{s,2}(\Omega)$ onto X_l, with respect to the $W_0^{s,2}(\Omega)$ inner product. Using successively (4.14), assumptions (2.2)–(2.4), (2.5) and (2.8)–(2.9), Hölder and Sobolev inequalities, we have for $d \neq 2$ (for $d = 2$ is easier)

$$\begin{aligned}\left|\int_\Omega \partial_t k^{j,l}(t)\phi\, dx\right| \leq C\Big[&\|k^{j,l}(t)\|_2\|\nabla \boldsymbol{u}^{j,l}(t)\|_2\|P^l(\phi)\|_d\\ &+ \left(\|\nabla k^{j,l}(t)\|_2 + \|\mathbf{D}(\boldsymbol{u}^{j,l}(t))\|_2\right)\|P^l(\phi)\|_2\\ &+ \|\boldsymbol{u}^{j,l}(t)\|_{r_u}^\beta \|P^l(\phi)\|_{\left(\frac{r_u}{\beta}\right)'} + \|k^{j,l}(t)\|_{\rho_k}^{\vartheta+1}\|P^l(\phi)\|_{\left(\frac{\rho_k}{\vartheta+1}\right)'}\Big],\end{aligned}$$

for some positive constant $C = C(d, C_T, C_D, C_P, C_\xi)$. Observing that

$$\left\langle \partial_t k^{j,l}(t), P^l(\phi)\right\rangle = \left\langle \partial_t k^{j,l}(t), \phi\right\rangle \quad \forall \phi \in W_0^{1,s}(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product over $W^{-s,2}(\Omega)\times W_0^{s,2}(\Omega)$, and that

$$\|P^l(\phi)\|_d,\ \|P^l(\phi)\|_2,\ \|P^l(\phi)\|_{\left(\frac{r_u}{\beta}\right)'},\ \|P^l(\phi)\|_{\left(\frac{\rho_k}{\vartheta+1}\right)'} \leq C\|P^l(\phi)\|_{W_0^{s,2}(\Omega)} \leq C\|\phi\|_{W_0^{s,2}(\Omega)},$$

we can use the above inequality, together with (4.7)–(4.8) and (4.9)–(4.10), to show that

$$\int_0^T \left\|\partial_t k^{j,l}(t)\right\|_{W^{-1,s}(\Omega)}^\rho dt \leq C(j), \quad \rho := \min\left\{\frac{r_u}{\beta}, \frac{\rho_k}{\vartheta+1}\right\}, \tag{4.15}$$

for some positive constant $C = C(j, C_D, C_T, C_P, C_\varepsilon, \Theta_1, \Theta_2, \Xi_1)$, and for r_u and ρ_k defined at (2.7).

5. PASSING TO THE LIMIT AS $l \to \infty$

In this part and for the sake of simplifying the notations, we still relabel all the considered subsequences to the same indices of the original ones.

From the estimates (4.7), (4.8) and (4.13), we can use the Banach-Alaoglu theorem to extract some subsequences such that

$$\boldsymbol{u}^{j,l} \xrightarrow[l\to\infty]{*} \boldsymbol{u}^j \quad \text{in } L^\infty(0,T;\mathbf{H}),$$

$$\boldsymbol{u}^{j,l} \xrightarrow[l\to\infty]{} \boldsymbol{u}^j \quad \text{in } L^2(0,T;\mathbf{V}) \cap L^{r_u}(0,T;L^{r_u}(\Omega)^d), \tag{5.1}$$

$$\boldsymbol{a}^{j,l} \xrightarrow[l\to\infty]{} \boldsymbol{a}^j \quad \text{in } W^{1,2}(0,T), \tag{5.2}$$

where r_u is given in (2.7). Since $W^{1,2}(0,T) \hookrightarrow\hookrightarrow C[0,T])$, (5.2) allows us to extract another subsequence such that

$$\boldsymbol{a}^{j,l} \xrightarrow[l\to\infty]{} \boldsymbol{a}^j \quad \text{in } C[0,T],$$

which in turn, together with (3.1), implies

$$\nabla \boldsymbol{u}^{j,l} \xrightarrow[l\to\infty]{\text{uniformly}} \nabla \boldsymbol{u}^j \quad \text{in } Q_T. \tag{5.3}$$

$$\tag{5.4}$$

On the other hand, from (4.9), (4.10) and (4.15) , we have

$$k^{j,l} \xrightarrow[l\to\infty]{*} k^j \qquad \text{in } L^\infty(0,T;L^2(\Omega)),$$

$$k^{j,l} \xrightarrow[l\to\infty]{} k^j \qquad \text{in } L^2(0,T;H_0^1(\Omega)) \cap L^{\rho_k}(0,T;L^{\rho_k}(\Omega)), \tag{5.5}$$

$$\partial_t k^{j,l} \xrightarrow[l\to\infty]{} \partial_t k^j \quad \text{in } L^\rho(0,T;W^{-s,2}(\Omega)), \tag{5.6}$$

where ρ_k is given in (2.7) and ρ in (4.15). Using the estimates (4.7)–(4.8) and (4.12), by one hand, and (4.9)–(4.10) and (4.15), on the other, together with the Aubin–Lions–Simon compactness lemma (see [30, Corollary 6]), we also have

$$\boldsymbol{u}^{j,l} \xrightarrow[l\to\infty]{} \boldsymbol{u}^j \quad \text{in } L^q(0,T;L^q(\Omega)^d) \quad \forall q:\ 1 \le q < r_u, \tag{5.7}$$

$$k^{j,l} \xrightarrow[l\to\infty]{} k^j \quad \text{in } L^q(0,T;L^q(\Omega)) \quad \forall q:\ 1 \le q < \rho_k. \tag{5.8}$$

Thus, in view of (5.7)–(5.8), by the Riesz–Fischer theorem there exist another subsequences such that

$$\boldsymbol{u}^{j,l} \xrightarrow[l\to\infty]{} \boldsymbol{u}^j \quad \text{a.e. in } Q_T, \tag{5.9}$$

$$k^{j,l} \xrightarrow[l\to\infty]{} k^j \quad \text{a.e. in } Q_T. \tag{5.10}$$

All the terms in the approximate mean flow equation (3.2), with the exception of the ones involving the Darcy and Forchheimer drag forces and the turbulent viscosity, can be proven to converge, as in the classical Navier–Stokes equations in a general space dimension. With respect to the drag forces, just the Forchheimer term needs some justification. By (5.9), one has

$$|\boldsymbol{u}^{j,l}|^{\alpha-2}\boldsymbol{u}^{j,l} \underset{l\to\infty}{\longrightarrow} |\boldsymbol{u}^{j}|^{\alpha-2}\boldsymbol{u}^{j} \quad \text{a.e. in } Q_T. \tag{5.11}$$

On the other hand, due to (4.7), we have

$$\int_0^T \| \, |\boldsymbol{u}^{j,l}(t)|^{\alpha-2}\boldsymbol{u}^{j,l}(t)\|_{\alpha'}^{\alpha'} dt \leq \Theta_4 \tag{5.12}$$

for some positive constant $\Theta_4 = C(c_{Fo}, \Theta_1)$. As a consequence of (5.11) and (5.12) there holds

$$|\boldsymbol{u}^{j,l}|^{\alpha-2}\boldsymbol{u}^{j,l} \underset{l\to\infty}{\rightharpoonup} |\boldsymbol{u}^{j}|^{\alpha-2}\boldsymbol{u}^{j} \ \text{ in } \ L^{\alpha'}(0,T;L^{\alpha'}(\Omega)^d). \tag{5.13}$$

For the turbulent viscosity term, we first observe that, by (2.1), (5.3) and (5.10), we have

$$\nu_T(k^{j,l})\mathbf{D}(\boldsymbol{u}^{j,l}) \underset{l\to\infty}{\longrightarrow} \nu_T(k^{j})\mathbf{D}(\boldsymbol{u}^{j}) \quad \text{a.e. in } Q_T. \tag{5.14}$$

Moreover, from assumption (2.2) and (4.7), one has

$$\int_0^T \|\nu_T(k^{j,l})\mathbf{D}(\boldsymbol{u}^{j,l}(t))\|_2^2 dt \leq C,$$

for some independent of j positive constant C. As a consequence,

$$\nu_T(k^{j,l})\mathbf{D}(\boldsymbol{u}^{j,l}) \underset{l\to\infty}{\rightharpoonup} \nu_T(k^{j})\mathbf{D}(\boldsymbol{u}^{j}) \ \text{ in } \ L^2(0,T;L^2(\Omega)^{d\times d}). \tag{5.15}$$

With respect to the regularized convective term, we can use (2.25) and (5.9) so that

$$\Phi_n(|\boldsymbol{u}^{j,l}|^2)\boldsymbol{u}^{j,l}\otimes\boldsymbol{u}^{j,l} \underset{l\to\infty}{\longrightarrow} \Phi_n(|\boldsymbol{u}^{j}|^2)\boldsymbol{u}^{j}\otimes\boldsymbol{u}^{j} \quad \text{a.e. in } Q_T.$$

And from (2.24) and (4.8), we can show that

$$\int_0^T \left\|\Phi_n(|\boldsymbol{u}^{j,l}(t)|^2)\boldsymbol{u}^{j,l}\otimes\boldsymbol{u}^{j,l}\right\|_{\frac{r_u}{2}}^{\frac{r_u}{2}} dt \leq C,$$

for some independent of j positive constant C, and for r_u given at (2.7). This yields

$$\Phi_n(|\boldsymbol{u}^{j,l}|^2)\boldsymbol{u}^{j,l}\otimes\boldsymbol{u}^{j,l} \underset{l\to\infty}{\rightharpoonup} \Phi_n(|\boldsymbol{u}^{j}|^2)\boldsymbol{u}^{j}\otimes\boldsymbol{u}^{j} \ \text{ in } \ L^{\frac{r_u}{2}}(0,T;L^{\frac{r_u}{2}}(\Omega)^{d\times d}). \tag{5.16}$$

Using the convergence results (5.1), (5.2), (5.13), (5.15) and (5.16), we can pass to the limit $l \to \infty$ in the approximate weak formulation (3.2) to obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \boldsymbol{u}^j(t)\cdot\boldsymbol{v}_i\,dx - \int_\Omega \Phi_n(|\boldsymbol{u}^j(t)|^2)\boldsymbol{u}^j(t)\otimes\boldsymbol{u}^j(t):\nabla\boldsymbol{v}_i\,dx \\ &+\int_\Omega \nu_T(k^j(t))\,\mathbf{D}(\boldsymbol{u}^j(t)):\mathbf{D}(\boldsymbol{v}_i)\,dx \\ &+\int_\Omega \left(c_{Da}+c_{Fo}|\boldsymbol{u}^j(t)|^{\alpha-2}\right)\boldsymbol{u}^j(t)\cdot\boldsymbol{v}_i\,dx = \int_\Omega \boldsymbol{g}(t)\cdot\boldsymbol{v}_i\,dx \quad \forall i\in\{1,\dots,j\}. \end{aligned} \tag{5.17}$$

Regarding the approximate TKE equation (3.3), let us just comment on the terms of turbulent diffusion, dissipation, viscosity and production. Arguing as we did for (5.15), but now using (2.3), (4.9) and (5.10), we can prove that

$$\nu_D(k^{j,l})\nabla k^{j,l} \underset{l\to\infty}{\rightharpoonup} \nu_D(k^j)\nabla k^j \quad \text{in} \quad L^2(0,T;L^2(\Omega)^d). \tag{5.18}$$

Due to (2.1) and (5.10), there holds

$$\varepsilon(k^{j,l}) \underset{l\to\infty}{\longrightarrow} \varepsilon(k^j) \quad \text{a.e. in} \quad Q_T, \tag{5.19}$$

and by using assumption (2.5), together with (4.9), we can show that

$$\int_0^T \|\varepsilon(k^{j,l}(t))\|_{\frac{\vartheta+2}{\vartheta+1}}^{\frac{\vartheta+2}{\vartheta+1}}dt \leq \Xi_4(j) \tag{5.20}$$

for some positive constant $\Xi_4 = C(j,\vartheta,C_\varepsilon,\Xi_2)$. Hence, (5.19) and (5.20) assure that

$$\varepsilon(k^{j,l}) \underset{l\to\infty}{\rightharpoonup} \varepsilon(k^j) \quad \text{in} \quad L^{\frac{\vartheta+2}{\vartheta+1}}(0,T;L^{\frac{\vartheta+2}{\vartheta+1}}(\Omega)). \tag{5.21}$$

Arguing as we did for (5.14), we also have

$$\nu_T(k^{j,l})|\mathbf{D}(\boldsymbol{u}^{j,l})|^2 \underset{l\to\infty}{\longrightarrow} \nu_T(k^j)\left|\mathbf{D}(\boldsymbol{u}^j)\right|^2 \quad \text{a.e. in} \quad Q_T.$$

From (4.3), we can show that

$$\limsup_{l\to\infty}\int_{Q_T}\nu_T(k^{j,l})|\mathbf{D}(\boldsymbol{u}^{j,l})|^2\,dxdt \leq C,$$

for some independent of l (and j) positive constant C. Therefore, in view of the Vitali–Hahn–Saks theorem,

$$\nu_T(k^j)|\mathbf{D}(\boldsymbol{u}^{j,l})|^2 \underset{l\to\infty}{\longrightarrow} \nu_T(k^j)|\mathbf{D}(\boldsymbol{u}^j)|^2 \quad \text{in} \quad L^1(0,T;L^1(\Omega)). \tag{5.22}$$

On the other hand, from (2.1) and (5.9)–(5.10), one has

$$\nu_P(k^{j,l})|\boldsymbol{u}^{j,l}|^\beta \xrightarrow[l\to\infty]{} \nu_P(k^j)|\boldsymbol{u}^j|^\beta \quad \text{a.e. in } \; Q_T. \tag{5.23}$$

Using (2.4) and (4.8), we can show that

$$\int_{Q_T} |\nu_P(k^{j,l})|\boldsymbol{u}^{j,l}|^\beta|^q dxdt \leq C \quad \forall q : 1 < q \leq \frac{r_u}{\beta}, \tag{5.24}$$

for some positive constant $C = C(C_P, \beta, q, c_T, \|\boldsymbol{g}\|_{2,L^2(\Omega)^d}, \|\boldsymbol{u}_0\|_2)$. Note that, due to the assumption (2.9), $\frac{r_u}{\beta} > 1$. As a consequence of (5.23) and (5.24), there holds

$$\nu_P(k^{j,l})|\boldsymbol{u}^{j,l}|^\beta \xrightarrow[l\to\infty]{} \nu_P(k^j)|\boldsymbol{u}^j|^\beta \;\; \text{in} \;\; L^q(0,T;L^q(\Omega)) \quad \forall q : 1 < q \leq \frac{r_u}{\beta}. \tag{5.25}$$

Finally, we use (5.5)–(5.6), (5.18), (5.21), (5.22) and (5.25), to pass to the limit $l \to \infty$ in the approximate weak formulation (3.3) to obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega k^j(t) w_i \, dx - \int_\Omega k^j(t)\boldsymbol{u}^j(t)\cdot\nabla w_i \, dx + \int_\Omega \nu_D(k^j(t))\nabla k^j(t)\cdot\nabla w_i \, dx \\ &+ \int_\Omega \varepsilon(k^j(t)) w_i \, dx \\ &= \int_\Omega \nu_T(k^j)|\mathbf{D}(\boldsymbol{u}^j)|^2 w_i \, dx + \int_\Omega \nu_P(k^j(t))|\boldsymbol{u}^j(t)|^\beta w_i \, dx \quad \forall i \in \mathbb{N}. \end{aligned} \tag{5.26}$$

By a standard procedure (see e.g. [31, Ch. III, §3.2]), we can combine (3.2) and (5.17) with $(3.4)_1$, and $(3.5)_1$, and with the convergence results (5.7), (5.13) and (5.15), to show that $\boldsymbol{u}^j(0) = \boldsymbol{u}_0^j$. Similarly, we can also combine (3.3) and (5.26) with $(3.4)_2$, and $(3.6)_1$ with the convergence results (5.8), (5.18), (5.21), (5.22) and (5.25), to prove that $k^j(0) = k_0^j$. In short, we have

$$\boldsymbol{u}^j(0) = \boldsymbol{u}_0^j \quad \text{and} \quad k^j(0) = k_0^j \quad \text{in } \Omega. \tag{5.27}$$

On the other hand, we can see that, due to (1.10),

$$e(k^j) \geq 0 \quad \text{a.e. in } Q_T. \tag{5.28}$$

Let us show now that

$$k^j \geq 0 \quad \text{a.e. in } Q_T. \tag{5.29}$$

For that, let us consider a couple of functions $(\boldsymbol{u}^j, k^j) \in L^2(0,T;\mathbf{V}) \times L^2(0,T;H_0^1(\Omega))$ satisfying (5.17) and (5.26). We start by decomposing k^j as $k^j = k_+^j - k_-^j$, where $k_+^j := \max\{0, k^j\}$ and $k_-^j := -\min\{0, k^j\}$. Since $k^j \in L^2(0,T;H_0^1(\Omega))$ implies that $k_-^j \in L^2(0,T;H_0^1(\Omega))$, we can take $w = -k_-^j$ in (5.26) so that, in view of the properties of k_+^j and k_-^j (see *e.g.* [12, p. 239]), and once that $\operatorname{div} \boldsymbol{u}^j = 0$ in $\operatorname{div} \boldsymbol{u}^j = 0$ in Q_T, one has, after integrating the resulting equation between $t = 0$ and $t = T$,

$$\int_{Q_T} \nu_D(k^j)|\nabla k_-^j|^2 dxdt - \int_{Q_T} \varepsilon(k^j)k_-^j \, dxdt$$
$$= -\int_{Q_T} \nu_T(k^j)|\mathbf{D}(\boldsymbol{u}^j)|^2 k_-^j \, dxdt - \int_{Q_T} \nu_P(k^j)|\boldsymbol{u}^j|^\beta k_-^j \, dxdt.$$

It is clear that the terms on the r.h.s. are non-positive, and due to (1.9), (5.28), and to the fact that $k_-^j k_+^j = 0$ in Q_T, the second l.h.s. term is non-negative. Gathering this information with (2.3), we have

$$\int_0^T \|\nabla k_-^j(t)\|_2^2 dt \leq 0.$$

Combining this with the Sobolev inequality, we get $k_-^j = 0$ a.e. in Q_T and, thus, (5.29) follows. Finally, as a consequence of (1.9), (5.28) and (5.29), one has

$$\varepsilon(k^j) \geq 0 \quad \text{a.e. in } Q_T. \tag{5.30}$$

6. ESTIMATES INDEPENDENT OF j

Let us first obtain estimates for $\boldsymbol{u}^j$, $\nabla \boldsymbol{u}^j$, and $\partial_t \boldsymbol{u}^j$ that are independent of j. By linearity and continuity, we can show from (5.17) that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \boldsymbol{u}^j(t)\cdot\boldsymbol{v}\,dx - \int_\Omega \Phi_n(|\boldsymbol{u}^j(t)|^2)\boldsymbol{u}^j(t)\otimes\boldsymbol{u}^j(t) : \nabla\boldsymbol{v}\,dx \\ &+ \int_\Omega \nu_T(k^j(t))\,\mathbf{D}(\boldsymbol{u}^j(t)) : \mathbf{D}(\boldsymbol{v})\,dx \\ &+ \int_\Omega \left(c_{Da} + c_{Fo}|\boldsymbol{u}^j(t)|^{\alpha-2}\right)\boldsymbol{u}^j(t)\cdot\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{g}(t)\cdot\boldsymbol{v}\,dx \end{aligned} \tag{6.1}$$

holds for all $t \in (0,T)$ and all $\boldsymbol{v} \in \mathbf{V} \cap L^\alpha(\Omega)^d$. At any time $t \in (0,T]$, we take $\boldsymbol{v} = \boldsymbol{u}^j(t)$ in (6.1) so that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}^j(t)\|_2^2 + \int_\Omega \nu_T(k^j(t))|\mathbf{D}(\boldsymbol{u}^j(t))|^2\,dx + \int_\Omega \left(c_{Da} + c_{Fo}|\boldsymbol{u}^j(t)|^{\alpha-2}\right)|\boldsymbol{u}^j(t)|^2\,dx \\ &= \int_\Omega \boldsymbol{g}(t)\cdot\boldsymbol{u}^j(t)\,dx \quad \forall t \in (0,T). \end{aligned} \tag{6.2}$$

Observe the reasoning used in (4.2), which can be used here as well to cancel the regularized convective term. Using (6.2) instead, we can see the estimates (4.7) and (4.8) also hold with $\boldsymbol{u}^j$ in the place of $\boldsymbol{u}^{j,l}$.

Now let $\boldsymbol{\psi} \in \mathbf{V}^s$ and let P^j denote the orthogonal projection from $\mathbf{V}^s$ onto $\mathbf{X}_j$, with respect to the $W_0^{s,2}(\Omega)^d$ inner product. Using (5.17), assumption (2.2), (2.24)–(2.25), and Hölder, Sobolev and Korn inequalities, one has for a.e. $t \in [0,T]$

$$\left|\int_\Omega \partial_t \boldsymbol{u}^j(t) \cdot P^j(\boldsymbol{\psi})\, dx\right| \leq C\|\boldsymbol{u}^j(t)\|^2_{\frac{r_u}{2}} \|\nabla P^j(\boldsymbol{\psi})\|_{\left(\frac{r_u}{2}\right)'} + \|\nabla \boldsymbol{u}^j(t)\|_2 \|\nabla P^j(\boldsymbol{\psi})\|_2$$
$$+ \left(\|\boldsymbol{u}^j(t)\|_2 + \|\boldsymbol{g}(t)\|_2\right) \|P^j(\boldsymbol{\psi})\|_2 + \|\boldsymbol{u}^j(t)\|_\alpha^{\alpha-1} \|P^j(\boldsymbol{\psi})\|_\alpha \big]$$

for some positive constant $C = C(C_K, C_T, c_{Da}, c_{Fo})$. In view of this, using assumption (2.6), estimate (4.7), and observing that

$$\|\nabla P^j(\boldsymbol{\psi})\|_{\left(\frac{r_u}{2}\right)'},\ \|\nabla P^j(\boldsymbol{\psi})\|_2,\ \|P^j(\boldsymbol{\psi})\|_2,\ \|P^j(\boldsymbol{\psi})\|_\alpha \leq C\|P^j(\boldsymbol{\psi})\|_{W_0^{s,2}(\Omega)^d} \leq C\|\boldsymbol{\psi}\|_{W_0^{s,2}(\Omega)^d},$$

and that

$$\left\langle \partial_t \boldsymbol{u}^j(t), P^j(\boldsymbol{\psi})\right\rangle = \left\langle \partial_t \boldsymbol{u}^j(t), \boldsymbol{\psi}\right\rangle \quad \forall \boldsymbol{\psi} \in \mathbf{V}^s,$$

where $\langle \cdot, \cdot \rangle$ denotes here the duality product over $W^{-s,2}(\Omega)^d \times W_0^{s,2}(\Omega)^d$, we obtain

$$\int_0^T \|\partial_t \boldsymbol{u}^j(t)\|^r_{\mathbf{V}^{s'}}\, dt \leq C, \quad r := \min\left\{2, \frac{r_u}{2}, \alpha'\right\}, \tag{6.3}$$

for some positive constant $C = C(C_T, \alpha, d, c_{Da}, c_{Fo}, \Theta_1, \Theta_2, \Omega, T)$, and $\mathbf{V}^{s'}$ denotes the dual space of $\mathbf{V}^s$.

We are now going to obtain estimates for k^j and ∇k^j that are independent of j. For the sake of text organization, and due to its importance in this work, the estimates obtained in this part will be listed in separate lemmas. By linearity and continuity, we can infer from (5.26) that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega k^j(t) w\, dx - \int_\Omega k^j(t)\boldsymbol{u}^j(t) \cdot \nabla w\, dx + \int_\Omega \nu_D(k^j(t))\nabla k^j(t) \cdot \nabla w\, dx \\ &+ \int_\Omega \varepsilon(k^j(t)) w\, dx \\ &= \int_\Omega \nu_T(k^j(t))|\mathbf{D}(\boldsymbol{u}^j(t))|^2 w\, dx + \int_\Omega \nu_P(k^j(t))|\boldsymbol{u}^j(t)|^\beta w\, dx \end{aligned} \tag{6.4}$$

holds for all $t \in (0,T)$ and all $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Note that the reasoning used to obtain (4.9), (4.10) and (4.15) is no longer valid here, because the estimates there depend on j. In the next lemmas we derive the appropriate estimates. The estimate established in the first next lemma results from testing (6.4) with $w = \mathcal{T}_1(k^j)$, where $\mathcal{T}_1(k^j)$ is the truncation of k^j, with $m = 1$, defined by $\mathcal{T}_m : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\mathcal{T}_m(k) = \begin{cases} k & \text{if } |k| \leq m, \\ \frac{m}{|k|}k & \text{if } |k| > m. \end{cases} \tag{6.5}$$

Lemma 6.1. *Assume the identity* (6.4) *is valid for* $\boldsymbol{u}^j$ *and* k^j *in the above conditions. If* (2.9) *holds, then there exists an independent of* j *positive constant* K_1 *such that*

$$\sup_{t\in[0,T]} \|k^j(t)\|_1 + \int_0^T \|k^j(t)\|_{\vartheta+1}^{\vartheta+1} dt \leq K_1. \tag{6.6}$$

Proof. Taking $w = \mathcal{T}_1(k^j)$ in (6.4), we get

$$\begin{aligned} &\frac{d}{dt}\|\mathcal{H}_1(k^j(t))\|_1 - \int_\Omega \boldsymbol{u}^j(t)\cdot\nabla\mathcal{T}_1(k^j(t))\,dx + \int_\Omega \nu_D(k^j(t))\nabla k^j(t)\cdot\nabla\big(\mathcal{T}_1(k^j(t))\big)\,dx \\ &+ \int_\Omega \varepsilon(k^j(t))\mathcal{T}_1(k^j(t))\,dx \\ &= \int_\Omega \nu_T(k^j(t))|\mathbf{D}(\boldsymbol{u}^j(t))|^2\mathcal{T}_1(k^j(t))\,dx + \int_\Omega \nu_P(k^j(t))|\boldsymbol{u}^j(t)|^\beta\mathcal{T}_1(k^j(t))\,dx, \end{aligned} \tag{6.7}$$

where $\mathcal{H}_1$ is the primitive function of $\mathcal{T}_1$,

$$\mathcal{H}_1(k) := \int_0^k \mathcal{T}_1(s)\,ds. \tag{6.8}$$

We observe that the second l.h.s. term of (6.7) vanishes, because $\boldsymbol{u}^j$ is also divergence free and has zero trace on the boundary $\partial\Omega$. Due to the definition of the truncation $\mathcal{T}_1$ (see (6.5)) and to (5.29), we can see the third l.h.s. term is nonnegative. For the fourth l.h.s. term, we can use the assumption (2.5), together with (5.29) and with the definition of the truncation $\mathcal{T}_1$, to show that

$$\int_\Omega k^j(t)^{\vartheta+1}\,dx - C \leq \int_\Omega k^j(t)^{\vartheta+1}\mathcal{T}_1(k^j(t))\,dx$$

for some positive constant $C = C(\Omega)$. Using again the definition of the truncation $\mathcal{T}_1$ together with the assumption (2.4), and with the fact that $\mathcal{T}_1(k^j(t)) \leq 1$, we obtain

$$\begin{aligned} &\frac{d}{dt}\|\mathcal{H}_1(k^j(t))\|_1 + c_\varepsilon\left(\int_\Omega k^j(t)^{\vartheta+1}\,dx - C\right) \\ &\leq \int_\Omega \nu_T(k^j(t))|\mathbf{D}(\boldsymbol{u}^j(t))|^2 dx + C_P\int_\Omega |\boldsymbol{u}^j(t)|^\beta\,dx. \end{aligned} \tag{6.9}$$

Integrating (6.9) between 0 and $t \in (0,T)$, using $(5.27)_2$ and (4.7), with $\boldsymbol{u}^j$ and k^j in the places of $\boldsymbol{u}^{j,l}$ and $k^{j,l}$, and taking the supreme in the interval $[0,T]$ of the resulting inequality, we obtain

$$\begin{aligned}\sup_{t\in[0,T]} \|\mathcal{H}_1(k^j(t))\|_1 + c_\varepsilon \int_0^T \|k^j(t)\|_{\vartheta+1}^{\vartheta+1} dt \leq \|\mathcal{H}_1(k_0^j)\|_1 + C \\ + C_P \int_{Q_T} |\boldsymbol{u}^j|^\beta dxdt,\end{aligned} \tag{6.10}$$

where $C = C(\Omega, T, \Theta_1)$ is a positive constant. On the other hand, by the definition of the function $\mathcal{H}_1$, it can be easily proved the existence of two absolute positive constants C_1 and C_2 such that

$$k - C_1 \leq \mathcal{H}_1(k) \leq C_2 k \quad \forall k \in \mathbb{R}_0^+.$$

Using this fact in (6.10), we get

$$\sup_{t\in[0,T]} \|k^j(t)\|_1 + c_\varepsilon \int_0^T \|k^j(t)\|_{\vartheta+1}^{\vartheta+1} dt \leq C_1 \|k_0^j\|_1 + C_2 + C_3 \int_0^T \|\boldsymbol{u}^j(t)\|_\beta^\beta dt,$$

where $C_1, C_2 = C(\Omega, T, \Theta_1)$, $C_3 = C_P$ are positive constants. Observe that, analogously to (4.8), we can also use parabolic interpolation to show that

$$\int_0^T \|\boldsymbol{u}^j(t)\|_\beta^\beta dt \leq \Theta_2, \quad \beta \leq r_u, \tag{6.11}$$

for the positive constant Θ_2 given in (4.8), which may also depend here on β. The estimate (4.8), with $\boldsymbol{u}^j$ in the place of $\boldsymbol{u}^{j,l}$, together with (6.11), and with $(3.6)_2$, imply

$$\sup_{t\in[0,T]} \|k^j(t)\|_1 + c_\varepsilon \int_0^T \|k^j(t)\|_{\vartheta+1}^{\vartheta+1} dt \leq C, \tag{6.12}$$

where $C = C(C_P, \beta, r_u, \|k_0\|_1, \Theta_1, \Theta_2, \Omega, T)$ is a positive constant. Whence we get (6.6) for some positive constant $K_1 = C(\beta, r_u, c_\varepsilon, C_P, \|k_0\|_1, \Theta_1, \Theta_2, \Omega, T)$, which concludes the proof of Lemma 6.1. □

To obtain an estimate for ∇k^j, we consider the following special test function, in the spirit of [29],

$$\upsilon(k^j) := 1 - \frac{1}{(1+k^j)^\delta}, \quad \text{with } 0 < \delta \ll 1. \tag{6.13}$$

See also [7, 8] for the consideration of distinct test functions. Observe that $\upsilon(k^j)$ satisfies

$$0 \le \upsilon(k^j) \le 1, \quad \nabla\upsilon(k^j) = \delta\frac{\nabla k^j}{(1+k^j)^{1+\delta}} \tag{6.14}$$

and therefore $\upsilon(k^j) \in L^2(0,T;H_0^1(\Omega))$.

Lemma 6.2. *Assume we are in the conditions of Lemma 6.1. Then there exist independent of j positive constants K_2 and K_3 such that*

$$\int_0^T \|\nabla k^j(t)\|_q^q dt \le K_2 + \frac{K_3}{\delta} \quad \forall\,\delta>0 \ \ \textit{small}, \quad q<\frac{d+2}{d+1}. \tag{6.15}$$

Proof. Taking $w=\upsilon(k^j(t))$ in (6.4) so that, after integrating the resulting equation between 0 and $t\in(0,T)$, using $(5.27)_2$, and taking the supreme in the interval $[0,T]$, we get

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\Upsilon(k^j(t))\|_1 + \int_{Q_T}\boldsymbol{u}^j\cdot\nabla\Upsilon(k^j)\,dxdt \\
&+\delta\int_{Q_T}\nu_D(k^j)\frac{|\nabla k^j|^2}{(1+k^j)^{1+\delta}}\,dxdt + \int_{Q_T}\varepsilon(k^j)\upsilon(k^j)\,dxdt \\
&=\|\Upsilon(k_0^j)\|_1 + \int_{Q_T}\nu_T(k^j)|\mathbf{D}(\boldsymbol{u}^j)|^2\upsilon(k^j)\,dxdt + \int_{Q_T}\nu_P(k^j)|\boldsymbol{u}^j|^\beta\,\upsilon(k^j)\,dxdt,
\end{aligned} \tag{6.16}$$

where $\Upsilon(k)$ is the following primitive function of $\upsilon(k)$,

$$\Upsilon(k) := \int_0^k \upsilon(s)\,ds. \tag{6.17}$$

The second l.h.s. term of (6.16) vanishes, since $\boldsymbol{u}^j$ is divergence free and has zero trace on the boundary $\partial\Omega$. The fourth l.h.s. term is nonnegative due to (5.28) and $(6.14)_1$. In addition, since $\upsilon(k^j)\le 1$, we obtain from (6.16),

$$\delta\int_{Q_T}\nu_D(k^j)\frac{|\nabla k^j|^2}{(1+k^j)^{1+\delta}}\,dxdt \le \|\Upsilon(k_0^j)\|_1 + \Theta_1 + C_P\int_{Q_T}\nu_P(k^j)|\boldsymbol{u}^j|^\beta dxdt, \tag{6.18}$$

where Θ_1 is the positive constant from the counterpart estimate of (4.7). Using the assumptions (2.3) and (2.4), together with $(3.6)_2$, (6.17), and with the fact that $|\Upsilon(k)|\le|k|$ for all $k\in\mathbb{R}$, one gets from (6.18)

$$\delta c_D\int_{Q_T}\frac{|\nabla k^j|^2}{(1+k^j)^{1+\delta}}\,dxdt \le \|k_0\|_1 + \Theta_1 + C_P\int_{Q_T}|\boldsymbol{u}^j|^\beta dxdt.$$

Proceeding as we did for (6.12), we get

$$\delta c_D \int\limits_{Q_T} \frac{|\nabla k^j|^2}{(1+k^j)^{1+\delta}}\, dxdt \leq C \tag{6.19}$$

for some positive constant $C = C(\|k_0\|_1, \Theta_1, K_1)$.

Next, we apply the Young inequality, together with the estimate (6.19), so that

$$\begin{aligned}\int\limits_0^T \|\nabla k^j(t)\|_q^q dt = &\int\limits_{Q_T} \frac{|\nabla k^j|^q}{(1+k^j)^{(1+\delta)\frac{q}{2}}}(1+k^j)^{(1+\delta)\frac{q}{2}} dxdt \quad (q<2)\\ \leq &\delta c_D \int\limits_{Q_T} \frac{|\nabla k^j|^2}{(1+k^j)^{(1+\delta)}} dxdt + C_1' + C_2 \int\limits_{Q_T} |k^j|^{(1+\delta)\frac{q}{2-q}} dxdt \\ \leq &C_1 + C_2 \int\limits_0^T \|k^j(t)\|_{(1+\delta)\frac{q}{2-q}}^{(1+\delta)\frac{q}{2-q}} dt\end{aligned} \tag{6.20}$$

for some positive constants $C_1' = C(\delta, q, c_D, \Omega, T)$, $C_1 = C(C_1', \|k_0\|_1, \Theta_1, K_1)$ and $C_2 = C(\delta, q, c_D)$.

Let us now consider the following function related with the weight $(1+k^j)^{-\delta-1}$ of the first r.h.s. integral in (6.20),

$$\Lambda(k) := \int\limits_0^k (1+s)^{\frac{-\delta-1}{2}} ds. \tag{6.21}$$

It can be easily proved the existence of two positive constants C_1 and C_2 such that

$$C_1\left[(1+k)^{\frac{1-\delta}{2}} - 1\right] \leq \Lambda(k) \leq C_2(1+k)^{\frac{1-\delta}{2}} \quad \forall k \in \mathbb{R}_0^+. \tag{6.22}$$

Moreover, using (6.21) and the Sobolev inequality, together with the estimate (6.19), there holds

$$\begin{aligned}\int\limits_0^T \left(\|\Lambda(k^j(t))\|_2^2 + \|\nabla\Lambda(k^j(t))\|_2^2\right) dt \leq &C_1 \int\limits_0^T \left\|\nabla\big(\Lambda(k^j(t))\big)\right\|_2^2 dt \\ = &C_1 \int\limits_{Q_T} \frac{|\nabla k^j|^2}{(1+k^j)^{1+\delta}}\, dxdt \leq \frac{C_2}{\delta}\end{aligned} \tag{6.23}$$

for some positive constants $C_1 = C(d, \Omega)$ and $C_2 = C(d, \Omega, c_D, \|k_0\|_1, \Theta_1, K_1)$.

Then, we use interpolation so that

$$\|k^j\|_{(1+\delta)\frac{q}{2-q}} \leq \|k^j\|_1^{1-\lambda}\|k^j\|_{(1-\delta)\frac{\sigma}{2}}^{\lambda}, \quad \lambda = \frac{\sigma}{q}\frac{(1-\delta)[(2+\delta)q-2]}{(1+\delta)[(1-\delta)\sigma-2]}, \tag{6.24}$$

where σ denotes the Sobolev conjugate of 2. Next, we use (6.22), the Sobolev inequality and (6.23) so that

$$\begin{aligned}\int_0^T \|k^j(t)\|^{1-\delta}_{(1-\delta)\frac{\sigma}{2}} dt < \int_0^T \left(\int_\Omega (1+k^j)^{\frac{1-\delta}{2}\sigma} dx\right)^{\frac{2}{\sigma}} dt \leq \frac{1}{C_1}\int_0^T \|\Lambda(k(t))+1\|^2_\sigma\, dt \\ \leq C_2 \int_0^T \left\|\nabla\big(\Lambda(k^j(t))\big)\right\|_2^2 dt \leq \frac{C_3}{\delta},\end{aligned} \tag{6.25}$$

where C_1 is the corresponding constant from (6.22) and $C_2 = C(C_1, d, \Omega)$, $C_3 = C(C_1, d, \Omega, c_D, \|k_0\|_1, \Theta_1, K_1)$ are positive constants. We raise (6.24) to the power $(1+\delta)\frac{q}{2-q}$, then we integrate the resulting inequality between 0 and $t \in (0,T)$ and take the supreme in $[0,T]$, which, in view of (6.6), implies

$$\begin{aligned}\int_0^T \|k^j(t)\|^{(1+\delta)\frac{q}{2-q}}_{(1+\delta)\frac{q}{2-q}} dt \leq \sup_{t\in[0,T]} \|k^j(t)\|_1^{(1-\lambda)(1+\delta)\frac{q}{2-q}} \int_0^T \|k^j(t)\|^{\lambda(1+\delta)\frac{q}{2-q}}_{(1-\delta)\frac{\sigma}{2}} dt \\ \leq C \int_0^T \|k^j(t)\|^{\lambda(1+\delta)\frac{q}{2-q}}_{(1-\delta)\frac{\sigma}{2}} dt\end{aligned} \tag{6.26}$$

for some positive constant $C = C(\delta, q, d, K_1)$. In order to use (6.25), we choose q so that

$$\lambda(1+\delta)\frac{q}{2-q} = 1-\delta \Leftrightarrow q = \frac{2\sigma(2-\delta)-4}{3\sigma-2} \Leftrightarrow \begin{cases} q = \frac{d(1-\delta)+2}{d+1}, & d \neq 2, \\ q < \frac{2(2-\delta)}{3}, & d = 2. \end{cases} \tag{6.27}$$

Hence, combining (6.25) with (6.26), we have

$$\int_0^T \|k^j(t)\|^{(1+\delta)\frac{q}{2-q}}_{(1+\delta)\frac{q}{2-q}} dt \leq C_1 \int_0^T \|k^j(t)\|^{1-\delta}_{(1-\delta)\frac{\sigma}{2}} dt \leq \frac{C_2}{\delta} \tag{6.28}$$

for some positive constants $C_1 = C(\delta, q, d, K_1)$ and $C_2 = C(\delta, q, d, \Omega, c_D, \|k_0\|_1, \Theta_1, K_1)$.

Note that, in view of (6.20) and (6.27),

$$q < 2 \Leftrightarrow 1-\delta < 2$$

which is true for our choice of δ in (6.13). Plugging (6.28) into (6.20), and observing the requirements for the exponent q declared at (6.24) and (6.27), we prove that (6.15) holds true for some positive constants $K_2 = C(\delta, q, c_D, \|k_0\|_1, \Theta_1, K_1, \Omega, T)$ and $K_3 = C(\delta, q, c_d, d, \|k_0\|_1, \Theta_1, K_1, \Omega)$. This concludes the proof of Lemma 6.2. □

Combining Lemmas 6.1 and 6.2, we can now establish the following result. Note that the estimate (4.9) is not an alternative here, because it depends on j.

Lemma 6.3. *Assume we are in the conditions of Lemmas 6.1-6.2. Then, there exists an independent of j positive constant K_4 such that*

$$\int_0^T \|k^j(t)\|_q^q dt \leq \frac{K_4}{\delta} \quad \forall\, \delta > 0, \quad q < 1 + \frac{2}{d}. \tag{6.29}$$

Proof. For any $\delta : 0 < \delta << 1$, we can use interpolation so that

$$\|k^j(t)\|_q \leq \|k^j(t)\|_1^{1-\lambda} \|k^j(t)\|_{(1-\delta)\frac{\sigma}{2}}^{\lambda}, \quad \lambda = \frac{(q-1)(1-\delta)\sigma}{q[\sigma(1-\delta)-2]}, \tag{6.30}$$

where σ is the Sobolev conjugate of 2. Raising (6.30) to the power q, and then, in order to use (6.25) again, we choose q so that

$$q\lambda = 1 - \delta \Leftrightarrow q = 2 - \delta - \frac{2}{\sigma} \Rightarrow q : \begin{cases} = 1 - \delta + \frac{2}{d}, & d \neq 2, \\ < 2 - \delta, & d = 2. \end{cases} \tag{6.31}$$

Next, we integrate the resulting inequality between 0 and $t \in [0, T]$, to get, after the application of the estimates (6.6) and (6.25) in the final part,

$$\int_0^T \|k^j(t)\|_q^q dt \leq \sup_{t \in [0,T]} \|k^j(t)\|_1^{\frac{\sigma-2}{\sigma}} \int_0^T \|k^j(t)\|_{(1-\delta)\frac{\sigma}{2}}^{1-\delta} \leq \frac{C}{\delta}, \tag{6.32}$$

where $C = C(\sigma, d, c_D, \|k_0\|_1, \Theta_1, K_1, \Omega)$ is a positive constant. Hence, (6.29), is now a direct consequence of (6.31) and (6.32), where $K_4 = C(\delta, d, c_D, \|k_0\|_1, \Theta_1, K_1, \Omega)$ is a positive constant. The proof of Lemma 6.3 is thus concluded. ☐

Remark 6.4. Note that, due to (6.6), (6.29) also holds true for $q = \vartheta + 1$, but, in view of the assumption (2.8), this estimate is worse.

In this part, we can proceed as we did for (4.15), but now using the identity (6.4). By combining the Hölder inequality with the estimate (4.7), that still holds with $\boldsymbol{u}^j$ and k^j in the places of $\boldsymbol{u}^{j,l}$ and $k^{j,l}$, one has for any $q > d$

$$\int_0^T \left\| \nu_T(k^j(t)) |\mathbf{D}(\boldsymbol{u}^j(t))|^2 \right\|_{W^{-1,q'}(\Omega)} dt \leq \int_0^T \left\| \sqrt{\nu_T(k^j(t))} \mathbf{D}(\boldsymbol{u}^j(t)) \right\|_2^2 dt \leq \aleph_1, \tag{6.33}$$

and for some positive constant $\aleph_1 = C(\Theta_1, T)$.

Besides the estimates of Lemmas 6.1-6.3, we also need to obtain independent of j estimates for the turbulent diffusion term, as well as for the terms of turbulence transport and turbulence production.

We start by estimating the turbulent diffusion term.

Lemma 6.5. *Assume we are in the conditions of Lemmas 6.1-6.3. Then, there exists an independent of j positive constant K_6 such that*

$$\int_0^T \left\| \nu_D(k^j(t)) \nabla k^j(t) \right\|_\rho^\rho dt \leq \frac{K_5}{\delta} \quad \forall\, \delta > 0 \;\; \textit{small}, \quad \rho < \frac{d+2}{d+1}. \tag{6.34}$$

Proof. By using the assumption (2.3), together with the Hölder inequality and the estimate (6.19), we can show that

$$\int_{Q_T} \left|\nu_D(k^j)\nabla k_j\right|^\rho dxdt \le C_D^\rho \int_{Q_T} |\nabla k_j|^\rho dxdt = C_D^\rho \int_{Q_T} (1+k^j)^{(1+\delta)\frac{\rho}{2}} \frac{|\nabla k_j|^\rho}{(1+k^j)^{(1+\delta)\frac{\rho}{2}}} dxdt$$
$$\le C_D^\rho \left(\int_{Q_T} \frac{|\nabla k_j|^2}{(1+k^j)^{1+\delta}} dxdt\right)^{\frac{\rho}{2}} \left(\int_{Q_T} (1+k^j)^{(1+\delta)\frac{\rho}{2-\rho}} dxdt\right)^{\frac{2-\rho}{\rho}} \qquad (\rho < 2)$$
$$\le \frac{C}{\delta}\left[1 + \left(\int_0^T \|k_j(t)\|_{(1+\delta)\frac{\rho}{2-\rho}}^{(1+\delta)\frac{\rho}{2-\rho}} dt\right)^{\frac{2-\rho}{2}}\right]. \tag{6.35}$$

for some positive constant $C = C(c_D, C_D, \delta, \rho, \|k_0\|_1, \Theta_1, K_1, \Omega, T)$. Having in mind the estimate (6.32), with q given there by (6.31), we choose ρ so that

$$(1+\delta)\frac{\rho}{2-\rho} = 2 - \delta - \frac{2}{\sigma} \Leftrightarrow \rho = \frac{4(\sigma-1)}{3\sigma-2} - \frac{2\sigma}{3\sigma-2}\delta, \tag{6.36}$$

where σ denotes the Sobolev conjugate of 2. Since $0 < \delta << 1$, we have

$$\rho < \frac{4(\sigma-1)}{3\sigma-2} \Leftrightarrow \begin{cases} \rho < \frac{d+2}{d+1}, & d \ne 2, \\ \rho < \frac{4}{3}, & d = 2. \end{cases} \tag{6.37}$$

As a consequence of (6.29), we can readily see that (6.35) and (6.37) imply (6.34) for some positive constant $K_5 = C(\delta, d, c_D, C_D, \|k_0\|_1, \Theta_1, K_1, \Omega, T)$. Thus the proof of Lemma 6.5 is concluded. □

Remark 6.6. Note that any ρ in the conditions of (6.37) also satisfies $\rho < 2$, as required by (6.35).

Now, we can use (6.34) to show that for any $q > \rho' > \left(\frac{d+2}{d+1}\right)' = d+2$, one has

$$\int_0^T \left\|\mathrm{div}(\nu_D(k^j(t))\nabla k^j(t))\right\|_{W^{-1,q'}(\Omega)} dt \le \int_0^T \left\|\nu_D(k^j(t))\nabla k^j(t)\right\|_{q'} dt$$
$$\le C_1 \left(\int_0^T \left\|\nu_D(k^j(t))\nabla k^j(t)\right\|_\rho^\rho dt\right)^{\frac{1}{\rho}} \le \frac{\aleph_2}{\delta}, \tag{6.38}$$

where ρ is given in (6.34) (see also (6.36)) and $C_1 = C(q, \rho, \Omega, T)$, $\aleph_2 = C(C_1, \rho, K_5)$ are positive constants.

Next, we estimate the term of turbulence transport.

Lemma 6.7. *Assume we are in the conditions of Lemmas 6.1–6.5. Then, there exists an independent of j positive constant K_6 such that*

$$\int_0^T \left\| k^j(t)\boldsymbol{u}^j(t) \right\|_\rho^\rho dt \le \frac{K_6}{\delta} \quad \forall\, \delta > 0 \ \ \textit{small}, \quad \rho < \max\left\{ \frac{2(d+2)}{3d}, \frac{\alpha(d+2)}{d(\alpha+1)+2} \right\}. \tag{6.39}$$

Note that

$$\frac{2(d+2)}{3d} > 1 \Leftrightarrow d < 4, \qquad \frac{\alpha(d+2)}{d(\alpha+1)+2} > 1 \Leftrightarrow \alpha > \frac{d+2}{2},$$

which means that it must be $2 \le d \le 3$ if $\alpha \le \frac{d+2}{2}$ and $\alpha > \frac{d+2}{2}$ if $d = 4$.

Proof of Lemma 6.7. Using the Hölder inequality, one has

$$\int_{Q_T} |k^j \boldsymbol{u}^j|^\rho dxdt \le \left(\int_0^T \|k^j(t)\|_q^q dt \right)^{\frac{\rho}{q}} \left(\int_0^T \|\boldsymbol{u}^j(t)\|_{r_u}^{r_u} dt \right)^{\frac{\rho}{r_u}}, \tag{6.40}$$

where, for q given in (6.31),

$$\frac{1}{\rho} = \frac{1}{r_u} + \frac{1}{q} \Leftrightarrow \rho = \frac{r_u[\sigma(2-\delta)-2]}{\sigma(2-\delta)-2+r_u\sigma}$$
$$\Leftrightarrow \rho = \frac{2(\sigma-1)r_u}{2(\sigma-1)+r_u\sigma} - \tau\delta,$$

$$\tau := \frac{r_u^2\sigma^2}{[\sigma(2-\delta)-2+r_u\sigma][2(\sigma-1)+r_u\sigma]},$$

and where r_u is given by (2.7). Recall that σ denotes the Sobolev conjugate of 2. Since $0 < \delta << 1$ and $\sigma > 2$, we have $\tau > 0$, which implies that $\tau\delta$ is very small as well. Hence,

$$\rho < \rho_0 := \begin{cases} \frac{2(d+2)}{3d} & \text{if } \ r_u = \frac{2(d+2)}{d}, \\ \frac{\alpha(d+2)}{d(\alpha+1)+2} & \text{if } \ r_u = \alpha. \end{cases} \tag{6.41}$$

Note that $\rho_0 \ge 1$ in any case. Plugging (4.8) and (6.29), the first with $\boldsymbol{u}^j$ in the place of $\boldsymbol{u}^{j,l}$, into (6.40), we prove (6.39), and where $K_7 = C(\delta, d, \rho, r_u, \Theta_2, K_5)$ is a positive constant. This proves Lemma 6.7. □

Remark 6.8. Estimate (6.39) already gives us a condition depending on the power-law index characterizing the Darcy–Forchheimer drag forces. In particular, the values of interest of r_k in (6.41), from the point of view of physics, are

$$\rho_0 = \begin{cases} \max\left\{ \frac{10}{9}, \frac{5\alpha}{3\alpha+5} \right\}, & d = 3, \\ \max\left\{ \frac{4}{3}, \frac{2\alpha}{\alpha+2} \right\}, & d = 2. \end{cases}$$

Now, in view of (6.39), we can use the Hölder inequality to show that for any

$$q > \min\left\{\frac{2(d+2)}{4-d}, \frac{\alpha(d+2)}{2\alpha-d-2}\right\},$$

whenever $2 \leq d \leq 3$ if $\alpha \leq \frac{d+2}{2}$ and $\alpha > \frac{d+2}{2}$ if $d = 4$, one has

$$\int_0^T \|\operatorname{div}\left(k^j(t)\boldsymbol{u}^j(t)\right)\|_{W^{-1,q'}(\Omega)} dt \leq \int_0^T \|k^j(t)\boldsymbol{u}^j(t)\|_{q'} dt \leq C_1 \left(\int_0^T \left\|k^j(t)\boldsymbol{u}^j(t)\right\|_\rho^\rho dt\right)^{\frac{1}{\rho}}$$
$$\leq \frac{\aleph_3}{\delta} \quad \forall\, \delta > 0 \ \text{ small}, \tag{6.42}$$

for some positive constants $C_1 = C(\rho, q, \Omega, T)$ and $\aleph_3 = C(C_1, K_6)$.

It last to obtain an estimate for the turbulence production term.

Lemma 6.9. *Assume we are in the conditions of Lemmas 6.1–6.7. Then, there exists an independent of j positive constant K_7 such that*

$$\int_0^T \left\|\nu_P(k^j(t))|\boldsymbol{u}^j|^\beta\right\|_\rho^\rho dt \leq \frac{K_7}{\delta} \quad \forall\, \delta > 0, \quad \rho < \max\left\{\frac{2(d+2)}{\beta d}, \frac{\alpha}{\beta}\right\}. \tag{6.43}$$

Proof. Using the assumptions (2.4) and (2.9), and the Hölder inequality, one has

$$\int_{Q_T} \left|\nu_P(k^j)|\boldsymbol{u}^j|^\beta\right|^\rho dxdt \leq C_P^\rho \int_{Q_T} |\boldsymbol{u}^j|^{\beta\rho} dxdt \leq C\left(\int_0^T \|\boldsymbol{u}^j(t)\|_{r_u}^{r_u} dt\right)^{\frac{\rho\beta}{r_u}} \tag{6.44}$$

for some positive constant $C = C(C_P, \beta, \rho, r_u, \Theta_2, \Omega, T)$ and where

$$\frac{\rho\beta}{r_u} \leq 1 \Leftrightarrow \rho \leq \rho_1 := \frac{r_u}{\beta} = \begin{cases} \frac{2(d+2)}{\beta d} & \text{if } r_u = 2\frac{d+2}{d}, \\ \frac{\alpha}{\beta} & \text{if } r_u = \alpha. \end{cases} \tag{6.45}$$

Note that assumption (2.9) assures us that $r_k > 1$ in any case. Using the estimate (4.8), this with $\boldsymbol{u}^j$ in the place of $\boldsymbol{u}^{j,l}$, we can infer from (6.44) that (6.43) holds true for some positive constants $K_7 = C(\beta, \delta, d, c_D, C_P, \rho, r_u, \|k_0\|_1, \Theta_1, K_1, \Omega)$. This proves Lemma 6.9. □

Then, combining the Sobolev and Hölder inequalities with (6.44), we can see that for $q \geq \frac{d\rho'}{d+\rho'}$, where ρ' is the Hölder conjugate of $\rho > 1$ satisfying (6.45), one has

$$\int_0^T \left\|\nu_P(k^j(t))|\boldsymbol{u}^j(t)|^\beta\right\|_{W^{-1,q'}(\Omega)} dt \leq \int_0^T \left\|\nu_P(k^j(t))|\boldsymbol{u}^j(t)|^\beta\right\|_{(q^*)'} dt$$
$$\leq C_1\left(\int_0^T \left\|\nu_P(k^j(t))|\boldsymbol{u}^j(t)|^\beta\right\|_\rho^\rho dt\right)^{\frac{1}{\rho}} \leq \frac{\aleph_4}{\delta}, \tag{6.46}$$

for some positive constants $C_1 = C(q, \rho, \Omega, T)$ and $\aleph_5 = C(C_1, K_7)$, and where $(q^*)'$ denotes the Hölder conjugate of the Sobolev conjugate of q.

We estimate the turbulence dissipation term by using the assumption (2.5), together with the estimate (6.29), so that, for $q \geq d$, one has

$$\int_0^T \left\|\varepsilon(k^j(t))\right\|_{W^{-1,q'}(\Omega)} dt \leq C \int_0^T \left\|k^j(t)\right\|_{\vartheta+1} dt \leq \frac{\aleph_5}{\delta}, \tag{6.47}$$

for some positive constant $\aleph_5 = C(C_\varepsilon, \vartheta, \Omega, T, K_4)$.

Finally, combining (5.30) with (6.33), (6.38), (6.42), (6.46) and (6.47), one has

$$\int_0^T \left\|\partial_t k^j(t)\right\|_{W^{-1,q'}(\Omega)} dt \leq C_1 + \frac{C_2}{\delta} \quad \forall\, \delta > 0 \ \text{ small}, \tag{6.48}$$

for

$$q > q_0 := \max\left\{d+2, \frac{2(d+2)}{4-d}, \frac{\alpha(d+2)}{2\alpha - d - 2}, \frac{d\rho'}{d+\rho'}\right\} \tag{6.49}$$

and some positive constants $C_1 = C(\aleph_1)$ and $C_2 = (\aleph_2, \aleph_3, \aleph_4, \aleph_5)$.

7. PASSING TO THE LIMIT AS $j \to \infty$

In this section, we also relabel all the considered subsequences to the same indices of the original ones.

As observed in the previous section, the estimates (4.7) and (4.8) do not depend on j and therefore also hold with $\boldsymbol{u}^j$ and k^j in the places of $\boldsymbol{u}^{j,l}$ and $k^{j,l}$. In view of this, and of the estimate (6.3), which also does not depend on j, we can apply the Banach–Alaoglu theorem so that, for some subsequences,

$$\boldsymbol{u}^j \xrightarrow[j\to\infty]{*} \boldsymbol{u} \ \text{ in } \ L^\infty(0,T;\mathbf{H}),$$

$$\boldsymbol{u}^j \xrightarrow[j\to\infty]{} \boldsymbol{u}, \ \text{ in } L^2(0,T;\mathbf{V}) \cap L^{r_u}(0,T;L^{r_u}(\Omega)^d), \tag{7.1}$$

$$\partial_t \boldsymbol{u}^j \xrightarrow[j\to\infty]{} \partial_t \boldsymbol{u} \ \text{ in } \ L^r(0,T;\mathbf{V}^{s'}), \tag{7.2}$$

for r given at (6.3).

On the other hand, from (6.6), (6.15), (6.29) and (6.48), we can deduce that, up to some subsequences,

$$k^j \xrightarrow[j\to\infty]{*} k \ \text{ in } \ L^\infty(0,T;\mathcal{M}(\Omega)),$$

$$k^j \xrightarrow[j\to\infty]{} k \ \text{ in } \ L^q(0,T;W_0^{1,q}(\Omega)), \quad 1 < q < \frac{d+2}{d+1}, \tag{7.3}$$

$$k^j \xrightarrow[j\to\infty]{} k \ \text{ in } \ L^q(0,T;L^q(\Omega)) \cap L^{\vartheta+1}(0,T;L^{\vartheta+1}(\Omega)), \quad 1 < q < \frac{d+2}{d} \tag{7.4}$$

$$\partial_t k^j \xrightarrow[j\to\infty]{} \partial_t k \ \text{ in } \ \mathcal{M}(0,T;W^{-1,q'}(\Omega)), \quad q > q_0, \quad \text{for } q_0 \text{ is given in (6.49)}, \tag{7.5}$$

where $\mathcal{M}(\Omega)$ denotes the space of Radon measures σ over Ω.

By using the estimates (4.7)–(4.8), with $\boldsymbol{u}^j$ and k^j in the places of $\boldsymbol{u}^{j,l}$ and $k^{j,l}$, together with (6.3) and the Aubin–Lions–Simon compactness lemma (see [30, Corollary 6]), we have

$$\boldsymbol{u}^j \xrightarrow[j\to\infty]{} \boldsymbol{u} \ \text{ in } \ L^q(0,T;L^q(\Omega)^d), \quad 1 \le q < r_u, \tag{7.6}$$

where r_u is given in (2.7). By the same arguing, from the estimates (6.6), (6.15), (6.29) and (6.48), we have

$$k^j \xrightarrow[j\to\infty]{} k \ \text{ in } \ L^q(0,T;L^q(\Omega)), \quad 1 \le q < r_k \tag{7.7}$$

for r_k given in (2.7). Now, in view of (7.6) and (7.7) and the Riesz–Fischer theorem, there exist another subsequences such that

$$\boldsymbol{u}^j \xrightarrow[j\to\infty]{} \boldsymbol{u} \quad \text{a.e. in } \ Q_T, \tag{7.8}$$

$$k^j \xrightarrow[j\to\infty]{} k \quad \text{a.e. in } \ Q_T. \tag{7.9}$$

Reasoning as we did for (5.13), (5.15) and (5.22), but using, in the present case, (7.1), (7.8) and (7.9), we have

$$\Phi_n(|\boldsymbol{u}^j|^2)\boldsymbol{u}^j \otimes \boldsymbol{u}^j \xrightharpoonup[j\to\infty]{} \Phi_n(|\boldsymbol{u}|^2)\boldsymbol{u} \otimes \boldsymbol{u} \ \text{ in } \ L^{\frac{d+2}{d}}(0,T;L^{\frac{d+2}{d}}(\Omega)^d), \tag{7.10}$$

$$|\boldsymbol{u}^j|^{\alpha-2}\boldsymbol{u}^j \xrightharpoonup[j\to\infty]{} |\boldsymbol{u}|^{\alpha-2}\boldsymbol{u} \ \text{ in } \ L^{\alpha'}(0,T;L^{\alpha'}(\Omega)^d), \tag{7.11}$$

$$\nu_T(k^j)\mathbf{D}(\boldsymbol{u}^j) \xrightharpoonup[j\to\infty]{} \nu_T(k)\mathbf{D}(\boldsymbol{u}) \ \text{ in } \ L^2(0,T;L^2(\Omega)^{d\times d}). \tag{7.12}$$

The convergence results (7.1)–(7.2) and (7.10)–(7.12) are sufficient to pass the equation (5.17) to the limit $j \to \infty$. In view of this, and by means of linearity and continuity, we can see that (2.26) holds for all $\boldsymbol{v} \in \mathbf{V} \cap L^\alpha(\Omega)^d$. Observe that, in the meantime, we have discarded the subscript n.

Similarly to (7.12), we also can prove that

$$\sqrt{\nu_T(k^j)}\mathbf{D}(\boldsymbol{u}^j) \xrightharpoonup[j\to\infty]{} \sqrt{\nu_T(k)}\mathbf{D}(\boldsymbol{u}) \ \text{ in } \ L^2(0,T;L^2(\Omega)^{d\times d}). \tag{7.13}$$

Taking $\boldsymbol{v} = \boldsymbol{u}(t)$ in (2.26), integrating the resulting identity between 0 and T, using (1.2) and $(1.4)_1$, and arguing as we did for (4.3), we have

$$\begin{aligned} &\int_0^T \left\| \sqrt{\nu_T(k(t))}\mathbf{D}(\boldsymbol{u}(t)) \right\|_2^2 dt + c_{Fo} \int_0^T \|\boldsymbol{u}(t)\|_\alpha^\alpha dt \\ &= -\frac{1}{2}\|\boldsymbol{u}(T)\|_2^2 + \frac{1}{2}\|\boldsymbol{u}_0\|_2^2 - c_{Da}\int_0^T \|\boldsymbol{u}(t)\|_2^2 dt + \int_0^T\int_\Omega \boldsymbol{g}\cdot\boldsymbol{u}\,dxdt. \end{aligned} \tag{7.14}$$

Integrating (6.2) between 0 and T, and using $(5.27)_1$, one has

$$\int_0^T \left\|\sqrt{\nu_T(k^j(t))}\mathbf{D}(\boldsymbol{u}^j(t))\right\|_2^2 dt + c_{Fo}\int_0^T \|\boldsymbol{u}^j(t)\|_\alpha^\alpha dt$$
$$= -\frac{1}{2}\|\boldsymbol{u}^j(T)\|_2^2 + \frac{1}{2}\|\boldsymbol{u}_0^j\|_2^2 - c_{Da}\int_0^T \|\boldsymbol{u}^j(t)\|_2^2 dt + \int_0^T\int_\Omega \boldsymbol{g}\cdot\boldsymbol{u}\,dxdt.$$

Now, shifting the first term in the r.h.s. to the left, letting $j\to\infty$ and using the convergence results $(3.5)_2$, (7.1) and (7.6), we obtain

$$\limsup_{j\to\infty}\left(\int_0^T \left\|\sqrt{\nu_T(k^j(t))}\mathbf{D}(\boldsymbol{u}^j(t))\right\|_2^2 dt + c_{Fo}\int_0^T \|\boldsymbol{u}^j(t)\|_\alpha^\alpha dt\right) + \limsup_{j\to\infty}\left(\frac{1}{2}\|\boldsymbol{u}^j(T)\|_2^2\right)$$
$$\leq \frac{1}{2}\|\boldsymbol{u}_0\|_2^2 - c_{Da}\int_0^T \|\boldsymbol{u}(t)\|_2^2 dt + \int_0^T\int_\Omega \boldsymbol{g}\cdot\boldsymbol{u}\,dxdt. \tag{7.15}$$

Observing that, due to (7.6) and the lower semi-continuity of the norm,

$$\frac{1}{2}\|\boldsymbol{u}(T)\|_2^2 \leq \limsup_{j\to\infty}\left(\frac{1}{2}\|\boldsymbol{u}^j(T)\|_2^2\right),$$

we can plug (7.14) into (7.15), so that

$$\begin{aligned}&\limsup_{j\to\infty}\left(\int_0^T \left\|\sqrt{\nu_T(k^j(t))}\mathbf{D}(\boldsymbol{u}^j(t))\right\|_2^2 dt + c_{Fo}\int_0^T \|\boldsymbol{u}^j(t)\|_\alpha^\alpha dt\right)\\ &\leq \int_0^T \left\|\sqrt{\nu_T(k(t))}\mathbf{D}(\boldsymbol{u}(t))\right\|_2^2 dt + c_{Fo}\int_0^T \|\boldsymbol{u}(t)\|_\alpha^\alpha dt.\end{aligned} \tag{7.16}$$

On the other hand, by (7.11), (7.13) and the lower semi-continuity of the norms, there holds

$$\begin{aligned}&\int_0^T \left\|\sqrt{\nu_T(k(t))}\mathbf{D}(\boldsymbol{u}(t))\right\|_2^2 dt + c_{Fo}\int_0^T \|\boldsymbol{u}(t)\|_\alpha^\alpha dt\\ &\leq \liminf_{j\to\infty}\left(\int_0^T \left\|\sqrt{\nu_T(k^j(t))}\mathbf{D}(\boldsymbol{u}^j(t))\right\|_2^2 dt + c_{Fo}\int_0^T \|\boldsymbol{u}^j(t)\|_\alpha^\alpha dt\right).\end{aligned} \tag{7.17}$$

Then, combining (7.16) with (7.17), we obtain

$$\nu_T(k^j)|\mathbf{D}(\boldsymbol{u}^j)|^2 + c_{Fo}|\boldsymbol{u}^j|^\alpha \xrightarrow[j\to\infty]{} \nu_T(k)|\mathbf{D}(\boldsymbol{u})|^2 + c_{Fo}|\boldsymbol{u}|^\alpha \quad\text{in}\quad L^1(0,T;L^1(\Omega)),$$

which implies, by the uniqueness of the limit, that

$$\begin{aligned}
&\nu_T(k^j)|\mathbf{D}(\boldsymbol{u}^j)|^2 \xrightarrow[j\to\infty]{} \nu_T(k)|\mathbf{D}(\boldsymbol{u})|^2 \quad \text{in} \quad L^1(0,T;L^1(\Omega)), \\
&|\boldsymbol{u}^j|^\alpha \xrightarrow[j\to\infty]{} |\boldsymbol{u}|^\alpha \quad \text{in} \quad L^1(0,T;L^1(\Omega)).
\end{aligned} \tag{7.18}$$

With respect to the turbulent diffusion term, by the estimate (6.34), we can infer the existence of $\varpi \in L^\rho(0,T;L^\rho(\Omega))$ such that

$$\nu_D(k^j)\nabla k^j \xrightarrow[j\to\infty]{} \varpi \quad \text{in} \quad L^\rho(0,T;L^\rho(\Omega)), \qquad \rho < \frac{d+2}{d+1}. \tag{7.19}$$

Then, we observe that from (6.21) and (6.23) one has

$$\Lambda(k^j) \xrightarrow[j\to\infty]{} \Lambda(k) \quad \text{in} \quad L^2(0,T;W_0^{1,2}(\Omega)). \tag{7.20}$$

Combining the assumptions (2.1) and (2.3) with (7.9), one has

$$\nu_D(k^j)\left(1+k^j\right)^{\frac{1+\delta}{2}} \xrightarrow[j\to\infty]{} \nu_D(k)\left(1+k\right)^{\frac{1+\delta}{2}} \quad \text{a.e. in} \quad Q_T. \tag{7.21}$$

Next, we can use (2.3) together with the Sobolev inequality and (6.15), with q given by (6.27), so that

$$\begin{aligned}
&\int_0^T \left\| \nu_D(k^j(t))\left(1+k^j(t)\right)^{\frac{1+\delta}{2}} \right\|_2^2 dt \leq C_1' \left(1 + \int_0^T \left\|k^j(t)\right\|_{1+\delta}^{1+\delta} dt\right) \\
&\leq C_2' \left(1 + \int_0^T \left\|\nabla k^j(t)\right\|_q^{1+\delta} dt\right) \leq C_1 + \frac{C_2}{\delta} \quad \forall\, \delta > 0 \ \text{small},
\end{aligned} \tag{7.22}$$

and for some positive constants $C_1' = C(C_D,\delta,\Omega,T)$, $C_2' = C(C_D,q,\delta,d,\Omega,T)$, $C_1 = C(C_2',K_2)$ and $C_2 = C(C_2',K_3)$. Note that for $\delta > 0$ sufficiently small, $1+\delta \leq q^*$, where q^* is the Sobolev conjugate of q given by (6.27). Due to the Vitali–Hahn–Saks theorem, (7.21) and (7.22) imply

$$\nu_D(k^j)\left(1+k^j\right)^{\frac{1+\delta}{2}} \xrightarrow[j\to\infty]{} \nu_D(k)\left(1+k\right)^{\frac{1+\delta}{2}} \quad \text{in} \quad L^2(0,T;L^2(\Omega)). \tag{7.23}$$

As a consequence of (6.21), (7.20) and (7.23), we can justify that

$$\begin{aligned}
&\int_{Q_T} \nu_D(k^j)\nabla k^j \cdot \boldsymbol{\omega}\, dxdt = \int_{Q_T} \nu_D(k^j)\left(1+k^j\right)^{\frac{1+\delta}{2}} \nabla\Lambda(k^j)\cdot\boldsymbol{\omega}\, dxdt \\
&\xrightarrow[j\to\infty]{} \int_{Q_T} \nu_D(k)\left(1+k\right)^{\frac{1+\delta}{2}} \nabla\Lambda(k)\cdot\boldsymbol{\omega}\, dxdt = \int_{Q_T} \nu_D(k)\nabla k\cdot\boldsymbol{\omega}\, dxdt
\end{aligned} \tag{7.24}$$

for all $\boldsymbol{\omega} \in C_0^\infty((0,T)\times\Omega)^d$. Hence, by virtue of the convergence (7.24) we can readily see that in (7.19) it must be $\varpi = \nu_D(k)\nabla k$, i.e.

$$\nu_D(k^j)\nabla k^j \xrightharpoonup[j\to\infty]{} \nu_D(k)\nabla k \quad \text{in} \quad L^\rho(0,T;L^\rho(\Omega)), \quad \rho < \frac{d+2}{d+1}. \tag{7.25}$$

On the other hand, we can combine (7.8) and (7.9) with (6.39) so that

$$k^j\boldsymbol{u}^j \xrightharpoonup[j\to\infty]{} k\boldsymbol{u} \quad \text{in} \quad L^\rho(0,T;L^\rho(\Omega)), \quad \rho < \max\left\{\frac{2(d+2)}{3d}, \frac{\alpha(d+2)}{d(\alpha+1)+2}\right\}. \tag{7.26}$$

Regarding the turbulence dissipation term, we can deduce from (2.1) and (7.9) that

$$\varepsilon(k^j) \xrightarrow[j\to\infty]{} \varepsilon(k) \quad \text{a.e. in} \quad Q_T. \tag{7.27}$$

And by using the assumption (2.5), together with the estimate (6.29), we can prove that

$$\int\limits_{Q_T} \left|\varepsilon(k^j)\right|^q dxdt \le C_1 \int\limits_{Q_T} \left|k^j\right|^{q(\vartheta+1)} dxdt \le \frac{C_2}{\delta} \quad \forall\,\delta>0 \text{ small}, \quad q < \frac{d+2}{d(\vartheta+1)}, \tag{7.28}$$

for some positive constants $C_1 = C(C_\varepsilon,\vartheta)$ and $C_2 = C(C_1,\delta,d,K_4,\Omega,T)$. Then, by the Vitali–Hahn–Saks theorem, we can see that (7.27) and (7.28) imply

$$\varepsilon(k^j) \xrightarrow[j\to\infty]{} \varepsilon(k) \quad \text{in} \quad L^q(0,T;L^q(\Omega)), \quad q < \frac{d+2}{d(\vartheta+1)}. \tag{7.29}$$

Note that, due to the assumption $(2.8)_2$, $\frac{d+2}{d(\vartheta+1)} > 1$.

Lastly, we can also combine (2.1), (2.9), (6.5) and (6.43), with (7.8)–(7.9), to show that

$$\nu_P(k^j)|\boldsymbol{u}^j|^\beta \xrightarrow[l\to\infty]{} \nu_P(k)|\boldsymbol{u}|^\beta \quad \text{a.e. in} \quad Q_T, \tag{7.30}$$

$$\int\limits_{Q_T} |\nu_P(k^j)|\boldsymbol{u}^j|^\beta|^\rho dxdt \le C, \quad \rho < \rho_1, \tag{7.31}$$

for some positive constant C not depending on j, and for ρ_1 is given in (6.45). In view of (7.30) and (7.31), we can use Vitali–Hahn–Saks theorem again so that

$$\nu_P(k^j)|\boldsymbol{u}^j|^\beta \xrightarrow[j\to\infty]{} \nu_P(k)|\boldsymbol{u}|^\beta \quad \text{in} \quad L^\rho(0,T;L^q(\Omega)), \quad \rho < \frac{r_u}{\beta}. \tag{7.32}$$

Now, using the convergence results (7.3)–(7.4), (7.5), (7.6), (7.18), (7.25), (7.26), (7.29) and (7.32), we can pass (6.4) to the limit $j\to\infty$ so that (2.27) is valid for all $w \in W_0^{1,\infty}(\Omega)$.

Moreover, reasoning as we did for (5.27), we also can show that

$$\boldsymbol{u}(0)=\boldsymbol{u}_{n,0} \quad \text{and} \quad k(0)=k_{n,0} \quad \text{in } \Omega. \tag{7.33}$$

The proof of Proposition 2.3 is thus concluded. □

8. CONCLUDING THE PROOF OF THEOREM 2.2

Note that, right at the beginning of the proof of Proposition 2.3, we discarded the subscript n, which will now be recovered so that we can proceed for the proof of Theorem 2.2.

In the previous sections, we have proven that for each $n \in \mathbb{N}$ there exists a couple of functions $(\boldsymbol{u}_n, k_n)$ satisfying (2.26) and (2.27). Using continuity arguments, integration in-time of (2.26) and (2.27), and (7.33), we can see that

$$\begin{aligned}
&-\int_0^T\int_\Omega \boldsymbol{u}_n\cdot\partial\boldsymbol{\psi}\,dxdt-\int_0^T\int_\Omega \Phi_n(|\boldsymbol{u}_n|^2)\boldsymbol{u}_n\otimes\boldsymbol{u}_n:\nabla\boldsymbol{\psi}\,dxdt\\
&+\int_0^T\int_\Omega \nu_T(k_n)\,\mathbf{D}(\boldsymbol{u}_n):\nabla\boldsymbol{\psi}\,dxdt+\int_0^T\int_\Omega \left(c_{Da}+c_{Fo}|\boldsymbol{u}_n|^{\alpha-2}\right)\boldsymbol{u}_n\cdot\boldsymbol{\psi}\,dxdt\\
&=\int_\Omega \boldsymbol{u}_{n,0}\cdot\boldsymbol{\psi}(0)\,dx+\int_0^T\int_\Omega \boldsymbol{g}\cdot\boldsymbol{\psi}\,dxdt
\end{aligned} \tag{8.1}$$

and

$$\begin{aligned}
&-\int_0^T\int_\Omega k_n\partial_t\varphi\,dxdt-\int_0^T\int_\Omega k_n\boldsymbol{u}_n\cdot\nabla\varphi\,dxdt\\
&+\int_0^T\int_\Omega \nu_D(k_n)\nabla k_n\cdot\nabla\varphi\,dxdt+\int_0^T\int_\Omega \varepsilon(k_n)\varphi\,dxdt\\
&=\int_\Omega k_{n,0}\varphi(0)\,dx+\int_0^T\int_\Omega \nu_T(k_n)|\mathbf{D}(\boldsymbol{u}_n)|^2\varphi\,dxdt+\int_0^T\int_\Omega \nu_P(k_n)|\boldsymbol{u}_n|^\beta\varphi\,dxdt
\end{aligned} \tag{8.2}$$

are verified for all $\boldsymbol{\psi}\in C^\infty(Q_T)$, with $\operatorname{div}\boldsymbol{\psi}=0$ in Q_T and $\operatorname{supp}\boldsymbol{\psi}\subset\subset\Omega\times[0,T)$, and for all $\varphi\in C^\infty(Q_T)$, with $\varphi\geq 0$ in Q_T and $\operatorname{supp}\varphi\subset\subset\Omega\times[0,T)$.

Using (2.26) and (2.27), and proceeding as we did in the previous sections, we can show that the estimates (4.7)–(4.8), (6.6), (6.15), (6.29), (6.34), (6.39), (6.43) and (6.48) hold for $\boldsymbol{u}_n$ and k_n. As a consequence, and in view of the Banach–Alaoglu theorem, we have for some subsequences

$$\boldsymbol{u}_n \xrightarrow[n\to\infty]{*} \boldsymbol{u} \ \text{ in } \ L^\infty(0,T;\mathbf{H}), \tag{8.3}$$

$$\boldsymbol{u}_n \xrightarrow[n\to\infty]{} \boldsymbol{u}, \ \text{ in } L^2(0,T;\mathbf{V})\cap L^{r_u}(0,T;L^{r_u}(\Omega)^d), \tag{8.4}$$

$$k_n \xrightarrow[n\to\infty]{*} k \ \text{ in } \ L^\infty(0,T;\mathcal{M}(\Omega)),$$

$$k_n \xrightarrow[n\to\infty]{} k \ \text{ in } \ L^q(0,T;W_0^{1,q}(\Omega)), \quad 1<q<\frac{d+2}{d+1}, \tag{8.5}$$

$$k_n \xrightarrow[n\to\infty]{} k \ \text{ in } \ L^q(0,T;L^q(\Omega))\cap L^{\vartheta+1}(0,T;L^{\vartheta+1}(\Omega)), \quad 1<q<\frac{d+2}{d}, \tag{8.6}$$

$$\partial_t k_n \xrightarrow[n\to\infty]{} \partial_t k \ \text{ in } \ \mathcal{M}(0,T;W^{-1,q'}(\Omega)), \tag{8.7}$$

$$q>\max\left\{d+2,\frac{2(d+2)}{4-d},\frac{\alpha(d+2)}{2\alpha-d-2},\frac{d\rho'}{d+\rho'}\right\},$$

From (8.7) one immediately has (2.20).

On the other hand, from (2.26) we can infer that for all $t\in(0,T)$

$$\begin{aligned}\partial_t\boldsymbol{u}_n(t) = &-\mathbf{div}\big(\Phi_n(|\boldsymbol{u}_n(t)|^2)\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t)\big)+\mathbf{div}\big(\nu_T(k_n(t))\,\mathbf{D}(\boldsymbol{u}_n(t))\big)\\ &-\big(c_{Da}+c_{Fo}|\boldsymbol{u}_n(t)|^{\alpha-2}\big)\,\boldsymbol{u}_n(t)+\mathbf{g}(t)\end{aligned} \tag{8.8}$$

holds in the distribution sense on $\mathbf{Y}'$, where $\mathbf{Y}'$ denotes the dual space of

$$\mathbf{Y}:=\mathbf{V}\cap L^\alpha(\Omega)^d.$$

Using (2.2), together with the counterparts of (4.7)–(4.8), we can prove that

$$\int_0^T \|\mathbf{div}\,(\nu_T(k_n(t))\,\mathbf{D}(\boldsymbol{u}_n(t)))\|^2_{W^{-1,2}(\Omega)^d}\,dt\le C, \tag{8.9}$$

for some positive constant $C=C(d,C_T,\Theta_1,\Omega,T)$, and where $W^{-1,s'}(\Omega)^d$ denotes the dual space of $W_0^{1,s}(\Omega)^d$. By using (2.24), the Hölder inequality and the counterpart of (4.7), we can show that

$$\begin{aligned}&\int_0^T \big\|\mathbf{div}\,\big(\Phi_n(|\boldsymbol{u}_n(t)|^2)\boldsymbol{u}_n(t)\otimes\boldsymbol{u}_n(t)\big)\big\|^{r_1}_{W^{-1,2}(\Omega)^d}\,dt\le C,\\ &r_1=\begin{cases}\frac{d}{3}, & \text{if } d\neq 2,\\ 1+\frac{\delta}{4+\delta},\ \delta>0, & \text{if } d=2,\end{cases}\end{aligned} \tag{8.10}$$

for some positive constant $C=C(d,\Theta_2,\Omega,T)$. Observe that assumption $d=2$ or $d=3$ (and therefore $d<4$) in the space dimension, assures that $r_1>1$. It can be proved that the body forces $\mathbf{g}$ and the Darcy term (which includes the Forchheimer term when $\alpha = 2$) of (2.26) are bounded in $L^2(0,T;W^{-1,2}(\Omega)^d)$. For the Forchheimer term, we can use the Hölder inequality together with the counterpart of (4.7), to show that

$$\int_0^T \big\| |\boldsymbol{u}_n(t)|^{\alpha-2}\boldsymbol{u}_n(t)\big\|^{r_2}_{W^{-1,2}(\Omega)^d}\,dt \leq C, \quad r_2 = \begin{cases} \frac{4}{3\alpha-8}, & \text{if } d=3,\\ \frac{2\delta}{(\alpha-2)(2+\delta)-\delta},\ \delta>0, & \text{if } d=2, \end{cases} \tag{8.11}$$

for some positive constant $C = C(s,\alpha,d,c_{Fo},\Theta_1,\Omega,T)$. Note that assumptions (2.16)–(2.17) assure that $r_2 > 1$.

Now, using (8.8), (8.9), (8.10) and (8.11), we prove that

$$\begin{gathered} \int_0^T \|\partial_t \boldsymbol{u}_n(t)\|^r_{W^{-1,2}(\Omega)^d}\,dt \leq C, \\ r = \begin{cases} \min\left\{\frac{4}{3}, \frac{4}{3\alpha-8}\right\}, & \text{if } d=3,\\ \min\left\{1+\frac{\delta}{4+\delta}, \frac{2(1+\delta)}{\alpha(3+\delta)-7-3\delta}\right\},\ \delta>0, & \text{if } d=2, \end{cases} \end{gathered} \tag{8.12}$$

holds for some positive constant that does not depend on n. Combining (8.12) with the Banach-Alaoglu theorem, we also have for some subsequence

$$\partial_t \boldsymbol{u}_n \xrightarrow[n\to\infty]{} \partial_t \boldsymbol{u} \ \text{ in } \ L^r(0,T;W^{-1,2}(\Omega)^d), \tag{8.13}$$

which proves (2.19). Taking into account (8.12), we can justify, with the same reasoning used for obtaining (7.6)–(7.7) and (7.8)–(7.9), the existence of subsequences such that

$$\boldsymbol{u}_n \xrightarrow[n\to\infty]{} \boldsymbol{u} \ \text{ in } \ L^q(0,T;L^q(\Omega)^d), \quad 1 \leq q < r_u,$$

$$k_n \xrightarrow[n\to\infty]{} k \ \text{ in } \ L^q(0,T;L^q(\Omega)), \quad 1 \leq q < r_k,$$

$$\boldsymbol{u}_n \xrightarrow[n\to\infty]{} \boldsymbol{u} \quad \text{a.e. in } \ Q_T, \tag{8.14}$$

$$k_n \xrightarrow[n\to\infty]{} k \quad \text{a.e. in } \ Q_T. \tag{8.15}$$

By the convergence results (8.4), (8.14) and (8.15), we also can show that

$$\Phi_n(|u_n|^2)\boldsymbol{u}_n \otimes \boldsymbol{u}_n \xrightarrow[n\to\infty]{} \boldsymbol{u}\otimes\boldsymbol{u} \ \text{ in } \ L^{\frac{d+2}{d}}(0,T;L^{\frac{d+2}{d}}(\Omega)^d), \tag{8.16}$$

$$|\boldsymbol{u}_n|^{\alpha-2}\boldsymbol{u}_n \xrightarrow[n\to\infty]{} |\boldsymbol{u}|^{\alpha-2}\boldsymbol{u} \ \text{ in } \ L^{\alpha'}(0,T;L^{\alpha'}(\Omega)^d), \tag{8.17}$$

$$\nu_T(k_n)\mathbf{D}(\boldsymbol{u}_n) \xrightarrow[n\to\infty]{} \nu_T(k)\mathbf{D}(\boldsymbol{u}) \ \text{ in } \ L^2(0,T;L^2(\Omega)^{d\times d}), \tag{8.18}$$

$$\sqrt{\nu_T(k_n)}\mathbf{D}(\boldsymbol{u}_n) \xrightarrow[n\to\infty]{} \sqrt{\nu_T(k)}\mathbf{D}(\boldsymbol{u}) \ \text{ in } \ L^2(0,T;L^2(\Omega)^{d\times d}). \tag{8.19}$$

Note that in the convergence result (8.16), we also have used the definition of the function Φ given at (2.24)–(2.25).

Then, passing the equation (8.1) to the limit $n \to \infty$, using for that purpose the convergence results (2.22), (8.4), (8.13) and (8.16)–(8.18), we prove the validity of (2.13).

By a standard procedure (see e.g. [31, Lemma III.1.2]), we can invoke (8.3)–(8.4), (8.13) and (8.19) to prove that $\boldsymbol{u} \in L^\infty(0,T;\mathbf{H})$ is weakly continuous with values in $\mathbf{H}$, i.e. (2.18) holds true, and hence $(1.4)_1$ is meaningful.

On the other hand, arguing exactly as we did for (7.25), (7.26), (7.29), and (7.32), using in this case (8.14)–(8.15) instead, we can show that

$$\nu_D(k_n)\nabla k_n \xrightarrow[n\to\infty]{} \nu_D(k)\nabla k \quad \text{in} \quad L^\rho(0,T;L^\rho(\Omega)), \quad \rho < \frac{d+2}{d+1}, \tag{8.20}$$

$$k_n\boldsymbol{u}_n \xrightarrow[n\to\infty]{} k\boldsymbol{u} \quad \text{in} \quad L^\rho(0,T;L^\rho(\Omega)), \quad \rho < \max\left\{\frac{2(d+2)}{3d}, \frac{\alpha(d+2)}{d(\alpha+1)+2}\right\}, \tag{8.21}$$

$$\varepsilon(k_n) \xrightarrow[n\to\infty]{} \varepsilon(k) \quad \text{in} \quad L^q(0,T;L^q(\Omega)), \quad q < \frac{d+2}{d(\vartheta+1)}, \tag{8.22}$$

$$\nu_P(k_n)|\boldsymbol{u}_n|^\beta \xrightarrow[n\to\infty]{} \nu_P(k)|\boldsymbol{u}|^\beta \quad \text{in} \quad L^\rho(0,T;L^q(\Omega)), \quad \rho < \frac{r_u}{\beta}. \tag{8.23}$$

Moreover, (8.19) and the weak lower semicontinuity of the norm imply

$$\int_0^T\int_\Omega \nu_T(k)|\mathbf{D}(\boldsymbol{u})|^2\varphi\,dxdt \le \liminf_{n\to\infty}\int_0^T\int_\Omega \nu_T(k_n)|\mathbf{D}(\boldsymbol{u}_n)|^2\varphi\,dxdt. \tag{8.24}$$

Finally, using the convergence results (2.22), (8.5), (8.7), (8.20)–(8.23) and (8.24), we can pass (8.2) to the limit $n \to \infty$ and we obtain (2.14).

9. ATTAINMENT OF THE INITIAL CONDITIONS

Proceeding as we did for $(5.27)_1$, by the same standard procedure, we can combine (2.26) and (8.1), now with (2.23) and $(7.33)_1$, and with (8.4) and (8.16)–(8.18), to show that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in the sense of (2.15).

For the next step, we need the following results

$$k_n \ge 0, \quad \varepsilon(k_n) \ge 0 \quad \text{a.e. in } Q_T, \tag{9.1}$$

that can be proved by repeating the same arguments used to show (5.29) and (5.30).

To prove that $k(0) = k_0$, also in the sense of (2.15), we start by integrating (2.27) between 0 and $t \in (0,T)$, and then we invoke the same arguing used to prove (6.9), together with $(7.33)_2$ and (9.1), so that

$$\|\mathcal{H}_1(k_n(t))\|_1 \le \|\mathcal{H}_1(k_{n,0})\|_1 + \int_0^t\int_\Omega \nu_T(k_n)|\mathbf{D}(\boldsymbol{u}_n)|^2 dxdt + \int_0^t\int_\Omega \nu_P(k_n)|\boldsymbol{u}_n|^\beta dxdt.$$

Recall that $\mathcal{H}_1$ is the function defined in (6.8). Combining the Fatou lemma with (2.22), (8.15), (8.23) and (8.24), we obtain

$$\|\mathcal{H}_1(k(t))\|_1 \leq \|\mathcal{H}_1(k_0)\|_1 + \int_0^t \int_\Omega \nu_T(k)|\mathbf{D}(\boldsymbol{u})|^2 dxdt + \int_0^t \int_\Omega \nu_P(k)|\boldsymbol{u}|^\beta dxdt.$$

Hence,

$$\limsup_{t\to 0^+} \|\mathcal{H}_1(k(t))\|_1 \leq \|\mathcal{H}_1(k_0)\|_1. \tag{9.2}$$

We now test (2.27) with

$$\omega = \mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^\lambda \phi, \quad \phi \in C_0^\infty(\Omega), \quad \phi \geq 0 \text{ a.e. in } \Omega, \tag{9.3}$$

where λ is a constant to be defined later on, $\mathcal{T}_1(k_n)$ is the truncation of k_n defined in (6.5) for $n = 1$, and $\mathcal{H}_1$ is the primitive function of $\mathcal{T}_1$ defined in (6.8). Note that, since $\phi \in C_0^\infty(\Omega)$ and

$$\mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^\lambda = \begin{cases} \frac{(k_n)^{1+2\lambda}}{2^\lambda}, & k_n < 1, \\ 0, & k_n \geq 1, \end{cases}$$

the function ω given by (9.3) is in fact an admissible test function for (2.27) whenever $\lambda \geq -\frac{1}{2}$. Integrating the resulting equation between 0 and t, we obtain after some calculations,

$$\begin{aligned}
&\frac{1}{\lambda+1}\int_\Omega \mathcal{H}_1(k_n(t))^{\lambda+1}\phi\, dx - \frac{1}{\lambda+1}\int_0^t\int_\Omega \mathcal{H}_1(k_n)^{\lambda+1}\boldsymbol{u}^j\cdot\nabla\phi\, dxd\tau \\
&+ \int_0^t\int_\Omega \nu_D(k_n)\Big[\mathcal{T}_1'(k_n)\mathcal{H}_1(k_n)^\lambda + \lambda\mathcal{T}_1(k_n)^2\mathcal{H}_1(k_n)^{\lambda-1}\Big]|\nabla k_n|^2\phi\, dxd\tau \\
&+ \int_0^t\int_\Omega \nu_D(k_n)\mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^\lambda\nabla k_n\cdot\nabla\phi\, dxd\tau + \int_0^t\int_\Omega \varepsilon(k_n)\mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^\lambda\phi\, dxd\tau \\
&= \frac{1}{\lambda+1}\int_\Omega \mathcal{H}_1(k_{n,0})^{\lambda+1}\phi\, dx + \int_0^t\int_\Omega \nu_T(k_n)|\mathbf{D}(\boldsymbol{u}^j)|^2\mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^\lambda\phi\, dxd\tau \\
&\quad + \int_0^t\int_\Omega \nu_P(k_n)|\boldsymbol{u}^j|^\beta\mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^\lambda\phi\, dxd\tau.
\end{aligned} \tag{9.4}$$

Observe that, by (6.5) and (6.8), we have

$$\mathcal{T}_1'(k_n)\mathcal{H}_1(k_n)^\lambda + \lambda\mathcal{T}_1(k_n)^2\mathcal{H}_1(k_n)^{\lambda-1} = \begin{cases} \frac{(k_n)^{2\lambda}}{2^\lambda}(1+2\lambda), & k_n < 1, \\ 0, & k_n \geq 1. \end{cases}$$

Due to $(9.1)_1$, we have $\mathcal{T}_1'(k_n)\mathcal{H}_1(k_n)^\lambda + \lambda\mathcal{T}_1(k_n)^2\mathcal{H}_1(k_n)^{\lambda-1} \leq 0$, whenever $\lambda \leq -\frac{1}{2}$. In view of this, and since (2.3) holds and $\phi \geq 0$, we can justify that the third l.h.s. term in (9.4) is non-positive. From the condition imposed on λ, we see that it must be $\lambda = -\frac{1}{2}$ in (9.3). For such λ, we readily see that $0 \leq \mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^\lambda \leq \sqrt{2}$, which together with (2.2)–(2.4) and with the fact that $\phi \geq 0$, we can justify that the last two r.h.s. terms of (9.4) are nonnegative. As a consequence, we obtain from (9.4),

$$\begin{aligned}
&\int_\Omega \mathcal{H}_1(k_n(t))^{\frac{1}{2}}\phi\,dx - \int_0^t\int_\Omega \mathcal{H}_1(k_n)^{\frac{1}{2}}\boldsymbol{u}^j\cdot\nabla\phi\,dxd\tau \\
&+\frac{1}{2}\int_0^t\int_\Omega \nu_D(k_n)\mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^{-\frac{1}{2}}\nabla k_n\cdot\nabla\phi\,dxd\tau \\
&+\frac{1}{2}\int_0^t\int_\Omega \varepsilon(k_n)\mathcal{T}_1(k_n)\mathcal{H}_1(k_n)^{-\frac{1}{2}}\phi\,dxd\tau \\
&\geq \int_\Omega \mathcal{H}_1(k_{n,0})^{\frac{1}{2}}\phi\,dx.
\end{aligned}$$

Using $(7.33)_2$ together with (2.22), (8.6) and (8.20)–(8.22), we get

$$\begin{aligned}
&\int_\Omega \mathcal{H}_1(k(t))^{\frac{1}{2}}\phi\,dx - \int_0^t\int_\Omega \mathcal{H}_1(k)^{\frac{1}{2}}\boldsymbol{u}\cdot\nabla\phi\,dxd\tau \\
&+\frac{1}{2}\int_0^t\int_\Omega \nu_D(k)\mathcal{T}_1(k)\mathcal{H}_1(k)^{-\frac{1}{2}}\nabla k\cdot\nabla\phi\,dxd\tau + \frac{1}{2}\int_0^t\int_\Omega \varepsilon(k)\mathcal{T}_1(k)\mathcal{H}_1(k)^{-\frac{1}{2}}\phi\,dxd\tau \\
&\geq \int_\Omega \mathcal{H}_1(k_0)^{\frac{1}{2}}\phi\,dx
\end{aligned}$$

for a.e. $t \in (0,T)$. Taking the lim inf, as $t \to 0^+$, and using a density argument, one has

$$\liminf_{t\to 0^+}\int_\Omega \mathcal{H}_1(k(t))^{\frac{1}{2}}\phi\,dx \geq \int_\Omega \mathcal{H}_1(k_0)^{\frac{1}{2}}\phi\,dx \quad \forall\phi\in\ L^2(\Omega), \tag{9.5}$$

with $\phi \geq 0$ a.e. in Ω.

Now, we can use (9.2) and (9.5), together with the properties of lim sup and lim inf, and with $(9.1)_1$ and (2.22), to prove that

$$\begin{aligned}
&\lim_{t\to 0^+}\left\|\mathcal{H}_1(k(t))^{\frac{1}{2}}-\mathcal{H}_1(k_0)^{\frac{1}{2}}\right\|_2^2\\
&=\lim_{t\to 0^+}\left(\|\mathcal{H}_1(k(t))\|_1+\|\mathcal{H}_1(k_0)\|_1-2\int_\Omega \mathcal{H}_1(k(t))^{\frac{1}{2}}\mathcal{H}_1(k_0)^{\frac{1}{2}}dx\right)\\
&\leq \limsup_{t\to 0^+}\|\mathcal{H}_1(k(t))\|_1+\|\mathcal{H}_1(k_0)\|_1-2\liminf_{t\to 0^+}\int_\Omega \mathcal{H}_1(k(t))^{\frac{1}{2}}\mathcal{H}_1(k_0)^{\frac{1}{2}}dx\\
&\leq \|\mathcal{H}_1(k_0)\|_1+\|\mathcal{H}_1(k_0)\|_1-2\int_\Omega \mathcal{H}_1(k_0)dx=0.
\end{aligned} \tag{9.6}$$

As a consequence of (9.6), we achieve to $k(0)=k_0$ in the sense of (2.15). This concludes the proof of Theorem 2.2. □

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REFERENCES

[1] B.V. Antohe, J.L. Lage, *A general two-equation macroscopic turbulence model for incompressible flow in porous media*, Int. J. Heat Mass Transf. **13** (1997), 3013–3024.

[2] S.N. Antontsev, J.I. Díaz, H.B. de Oliveira, *On the confinement of a viscous fluid by means of a feedback external field*, C. R. Méc. Acad. Sci. Paris **330** (2002), no. 12, 797–802.

[3] S.N. Antontsev, J.I. Díaz, H.B. de Oliveira, *Stopping a viscous fluid by a feedback dissipative field: I. The stationary Stokes problem*, J. Math. Fluid Mech. **6** (2004), no. 4, 439–461.

[4] S.N. Antontsev, J.I. Díaz, H.B. de Oliveira, *Stopping a viscous fluid by a feedback dissipative field: II. The stationary Navier–Stokes problem*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **15** (2004), no. 3-4, 257–461.

[5] S.N. Antontsev, H.B. de Oliveira, *Finite time localized solutions of fluid problems with anisotropic dissipation*, [in:] I.N. Figueiredo, J.F. Rodrigues, L. Santos (eds), Free Boundary Problems, International Series of Numerical Mathematics, vol. 154, Birkhäuser, Basel, 2006.

[6] E. Aulisa, L. Bloshanskaya, L. Hoang, A. Ibragimov, *Analysis of generalized Forchheimer flows of compressible fluids in porous media*, J. Math. Phys. **50** (2009), 103102.

[7] P. Baras, M. Pierre, *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18** (1984), no. 1–2, 111–149.

[8] L. Boccardo, T. Gallouët, *Strongly nonlinear elliptic equations having natural growth terms and L^1 data*, Nonlinear Anal. **19** (1992), no. 6, 573–579.

[9] M. Bulíček, E. Feireisl, J. Málek, *A Navier–Stokes–Fourier system for incompressible fluids with temperature dependent material coefficients*, Nonlinear Anal. Real World Appl. **10** (2009), no. 2, 992–1015.

[10] M. Bulíček, R. Lewandowski, J. Málek, *On evolutionary Navier–Stokes–Fourier type systems in three spatial dimensions*, Comment. Math. Univ. Carolin. **52** (2011), no. 1, 89–114.

[11] L. Caffarelli, R. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. Pure Appl. Math. **35** (1982), no. 6, 771–831.

[12] T. Chacón-Rebollo, R. Lewandowski, *Mathematical and Numerical Foundations of Turbulence Models and Applications*, Springer New York, 2014.

[13] E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[14] P. Dreyfuss, *Results for a turbulent system with unbounded viscosities: Weak formulations, existence of solutions, boundedness and smoothness*, Nonlinear Anal. **68** (2008), no. 6, 1462–1478.

[15] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations: Steady-State Problems*, Springer New York, 2011.

[16] T. Gallouët, J. Lederer, R. Lewandowski, F. Murat, L. Tartar, *On a turbulent system with unbounded eddy viscosities*, Nonlinear Anal. **52** (2003), no. 4, 1051–1068.

[17] J. Lederer, R. Lewandowski, *A RANS 3D model with unbounded eddy viscosities*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 3, 413–441.

[18] M.J.S. de Lemos, *Turbulence in Porous Media*, 2nd ed., Elsevier, Waltham, MA, 2012.

[19] R. Lewandowski, *The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier–Stokes equation with an eddy viscosity*, Nonlinear Anal. **28** (1997), no. 2, 393–417.

[20] B. Mohammadi, O. Pironneau, *Analysis of the k-epsilon Turbulence Model*, Wiley-Masson, Paris, 1994.

[21] A. Nakayama, F. Kuwahara, *A macroscopic turbulence model for flow in a porous medium*, J. Fluid Eng. **121** (1999), no. 2, 427–433.

[22] A. Mielke, J. Naumann, *Global-in-time existence of weak solutions to Kolmogorov's two-equation model of turbulence*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 4, 321–326.

[23] H.B. de Oliveira, A. Paiva, *On a one equation turbulent model with feedbacks*, [in:] S. Pinelas, Z. Došlá, O. Došlý, P. Kloeden (eds), Springer Proc. Math. Stat., vol. 164, Springer Cham, 2016, 51–61.

[24] H.B. de Oliveira, A. Paiva, *Existence for a one-equation turbulent model with strong nonlinearities*, J. Elliptic Parabol. Equ. **3** (2017), no. 1–2, 65–91.

[25] H.B. de Oliveira, A. Paiva, *A stationary one-equation turbulent model with applications in porous media*, J. Math. Fluid Mech. **20** (2018), no. 2, 263–287.

[26] H.B. de Oliveira, *A note on the existence for a model of turbulent flows through porous media*, [in:] S. Pinelas, T. Caraballo, P. Kloeden, J. Graef (eds), Springer Proc. Math. Stat., vol. 230, Springer Cham, 2018, 21–38.

[27] H.B. de Oliveira, A. Paiva, *Partial regularity of the solutions to a turbulent problem in porous media*, Proc. Amer. Math. Soc. **147** (2019), no. 9, 3961–3981.

[28] M.H.J. Pedras, M.J.S. de Lemos, *On the definition of turbulent kinetic energy for flow in porous media*, Int. Commun. Heat Mass Transfer **27** (2000), no. 2, 211–220.

[29] J.-M. Rakotoson, *Quasilinear elliptic problems with measures as data*, Differ. Integral Equ. **4** (1991), no. 3, 449–457.

[30] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. **146** (1987), no. 4, 65–96.

[31] R. Temam, *Navier–Stokes Equations*, 2nd ed., Elsevier North-Holland, New York, 1979.

[32] K. Vafai (ed.), *Handbook of Porous Media*, 2nd ed., CRC Press, Boca Raton, FL (2005).

[33] B. Wood, X. He, S.V. Apte, *Modeling turbulent flows in porous media*, Annu. Rev. Fluid Mech. **52** (2020), no. 1, 171–203.

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