Int. J. Appl. Math. Comput. Sci., 2021, Vol. 31, No. 2, 187–194
DOI: 10.34768/amcs-2021-0013

ON A STABLE SOLUTION OF THE PROBLEM OF DISTURBANCE REDUCTION

VYACHESLAV I. MAKSIMOV [a]

[a] Krasovskii Institute of Mathematics and Mechanics
Ural Branch of the Russian Academy of Sciences
16 S. Kovalevskaya St., Yekaterinburg 620990, Russia
e-mail: maksimov@imm.uran.ru

We study the problem of active reduction of the influence of a disturbance on the output of a linear control system. We consider a system of linear differential equations under the action of an unknown disturbance and a control to be formed. Our goal is to design an algorithm for reducing the disturbance by means of an appropriate control on the basis of inaccurate measurements of the system phase coordinates. This algorithm should form a feedback control that would guarantee that the trajectory of a given system tracks the trajectory of the reference system, i.e., the system described by the same differential equations but with zero control and disturbance. We present an algorithm for solving this problem. The algorithm, based on the constructions of guaranteed control theory, is stable with respect to informational noises and computational errors.

Keywords: disturbance reduction, dynamical controlled system, guaranteed control theory.

1. Introduction

Control problems in the presence of unknown dynamical disturbances form an important part of control theory. To solve such problems, it is necessary to apply the principle of feedback control, which allows us to use all available current information about the system to make decisions on its control in real time. One of the actual problems is that of forming a control providing the reduction (compensation) of an unknown disturbance acting on the system.

There are a lot of approaches to investigate this problem. For example, within the framework of H_2-theory, the control is still a feedback that attenuates the effect of the disturbance with respect to a suitable cost functional (Kwakernaak, 2002). The active noise control uses an estimate of the disturbance acting on the system in order to remove its effect from the output (Gan and Kuo, 2002). In some applications, it is possible to measure directly all the noises and a feed-forward compensator can be used. Other applications require that, instead of this, the noises should be estimated based on their effect on the system. The problem of disturbance reduction with the information on future values of the disturbance has also been studied (see Willems, 1982). In addition, it is possible to construct a disturbance compensator that optimally shares the stationary state distribution satisfying given control specifications (see Falsone *et al.*, 2019). Recently, the active disturbance rejection control method has been intensively developed, e.g., by Yuan *et al.* (2019).

In this paper, we consider a control problem for a system of linear differential equations subject to the influence of an unknown disturbance. The problem consists in constructing an algorithm for forming a feedback control that would guarantee a given quality of the controlled process. Namely, the trajectory of a given system influenced by an unknown disturbance should track the trajectory of a reference system. The latter system is described by the same differential equations but with zero control and disturbance. Thus, we consider the problem of disturbance reduction.

In the present study, we investigate a specific issue: it is assumed that the phase coordinates of both systems are measured at discrete, frequent enough, time instants. Due to this assumption, it is impossible to solve the problem of disturbance reduction without errors, i.e., it is impossible to track accurately the trajectory of a reference system by that of a given controlled system. Taking into account this feature, we design an algorithm that is stable with respect to informational noises and computational errors. It is based on the constructions of the theory of recursive deconvolution and of feedback control.

The deconvolution problem is fundamental in applied sciences and has been studied in numerous papers. The idea of using a deconvolution technique in order to identify and then to cancel a disturbance has been employed in some applications (Gan and Kuo, 2002; Yu and Hu, 2001). It should be noted that all of these papers are oriented to applications. One of the approaches to solving the dynamical reconstruction problem has been developed by Osipov and Kryazhimskii (1995), Maksimov (2002; 2016), Favini *et al.* (2004), Maksimov and Mordukhovich (2017), Pandolfi (2007), Maksimov and Tröltzsch (2020) or Keesman and Maksimov (2008). The essence of this approach is that an input reconstruction algorithm is represented as a control algorithm for some artificial dynamical system (a model). Given current observations of the system, the model input is chosen in such a way that its realization in time obtained by a regularization principle guarantees the stability of the numerical method.

Situations in which it is necessary to decrease the influence of a disturbance or even to nullify its action arise in applied problems rather frequently. We note only two papers devoted to this topic; both are supplied with an extensive bibliography. In the work of Wasilewski *et al.* (2019), a new algorithm of adaptive control of torsional vibrations induced by switched nonlinear disturbances is suggested. In the paper by Cayero *et al.* (2019), on the basis of designing disturbance observers with decreasing their influence, a method for solving the problem of control of unmanned aerial vehicles is constructed.

2. Problem statement and the solution method

Consider the system of linear differential equations

$$\dot{x}(t) = Ax(t) + B(u(t) - v(t)) + f(t), \quad t \in T = [0, \vartheta], \tag{1}$$

with the initial state $x(0) = x_0$. Here, $0 < \vartheta < +\infty$, $x \in \mathbb{R}^n$, $u, v \in \mathbb{R}^r$, $f(\cdot) \in L_2(T; \mathbb{R}^n)$ is a given function, v is a disturbance, u is a control, A and B are matrices of the appropriate dimensions. The problem under consideration consists in the following. Some unknown disturbance $v(\cdot)$ acts on the system (1). At discrete, frequent enough, times

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m \quad (\tau_0 = 0, \tau_{i+1} = \tau_i + \delta, \tau_m = \vartheta),$$

a phase state $x(\tau_i) = x(\tau_i; x_0, u(\cdot), v(\cdot))$ of the system (1) is measured. Here and below, $x(\cdot; x_0, u(\cdot), v(\cdot))$ is the solution of the system (1) corresponding to the initial state x_0, control $u(\cdot)$, and disturbance $v(\cdot)$. The states $x(\tau_i), i \in [0 : m-1]$, are measured with errors. The measurements results, the vectors $\xi_i^h \in \mathbb{R}^n$, satisfy the inequalities

$$|x(\tau_i) - \xi_i^h|_n \le h. \tag{2}$$

Here, the number $h \in (0, 1)$ characterizes the accuracy of measurements, and the symbol $|\cdot|_n$ stands for the Euclidean norm in the space $\mathbb{R}^n$.

Our goal is to design an algorithm for the reduction of the unknown disturbance $v(\cdot) \in L_2(T; \mathbb{R}^r)$ by using a control $u(\cdot)$ on the basis of inaccurate measurements of $x(\tau_i)$. In other words, the task is to design a feedback algorithm that generates in real time a function $u^h = u^h(\cdot)$ such that the solution of the system (1) tracks the solution $x_1(\cdot) = x_1(\cdot; x_0, 0, 0)$ of the system

$$\dot{x}_1(t) = Ax_1(t) + f(t) \tag{3}$$

with the initial state $x_1(0) = x_0$ in the space $W^{1,2}(T; \mathbb{R}^n) = \{p(\cdot) \in L_2(T; \mathbb{R}^n) : \dot{p}(\cdot) \in L_2(T; \mathbb{R}^n)\}$.

Let us describe a method for solving the problem under consideration. We assume that the states $x_1(\tau_i)$, $i \in [0 : m-1]$, are measured with errors. The measurements results, the vectors $\psi_i^h \in \mathbb{R}^n$, satisfy the inequalities

$$|x_1(\tau_i) - \psi_i^h|_n \le \nu_i^h. \tag{4}$$

For any $h \in (0, 1)$, let us fix a family of partitions of the interval T by control moments of time $\tau_{h,i}$:

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = 0, \\ \tau_{h,m_h} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h), \quad \delta(h) \in (0, 1). \tag{5}$$

The control $u = u^h(\cdot)$ in the system (1) is defined by a control law $U(\cdot, \cdot, \cdot) : T \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^r$, which is constructed in such a way that the control $u^h(\cdot)$ of the form

$$u^h(t) = u_i^h = U(\tau_i, \xi_i^h, \psi_i^h) \text{ for a.a. } t \in [\tau_i, \tau_{i+1}) \\ (i \in [0 : m_h - 1], \quad \tau_i = \tau_{h,i}) \tag{6}$$

guarantees a small deviation of the solution $x^h(\cdot)$ of the system (1) from the solution $x_1(\cdot)$ of the system (3) in the metric of the space $W^{1,2}(T; \mathbb{R}^n)$. Here and below, $x^h(\cdot)$ is the solution of the system (1) generated by the control $u = u^h(\cdot)$ of the form (6), i.e., $x^h(\cdot) = x(\cdot; x_0, u^h(\cdot), v(\cdot))$ is the solution of the system

$$\dot{x}(t) = Ax(t) + B(u^h(t) - v(t)) + f(t), \quad t \in T. \tag{7}$$

3. Solution algorithm

Let us describe the solution algorithm for the above problem. Let a family Δ_h (see (5)) and a function $\alpha(h) : (0, 1) \to (0, 1)$ be fixed. Let $\mathcal{X}(t)$ be the fundamental matrix of the system $\dot{x}(t) = Ax(t)$. Then the inequality

$$|\mathcal{X}(t)| \le \exp\{|A|t\}, \quad t \ge 0, \tag{8}$$

is valid. Here, the symbol $|\cdot|$ stands for the Euclidean norm of a matrix. We assume that the matrix A is unknown, but its estimate, i.e., the value $\omega \geq |A|$, is known.

Before starting the work of the algorithm, we fix the value $h \in (0,1)$, a number $\alpha = \alpha(h)$ and a partition $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$ of the form (5). The work of the algorithm is decomposed into $m-1$ ($m = m_h$) steps. First, at the time τ_0, we select the vector u_0^h by the formula

$$u_0^h = U(\tau_0, \xi_0^h, \psi_0^h) = -\alpha^{-1}\exp\{-2\omega\tau_1\}(B'B)^+B'(\xi_0^h - \psi_0^h).$$

Here, the prime means transposition, while the symbol $(B'B)^+$ stands for the pseudo inverse matrix for the matrix $B'B$. Then, for all $t \in \delta_0 = [0, \delta)$, the control $u^h(t) = u_0^h$ is taken as the input of the system (7). At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $i \in [1 : m-1]$, $\tau_i = \tau_{h,i}$, the following actions take place. At the time τ_i, the vector u_i^h is calculated by the formula (6), in which

$$U(\tau_i, \xi_i^h, \psi_i^h) = -\alpha^{-1}\exp\{-2\omega\tau_{i+1}\}(B'B)^+B'(\psi_i^h - \xi_i^h). \tag{9}$$

Then, for all $t \in \delta_i$, the control $u^h(t)$ of the form (6), (9) is taken as the input of the system (7). As a result, under the action of such control and disturbance $v(\cdot)$, the system (7) passes from the state $x^h(\tau_i)$ to the state $x^h(\tau_{i+1})$. The procedure stops at time ϑ.

Let us show that the feedback $U(\cdot,\cdot,\cdot)$ of the form (9) solves the problem of disturbance reduction. Before proceeding to the proof of the theorem, we present the following three lemmas.

Lemma 1. (Maksimov, 2011) *Let a nonnegative function $\phi(t)$, $t \in T$, satisfy*

$$\phi(\tau_{i+1}) \leq \phi(\tau_i)(1 + q\delta) + \int_{\tau_i}^{\tau_{i+1}} |G(\tau)|_+ \,\mathrm{d}\tau$$

for all $i \in [0 : m-1]$, where $\tau_i \in \Delta$, $\delta = \tau_{i+1} - \tau_i$, $q = \text{const} > 0$, $G(\cdot) \in L_\infty(T;\mathbb{R})$. Then

$$\phi(\tau_i) \leq \Big(\phi(0) + \int_0^{\tau_i} |G(\tau)|_+ \,\mathrm{d}\tau\Big)\exp\{q\tau_i\},$$

$i \in [0 : m]$.

Lemma 2. (Discrete Gronwall inequality (Samarskii, 1971)) *Let $0 \leq \phi_j$, $0 \leq f_j$ for $j \in [0 : m]$ and $f_j \leq f_{j+1}$ for $j \in [0 : m-1]$. Then*

$$\phi_{j+1} \leq c_0\delta\sum_{i=1}^{j}\phi_i + f_j, \quad j \in [1 : m-1]$$

imply

$$\phi_{j+1} \leq f_j\exp\{c_0 j\delta\}, \quad j \in [0 : m-1],$$

if $c_0 = \text{const} > 0$, $\phi_1 \leq f_0$.

Lemma 3. (Maksimov, 2002, p. 47) *Let $u(\cdot) \in L_\infty(T_*;\mathbb{R}^n)$ and $v(\cdot) \in W(T_*;\mathbb{R}^n)$, $T_* = [a,b]$, $-\infty < a < b < +\infty$,*

$$\Big|\int_a^t u(\tau)\,\mathrm{d}\tau\Big|_n \leq \varepsilon, \quad |v(t)|_n \leq K, \quad \forall\, t \in T_*.$$

Then, for all $t \in T_*$, we have

$$\Big|\int_a^t (u(\tau), v(\tau))\,\mathrm{d}\tau\Big|_+ \leq \varepsilon(K + \mathrm{var}(T_*; v(\cdot))).$$

Here, the symbol $\mathrm{var}(T_*; v(\cdot))$ means the variation of the function $v(\cdot)$ over the interval T_*, the symbol $(\cdot,\cdot)$ means the scalar product in the corresponding finite-dimensional Euclidean space, the symbol $|\cdot|_+$ means the absolute value of a number, and the symbol $W(T_*;\mathbb{R}^n)$ means the set of functions $y(\cdot) : T_* \to \mathbb{R}^n$ of bounded variation.

We fix some constants $C_* \in (0,+\infty)$, $C_{**} \in (0,+\infty)$, $\varepsilon \in (0,1)$. We assume that the following condition holds.

Condition 1 *$\delta(h) = C_*h$, $\alpha(h) \to 0$ and $\delta(h)\alpha^{-2}(h) \leq C_{**}h^\varepsilon$ as $h \to 0$.*

Theorem 1. *There exist constants $d_1 > 0$ and $d_2 > 0$ such that the inequalities*

$$\max_{i \in [0:m_h]} |x_1(\tau_{h,i}) - x^h(\tau_{h,i})|_n \leq d_1\alpha^{1/2}(h), \tag{10}$$

$$\int_0^\vartheta |Bu^h(\tau)|_n^2 \,\mathrm{d}\tau \leq \int_0^\vartheta |Bv(\tau)|_n^2 \,d\tau + d_2h^\varepsilon \tag{11}$$

are fulfilled for any disturbance $v(\cdot) \in L_2(T;\mathbb{R}^r)$, any $h \in (0,1)$, any family Δ_h (see (5)), any realization $u^h(\cdot)$ of feedback $U(\cdot,\cdot,\cdot)$ of the form (9), any trajectory of the real system (1) $x^h(\cdot) = x(\cdot; x_0, u^h(\cdot), v(\cdot))$ (i.e., any solution of (7)), any measurement ψ_i^h with the property (4), and any measurement ξ_i^h with the property (2).

Proof. We estimate the change in the function

$$\varepsilon_h(t) = \lambda_h(t) + \alpha\int_0^t \{|Bu^h(\tau)|_n^2 - |Bv(\tau)|_n^2\}\,\mathrm{d}\tau, \tag{12}$$

where

$$\lambda_h(t) = \exp\{-2\omega t\}|x^h(t) - x_1(t)|_n^2.$$

By virtue of the Cauchy formula, we conclude that

$$x_1(t) = \mathcal{X}(t-\tau_i)x_1(\tau_i) + \int_{\tau_i}^{t} \mathcal{X}(t-\tau)f(\tau)\,\mathrm{d}\tau,$$

$$\begin{aligned} x^h(t) &= \mathcal{X}(t-\tau_i)x^h(\tau_i) \\ &\quad + \int_{\tau_i}^{t} \mathcal{X}(t-\tau)\{B(u^h(\tau)-v(\tau))f(\tau)\}\,\mathrm{d}\tau \end{aligned} \tag{13}$$

for all $t \in \delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$. Note that (see (8))

$$\begin{aligned} &|\mathcal{X}(\delta)\exp\{-\omega\tau_{i+1}\}| \\ &\quad \le \exp\{|A|\delta\}\exp\{-\omega\tau_{i+1}\} \\ &\quad \le \exp\{-\omega\tau_i\}, \quad \delta = \tau_{i+1}-\tau_i. \end{aligned}$$

Therefore,

$$\begin{aligned} &\exp\{-2\omega\tau_{i+1}\}|\mathcal{X}(\delta)(x_1(\tau_i)-x^h(\tau_i))|_n^2 \\ &\qquad \le \exp\{-2\omega\tau_i\}|x_1(\tau_i)-x^h(\tau_i)|_n^2. \end{aligned}$$

Then, using (13), it is easily seen that for all $i \in [0 : m-1]$ the estimate

$$\begin{aligned} \varepsilon_h(\tau_{i+1}) &\le \exp\{-2\omega\tau_i\}|x^h(\tau_i)-x_1(\tau_i)|_n^2 + \lambda_i + \mu_i \\ &\quad + \alpha\int_0^{\tau_{i+1}} \{|Bu^h(\tau)|_n^2 - |Bv(\tau)|_n^2\}\,\mathrm{d}\tau \end{aligned} \tag{14}$$

is valid. Here,

$$\lambda_i = 2\Big(S_i, \int_{\tau_i}^{\tau_{i+1}} \mathcal{X}(\tau_{i+1}-\tau)B\{u^h(\tau)-v(\tau)\}\,\mathrm{d}\tau\Big), \tag{15}$$

$$\begin{aligned} \mu_i &= \delta\exp\{-2\omega\tau_{i+1}\} \\ &\quad \times \int_{\tau_i}^{\tau_{i+1}} |\mathcal{X}(\tau_{i+1}-\tau)B\{u^h(\tau)-v(\tau)\}|_n^2\,\mathrm{d}\tau, \end{aligned} \tag{16}$$

$$S_i = \exp\{-2\omega\tau_{i+1}\}\mathcal{X}(\tau_{i+1}-\tau_i)\{x^h(\tau_i)-x_1(\tau_i)\}. \tag{17}$$

Using the inequality $\exp\{-2\omega\delta\} \le 1$ and (14), we derive the estimate

$$\begin{aligned} \varepsilon_h(\tau_{i+1}) &\le \varepsilon_h(\tau_i) + \lambda_i + \mu_i \\ &\quad + \alpha\int_{\tau_i}^{\tau_{i+1}} \{|Bu^h(\tau)|_n^2 - |Bv(\tau)|_n^2\}\,\mathrm{d}\tau. \end{aligned} \tag{18}$$

Note that the estimate

$$|\mathcal{X}(t) - I| \le C_0 t, \quad C_0 = C_0(\delta_*) \in (0,+\infty), \tag{19}$$

is valid for $t \in [0, \delta_*]$, $\delta_* \in (0,1)$. Here, I is the $n \times n$ identity matrix . Therefore, in virtue of (2), (19) and (4), we have

$$\begin{aligned} &|S_i - \exp\{-2\omega\tau_{i+1}\}s_i^h|_n \\ &\quad \le \exp\{-2\omega\tau_{i+1}\}|\mathcal{X}(\delta)\{s_i^h + (\psi_i^h - x_1(\tau_i)) \\ &\qquad + (x^h(\tau_i)-\xi_i^h)\} - s_i^h|_n \\ &\quad \le \delta C_0\exp\{-2\omega\tau_{i+1}\}|s_i^h|_n + C_{11}h \\ &\quad \le \delta C_0|s_i^h|_n + C_1 h, \end{aligned} \tag{20}$$

$$|S_i|_n \le C_{12}|s_i^h|_N + C_{13}h.$$

Here, $s_i^h = \xi_i^h - \psi_i^h$. Taking into account (20), we get

$$\begin{aligned} &|(S_i, \mathcal{X}(\delta)Bu) - \exp\{-2\omega\tau_{i+1}\}(s_i^h, Bu)| \\ &\quad \le |S_i|_n|\mathcal{X}(\delta) - I|_n|Bu|_n + |(S_i, Bu) \\ &\qquad - \exp\{-2\omega\tau_{i+1}\}(s_i^h, Bu)| \\ &\quad \le (\delta C_0|s_i^h|_n + C_{11}h)|Bu|_n + \delta C_0|s_i^h|_n|Bu|_n \\ &\quad \le C_{14}(h + \delta|s_i^h|_n)|Bu|_n. \end{aligned} \tag{21}$$

Note that

$$|s_i^h|_n \le \exp\{\omega\vartheta\}\lambda_h^{1/2}(\tau_i) + 2h. \tag{22}$$

Consider the function λ_i (see (15)). In turn, by virtue of (15), (21) and (22), we obtain the estimate

$$\begin{aligned} \lambda_i &\le 2\exp\{-2\omega\tau_{i+1}\} \\ &\quad \times \int_{\tau_i}^{\tau_{i+1}} (s_i^h, B\{u^h(\tau)-v(\tau)\})\,\mathrm{d}\tau + \rho_i, \end{aligned} \tag{23}$$

where

$$\rho_i = C_{15}(h + \delta\lambda_h^{1/2}(\tau_i))\int_{\tau_i}^{\tau_{i+1}} |B\{u^h(\tau)-v(\tau)\}|_n\,\mathrm{d}\tau.$$

Therefore,

$$\begin{aligned} \rho_i &\le C_{16}\Big\{\delta^2\lambda_h(\tau_i) + h^2 \\ &\quad + \delta\int_{\tau_i}^{\tau_{i+1}} \{|Bu^h(\tau)|_n^2 + |Bv(\tau)|_n^2\}\,\mathrm{d}\tau\Big\}. \end{aligned} \tag{24}$$

Then we get (cf. (16))

$$\mu_i \le C_{17}\delta\int_{\tau_i}^{\tau_{i+1}} \{|Bu^h(\tau)|_n^2 + |Bv(\tau)|_n^2\}\,\mathrm{d}\tau. \tag{25}$$

Note that the vector u_i^h (see (6), (9)) is found from the condition

$$\begin{aligned} u_i^h &= \arg\min\{2\exp\{-2\omega\tau_{i+1}\}(s_i^h, Bv) \\ &\qquad + \alpha|Bv|_n^2 : v \in \mathbb{R}^r\}. \end{aligned} \tag{26}$$

Consequently, from (23) and (26) we deduce that

$$\lambda_i + \alpha \int_{\tau_i}^{\tau_{i+1}} \{|Bu^h(\tau)|_n^2 - |Bv(\tau)|_n^2\}\,\mathrm{d}\tau \le \rho_i. \tag{27}$$

Hence, by virtue of (18), (25) and (27), we obtain

$$\varepsilon_h(\tau_{i+1}) \le \varepsilon_h(\tau_i) + C_{18}\Big\{\delta^2\lambda_h(\tau_i) + h^2 + \delta \int_{\tau_i}^{\tau_{i+1}} \{|Bu^h(\tau)|_n^2 + |Bv(\tau)|_n^2\}\,\mathrm{d}\tau\Big\}, \tag{28}$$

i.e., (see (12))

$$\lambda_h(\tau_{i+1}) \le (1 + C_{18}\delta^2)\lambda_h(\tau_i) + C_{18}\Big\{h^2 + \delta \int_{\tau_i}^{\tau_{i+1}} \{|Bu^h(\tau)|_n^2 + |Bv(\tau)|_n^2\}\,\mathrm{d}\tau\Big\} + \alpha \int_{\tau_i}^{\tau_{i+1}} |Bv(\tau)|_n^2\,\mathrm{d}\tau. \tag{29}$$

The rule for finding the control u_i^h implies the inequalities

$$\begin{aligned} |u_i^h|_r^2 &\le 2|(B'B)^+B'|^2 \exp\{-4\omega\tau_{i+1}\} \\ &\quad \times (\exp\{2\omega\vartheta\}\lambda_h(\tau_i) + 4h^2)\alpha^{-2} \\ &\le C_{19}(\lambda_h(\tau_i) + h^2)\alpha^{-2}. \end{aligned} \tag{30}$$

In addition, we have

$$\lambda_h(0) = 0. \tag{31}$$

Using the inequalities $\delta(h)\alpha^{-2}(h) \le C_{**}h^\varepsilon$, (29), (30) and

$$\begin{aligned} \delta \int_{\tau_i}^{\tau_{i+1}} |Bu^h(\tau)|_n^2\,\mathrm{d}\tau &\le C_{21}\delta^2(\lambda_h(\tau_i) + h^2)\alpha^{-2} \\ &\le C_{22}\delta(h)h^\varepsilon(\lambda_h(\tau_i) + h^2), \end{aligned} \tag{32}$$

we obtain the relation

$$\lambda_h(\tau_{i+1}) \le (1 + C_{23}\delta)\lambda_h(\tau_i) + C_{24}h^2 + \alpha \int_{\tau_i}^{\tau_{i+1}} |Bv(\tau)|_n^2\,\mathrm{d}\tau. \tag{33}$$

Hence, taking into account (31), (33) and Lemma 1, we get

$$\lambda_h(\tau_{i+1}) \le C_{25}(\alpha + h^2\delta^{-1}). \tag{34}$$

The inequality (10) follows from (34) and Condition 1. If $\delta(h) = C_*h$, then by virtue of (28), (32) and (34), we deduce that

$$\varepsilon(\tau_{i+1}) \le \varepsilon(\tau_i) + C_{26}\{\delta^{1+\varepsilon}(h^2\delta^{-1} + \alpha) + h^2 + \delta \int_{\tau_i}^{\tau_{i+1}} |Bv(\tau)|_n^2\,\mathrm{d}\tau\}.$$

Therefore, we get

$$\begin{aligned} \varepsilon(\tau_i) &\le C_{27}\{h^2\delta^{\varepsilon-1} + \delta^\varepsilon\alpha + h^2\delta^{-1} + \delta\} \\ &\le C_{28}(\alpha\delta^\varepsilon + h^2\delta^{-1} + \delta). \end{aligned} \tag{35}$$

Then, from (35), we have

$$\int_0^\vartheta |Bu^h(s)|_n^2\,\mathrm{d}s \le \int_0^\vartheta |Bv(s)|_n^2\,\mathrm{d}s + C_{28}\{\delta^\varepsilon(h) + h^2\alpha^{-1}(h)\delta^{-1}(h) + \delta(h)\alpha^{-1}(h)\}. \tag{36}$$

Using (36), we conclude that

$$\int_0^\vartheta |Bu^h(s)|_n^2\,\mathrm{d}s \le \int_0^\vartheta |Bv(s)|_n^2\,\mathrm{d}s + C_{29}h^\varepsilon.$$

The theorem is proved. ■

Let

$$\tilde v^h(t) = Bu^h(t), \quad \tilde v(t) = Bv(t), \quad t \in T.$$

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled. Then*

$$x^h(\cdot) \to x_1(\cdot) \textit{ in } \; W^{1,2}(T;\mathbb{R}^n) \quad \textit{as } h \to 0.$$

Proof. First, we establish the convergence

$$\tilde v^h(\cdot) \to \tilde v(\cdot) \quad \text{weakly in } L_2(T;\mathbb{R}^n) \text{ as } h \to 0. \tag{37}$$

Assuming the contrary, we conclude that there exists a subsequence $\tilde v^{h_j}(\cdot)$ ($h_j \to 0$ as $j \to \infty$) such that

$$\tilde v^{h_j}(\cdot) \to \tilde v_0(\cdot) \quad \text{weakly in } L_2(T;\mathbb{R}^n) \text{ as } j \to \infty, \tag{38}$$

$$\begin{aligned} \tilde v_0(t) &= Bv_0(t), \\ \tilde v_0(\cdot) &\ne \tilde v(\cdot). \end{aligned} \tag{39}$$

In this case, extracting, if necessary, a subsequence from $\{h_j\}_{j=1}^{+\infty}$, we have

$$x^{h_j}(\cdot) \to x_*(\cdot) \quad \text{in } C(T;\mathbb{R}^n) \text{ as } j \to +\infty,$$

where $x^{h_j}(\cdot) = x(\cdot; x_0, u^{h_j}(\cdot), v(\cdot)), x_*(\cdot)$ is the solution of the system

$$\dot{x}(t) = Ax(t) + B(v_0(t) - v(t)) + f(t), \quad t \in T,$$

with the initial state $x(0) = x_0$, i.e. $x_*(\cdot) = x(\cdot : x_0, v_0(\cdot), v(\cdot))$. Here, $C(T; \mathbb{R}^n)$ is the space of all continuous functions mapping the set T into the space $\mathbb{R}^n$. Let $C(T; \mathbb{R}^n)$ be equipped with the sup-norm. By virtue of Theorem 1 (cf. (10)),

$$x_*(\cdot) = x_1(\cdot).$$

Hence,

$$\tilde{v}_0(\cdot) = \tilde{v}(\cdot). \tag{40}$$

Indeed, if the relations (39) were valid, we would come to a contradiction. On the one hand, $\tilde{x}(t) = x_*(t) - x_1(t) = 0 \ \forall t \in T$, but on the other hand, $\dot{\tilde{x}}(t) = \tilde{v}_0(t) - \tilde{v}(t) \neq 0$ on some subset the interval T with a nonzero Lebesgue measure. The equality (40) contradicts the inequality (39). The convergence (37) is proved. Moreover, by virtue of the known property of the weak limit, from (37) we derive

$$\underline{\lim}_{h\to 0} |\tilde{v}^h(\cdot)|_{L_2} \geq |\tilde{v}(\cdot)|_{L_2}. \tag{41}$$

Here, the symbol $|\cdot|_{L_2}$ means the norm in the space $L_2(T; \mathbb{R}^n)$. In its turn, by virtue of (11), the inequality

$$|\tilde{v}^h(\cdot)|^2_{L_2} \leq |\tilde{v}(\cdot)|^2_{L_2} + d_2 h^{\varepsilon}$$

is valid. This implies

$$\overline{\lim}_{h\to 0} |\tilde{v}^h(\cdot)|_{L_2} \leq |\tilde{v}(\cdot)|_{L_2} \tag{42}$$

and (see (41), (42))

$$\overline{\lim}_{h\to 0} |\tilde{v}^h(\cdot)|_{L_2} \leq |\tilde{v}(\cdot)|_{L_2} \leq \underline{\lim}_{h\to 0} |\tilde{v}^h(\cdot)|_{L_2}.$$

Therefore, there exist a limit $\lim_{h\to 0} |\tilde{v}^h(\cdot)|_{L_2}$, and

$$\lim_{h\to 0} |\tilde{v}^h(\cdot)|_{L_2} = |\tilde{v}(\cdot)|_{L_2}. \tag{43}$$

Using (37) and (43), we conclude that

$$\tilde{v}^h(\cdot) \to \tilde{v}(\cdot) \quad \text{in } L_2(T; \mathbb{R}^n) \text{ as } h \to 0. \tag{44}$$

By virtue of the Cauchy formula, we get

$$|x^h(\cdot) - x_1(\cdot)|_{C(T;\mathbb{R}^n)} \leq d^{(1)} |\tilde{v}^h(\cdot) - \tilde{v}(\cdot)|_{L_2}.$$

Therefore,

$$|x^h(\cdot) - x_1(\cdot)|_{W^{1,2}(T;\mathbb{R}^n)} \leq d^{(2)} |\tilde{v}^h(\cdot) - \tilde{v}(\cdot)|_{L_2} \tag{45}$$

The statement of the theorem follows from (45) and (44). The theorem is proved. ∎

Under some additional conditions, we can obtain the convergence rate of the algorithm (see Theorem 3 below).

Theorem 3. *Let Condition* 1 *hold. Let also* $t \to Bv(t) \in W(T; \mathbb{R}^n)$. *Then*

$$|x_1(\cdot) - x^h(\cdot)|^2_{W^{1,2}(T;\mathbb{R}^n)} \leq K_0\{\alpha^{1/2}(h) + h^{\varepsilon}\}. \tag{46}$$

Here K_0 is a constant independent of h and α.

Proof. By the use of (13), it is easily seen that

$$\lambda_h^{1/2}(t) \leq c_1 \Big\{ \lambda_h^{1/2}(\tau_i) + \int_{\tau_i}^{t} (|Bu_i^h|_n + |Bv(\tau)|_n)\, \mathrm{d}\tau \Big\} \tag{47}$$

for $t \in [\tau_i, \tau_{i+1}]$, $i \in [0 : m-1]$. Here, c_j, $j = 1, 2, \dots$, are positive constants not depending on i, h and α. Also, we have

$$\int_{\tau_i}^{\tau_{i+1}} |Bv(s)|_n \mathrm{d}s \leq c_2 \delta^{1/2}(h) \leq c_3 h^{1/2}, \tag{48}$$

$$\lambda_h(\tau_i) \leq c_4 \alpha(h). \tag{49}$$

In turn, from (30), (48), and (49) we obtain

$$\begin{aligned} \int_{\tau_i}^{\tau_{i+1}} |Bu_i^h|_n \,\mathrm{d}\tau &\leq c_5 \delta \alpha^{-1}(h + \lambda_h^{1/2}(\tau_i)) \\ &\leq c_6 \delta \alpha^{-1/2}(h). \end{aligned} \tag{50}$$

In this case, from (47) and (50) we get for $t \in [\tau_i, \tau_{i+1}]$

$$|z_h(t)|_n \leq c_7 \alpha^{1/2}(h), \tag{51}$$

where $z_h(t) = x^h(t) - x_1(t)$. In addition, we see that

$$\begin{aligned} &\Big| \int_{t_1}^{t_2} (\tilde{v}(t) - \tilde{v}^h(t))\, \mathrm{d}t \Big|_n \\ &\leq c_8 \Big\{ |z_h(t_2) - z_h(t_1)|_n + \int_{t_1}^{t_2} |z_h(t)|_n \,\mathrm{d}t \Big\} \\ &\leq c_9 \alpha^{1/2}(h) \end{aligned}$$

for all $t_1, t_2 \in T$, $t_1 < t_2$. Using (11), from Lemma 3 we get the relations

$$\begin{aligned} &|\tilde{v}(\cdot) - \tilde{v}^h(\cdot)|^2_{L_2(T;\mathbb{R}^n)} \\ &\leq 2|\tilde{v}(\cdot)|^2_{L_2(T;\mathbb{R}^n)} \\ &\quad - 2\int_0^{\vartheta} (\tilde{v}(\tau), \tilde{v}^h(\tau))\, \mathrm{d}\tau + d_2 h^{\varepsilon} \\ &\leq c_{10}\{\alpha^{1/2} + h^{\varepsilon}\}. \end{aligned} \tag{52}$$

From (52) and (45) we derive the inequality (46). The theorem is proved. ∎

4. Numerical example

The algorithm from Section 4 was tested on a model example. A material particle of unit mass moves along a line under the action of a tractive force and an unknown disturbance. The gravity force is ignored. The displacement of the point is inaccurately measured at discrete, frequent enough, times. It is required to build an algorithm of reduction (in real time mode) of the unknown disturbance. According to the second Newton law, the motion is described by the equation

$$\ddot{X}(t) = u(t) - v(t) + f(t), \quad t \in [0, \vartheta], \tag{53}$$

where $u(t)$ is the outer force, $v(t)$ is the disturbance, $X(t)$ is the particle displacement. Assuming $\dot{X}(t) = Y(t)$, rewrite (53) in the form of the system (1),

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \quad x_1(0) = x_0, \\ \dot{x}_2(t) &= u(t) - v(t) + f(t), \quad x_2(0) = x_1, \end{aligned} \tag{54}$$

where $x_1 = X$, $x_2 = Y$. The system (3) has the form

$$\begin{aligned} \dot{x}_{11}(t) &= x_{12}(t), \quad x_{11}(0) = x_0, \\ \dot{x}_{12}(t) &= f(t), \quad x_{12}(0) = x_1. \end{aligned} \tag{55}$$

The systems (54) and (55) were solved using the Euler method with some integration step δ. The work of the algorithm was organized as follows. At the moments $\tau_i = i\delta$, $i \in [0 : m-1]$, the values u_i^h are calculated by the formulas (see (6) and (9))

$$u_i^h = \alpha^{-1} \exp\{-2\tau_{i+1}\}(\xi_i^h - \psi_i^h).$$

Then, a control of the form

$$u(t) = u^h(t) = u_i^h$$

was fed to the system (54). As the result of the action of this control and the disturbance $v(t)$ of the form given below, the system (54) passed from the state $\{x_1(\delta i), x_2(\delta i)\}$ to the state $\{x_1(\delta(i+1)), x_2(\delta(i+1)\}$. The algorithm was acting until the moment ϑ.

In the numerical experiment, we set $\vartheta = 2$, $x_0 = 2$, $x_1 = 2$, $v(t) = \sin t$, $\delta = 0.002$, $h = 0.001$, $f(t) = \cos(5t)$, $\xi_i^h = x_2(\tau_i) + h\cos(10t)$, $\psi_i^h = x_{12}(\tau_i) + h$. The simulation results are presented in Figs. 1 and 2. Figure 1 corresponds to the case $\alpha = 0.001$, whereas Fig. 2 to the case $\alpha = 0.01$. In the figures, the solid lines represent the derivative of the function $x_{12}(t)$, while the dotted lines represent the derivative of the function $x_2(t)$. We omitted the graphs of the functions $x_{12}(t)$ and $x_2(t)$ because they virtually coincide.

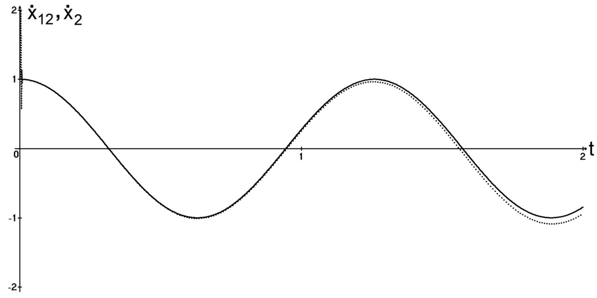


Fig. 1. Case $\alpha = 0.001$.

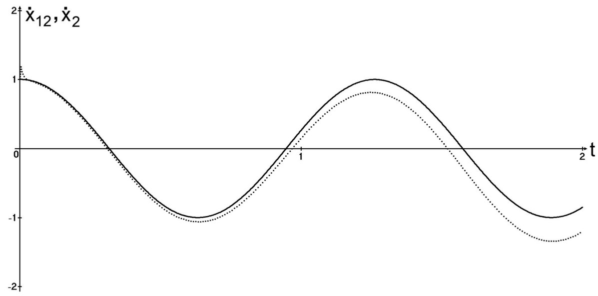


Fig. 2. Case $\alpha = 0.01$.

5. Conclusions

In the paper, the control problem for a linear system of differential equations under the influence of an unknown disturbance was investigated. An algorithm for reducing the disturbance by means of forming appropriate feedback control actions was designed. This algorithm, based on the constructions of guaranteed control theory, is stable with respect to informational noises and computational errors. The algorithm was tested with a model example.

References

Cayero, J., Rotondo, D., Marcego, B., and Puig, V. (2019). Optimal state observation using quadratic boundedness: Application to UAV disturbance estimation, *International Journal of Applied Mathematics and Computer Science* **29**(1): 99–109, DOI: 10.2478/amcs-2019-0008.

Falsone, A., Deori, L., Ioli, D., Garatti, S. and Prandini, M. (2019). Optimal disturbance compensation for constrained linear systems operating in stationary conditions: A scenario-based approach, *Automatica* **110**: 108537.

Favini, A., Maksimov, V. and Pandolfi, L. (2004). A deconvolution problem related to a singular system, *Journal of Mathematical Analysis and Applications* **292**(1): 60–72.

Gan, W.S. and Kuo, S.M. (2002). An integrated audio and active noise headsets, *IEEE Transactions on Consumer Electronics* **48**(2): 242–247.

Keesman, K.J. and Maksimov, V.I. (2008). On feedback identification of unknown characteristics: A bioreactor case study, *International Journal of Control* **81**(1): 134–145.

Kwakernaak, H. (2002). H_2-optimization—Theory and applications to robust control design, *Annual Reviews in Control* **26**(1): 45–56.

Maksimov, V.I. (2002). *Dynamical Inverse Problems of Distributed Systems*, VSP, Utrecht/Boston.

Maksimov, V. (2011). The tracking of the trajectory of a dynamical system, *Journal of Applied Mathematics and Mechanics* **75**(6): 667–674.

Maksimov, V.I. and Mordukhovich, B.S. (2017). Feedback design of differential equations of reconstruction for second-order distributed systems, *International Journal of Applied Mathematics and Computer Science* **27**(3): 467–475, DOI: 10.1515/amcs-2017-0032.

Maksimov, V. (2016). Game control problem for a phase field equation, *Journal of Optimization Theory and Applications* **170**(1): 294–307.

Maksimov, V. and Tröltzsch, F. (2020). Input reconstruction by feedback control for the Schlögl and FitzHugh–Hagumo model, *International Journal of Applied Mathematics and Computer Science* **30**(1): 5–22, DOI: 10.34768/amcs-2020-0001.

Osipov, Yu.S. and Kryazhimskii, A.V. (1995). *Inverse Problems for Ordinary Differential Equations: Dynamical Solutions*, Gordon and Breach, London.

Pandolfi, L. (2007). Adaptive recursive deconvolution and adaptive noise cancellation, *International Journal of Control* **80**(3): 403–415.

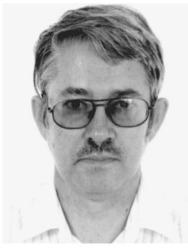
Samarskii, A.A. (1971). *Introduction to the Theory of Difference Schemes*, Nauka, Moscow, (in Russian).

Yu, S.-H. and Hu, J.-S. (2001). Controller design for active noise cancellation headphones using experimental raw data, *IEEE/ASME Transactions on Mechatronics* **6**(4): 483–490.

Wasilewski, M., Pisarski, D., Konowrocki, R., and Bajer, C.I. (2019). A new efficient adaptive control of torsional vibrations included by switched nonlinear disturbances, *International Journal of Applied Mathematics and Computer Science* **29**(2): 285–303, DOI: 10.2478/amcs-2019-0021.

Willems, J.C. (1982). Feedforward control, PID control laws, and almost invariant subspaces, *Systems and Control Letters* **1**(4): 277–282.

Yuan, Y., Wang, Z., Yu, Y., Guo, L., and Yang, H. (20019). Active disturbance rejection control for a pneumatic motion platform subject to actuator saturation: An extended state observer approach, *Automatica* **107**: 353–361.



Vyacheslav Maksimov graduated in mathematics and mechanics from Ural State University, Yekaterinburg, Russia, in 1972. He received his PhD and DSc degrees in physics and mathematics from the Institute of Mathematics and Mechanics (IMM), Ural Branch of the Russian Academy of Sciences, Yekaterinburg, in 1978 and 1992, respectively. Since 1972, he has been with the IMM. Since 1994, he has been the head of a department at that institute and a professor at the Chair of Controlled Systems Modeling of Ural Federal University. He is the author of more than 160 technical publications, including four monographs, and his research interests are primarily focused on control theory, distributed parameter systems, and mathematical modeling. Professor Maksimov is a member of the IFIP TC7 and WG7.2. He has also been a member of editorial boards of various journals.

Received: 24 January 2021
Revised: 2 April 2021
Accepted: 21 April 2021