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The use of simulated annealing method for optimization of fractional order PID controller

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The paper describes method of parameters selection for control system with fractional order $PI^{\lambda}D^{\mu}$ controller steering second order oscillated object. As a selection algorithm was used simulated annealing optimization method with random variant of cooling strategy. As a target function for optimization was selected Integral Squared Error (ISE).

KEYWORDS: fractional order controller, optimization, fractional calculus, simulated annealing

1. Introduction

Fractional order calculus becoming more widely use inter alia in nowadays automatics. There is increase of papers describing analysis and implementation of control systems based on fractional order derivative i. e. automatic

The principle of operation of fractional order controller is similar to the PID controller. The difference appears in fractional order of integration λ and differentiation μ. The addition of two new parameters cause that previously known method of PID regulator tuning no longer fulfill its role. Therefore, this article presents an analysis of the optimization controllers $PI^{\lambda}D^{\mu}$ with use of metaheuristic simulated annealing method.

2. Fractional differential calculus

Fractional order differential calculus expands basic definition of integral and derivative included in classis integral calculus. Such an extension a gives completely new opportunities in the mathematical analysis of objects and phenomena occurring in nature, control theory and its applications.

Currently there are three known basic definitions of differential-integral (generalized formula for the integral and differential fractional). The first, Riemann-Liouville definition can be derived from the formula for the integral multiple [6]:

$$ {}_aI_x^n f(x) = \int_a^x du_1 \int_a^{u_1} du_2 \ldots \int_a^{u_{n-1}} f(u_n) du_n = \frac{1}{\Gamma(n-1)} \int_a^x (x-u)^{n-1} f(u) du \quad (1) $$

where: n – integrations number ($n \in$ N), (a,x) – integral interval, $\Gamma(n\text{-}1)$ – Euler's gamma.

The extension of the integral formula for the row $\alpha \in$ R allows to define integral fractional according to the definition of the Riemann-Liouville described in formula [6, 7]:

$$ {}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u) du \quad (2) $$

where: α – fractional order of integration in set ($\alpha \in R^+$).

Generalization of equation (2) for differential-integral is given by

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(k-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{dt}} \right)^{\mathrm{k}} \int_a^x (x-u)^{k-\alpha-1} f(u) du \quad (3) $$

where : $\alpha \in$ R, $k \in$ N and $k\text{-}1 \leq \alpha \leq k$.

Positive order integral order ($\alpha > 0$) in equation (3) means integration, while negative order ($\alpha < 0$) means differentiation.

The issues related to the theory of control is most commonly used record mathematical model of controlled systems and controls to the plane of Laplace. For differential-integral Laplace transform is:

$$ \mathcal{L}\left\{ {}_aD_x^\alpha f(x) \right\} = \begin{cases} s^\alpha F(s) & \alpha < 0 \\ s^\alpha F(s) - \sum_{j=1}^{k} s^j \, {}_0D_x^{\alpha-j} f(0) & \alpha > 0 \end{cases} \quad (4) $$

With use of formula (4) becomes implementation problem, due to the requirement of knowledge of the initial conditions for fractional order derivative. The problem is that in most application we can't get physical definition of fractional order derivative. Hence more often used is Caputo definition of differential-integral:

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(k-\alpha)} \int_a^x \frac{f^{(k)}(u)}{(x-u)^{\alpha+1-k}} du \quad (5) $$

where: $k\text{-}1 < \alpha < k$.

Equation (5) differs from Reimann-Louville's differential-integral mainly in fractional order derivative place. In this case derivative of integer order $k:(k-1<a \wedge k \in N)$ is under integral. Thanks to this Laplace transform of differential - integral is given by:

$$\mathrm{L}\left\{{}_{a}D_{x}^{\alpha} f(x)\right\} = s^{\alpha} F(s) - \sum_{j=0}^{k-1} s^{\alpha-j-1} f^{(j)}(0) \tag{6}$$

Initial conditions are defined only for integral order derivative, so their physical interpretation is much easier.

Knowing definition of differential-integral form (6) let define transmittance *G(s)* for continuous fractional order system (assuming zero initial conditions) with formula:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}} \tag{7}$$

where: $Y(s)$, $U(s)$ – Laplace transform of output and input signal.

To obtain fractional order transfer function required is to use one of the approximation method of fractional systems. In paper [9] described are few well known methods of fractional order operator approximations such as: continued fraction expansions (CFE), Carlson's method, Matsuda's method and Oustaloup's method. In this paper Oustaloup's method was applied. Fractional order operator denoted as:

$$H(s) = s^{r}, \quad r \in \Re, \quad r \in [-1;1] \tag{8}$$

in frequency interval (ω_l, ω_h) as:

$$\widehat{H}(s) = C_o \prod_{k=-N}^{N} \frac{s + \omega_k'}{s + \omega_k} \tag{9}$$

where: N – approximation order, C_0 – gain described as:

$$C_o = \left(\frac{\omega_h}{\omega_l}\right)^{-\frac{r}{2}} \prod_{k=-N}^{N} \frac{\omega_k}{\omega_k'} \tag{10}$$

where: ω_l, ω_h – zeros end poles of transfer function:

$$\omega_k' = \omega_l \left(\frac{\omega_h}{\omega_l}\right)^{\frac{k+N+0,5(1-r)}{2N+1}}, \quad \omega_k = \omega_l \left(\frac{\omega_h}{\omega_l}\right)^{\frac{k+N+0,5(1+r)}{2N+1}} \tag{11}$$

One of the basic example of fractional order calculus application are fractional order $PI^{\lambda}D^{\mu}$ controllers which are equivalent to integral order PID controller. Such a controller have, besides three basic parameters – gains K_P, K_I, K_D, two new parameters – fractional order of integral λ (gdzie $0 < \lambda < 1$) and fractional order of derivative μ (gdzie $0 < \mu < 1$):

$$G(s) = K_P + K_I s^{-\lambda} + K_D s^{\mu} \tag{12}$$

3. Simulated annealing

The method of simulated annealing was described for the first time in from 1983 [2] and [1] from 1984. Both papers were develop independently to themselves. Described method of optimization is based on algorithm defined by N. Metropolis and published in [5] in 1953 year, for simulation of behavior of atoms in thermodynamic equilibrium at given temperature. Metropolis in his work found than for given temperature *T* probability of atoms energy increase for δE is defined as:

$$p(\delta E) = e^{\frac{-\delta E}{k*T}} \tag{13}$$

where: *k* – Boltzman's constant.

According to Metropolis probability of system energy change decreases with increase of energy change δE and decrease of temperature *T*. Primary algorithm rely on iterative draw of energy change δE. If drawn value of energy is higher than previous one than it is automatically accepted as new value of energy. Otherwise, drawn value is selected as new value of energy only with probability defined in (13) – there must be additional drawing. This is due to assumption than atoms energy can grow only with Metropolis probability [3].

A foregoing algorithm was modified in papers from 1983 and 1984 and adapted to issue of optimization. Authors of [2] defined analogies between calculating probability of energy change and looking for the objective function minimum. Instead of energy change introduced appellation of old and new objective function value. At the beginning algorithm run from start point $\boldsymbol{x_0}$, for which value of the objective function is equivalent to start atoms energy level $\boldsymbol{E}$. Now the energy is equivalent to value of objective function. Next new point x is drown and new value of objective function is calculated. If new value is better (smaller) than old value point $\boldsymbol{x}$ is automatically selected as solution. If new value is higher than old one, point $\boldsymbol{x}$ is selected as solution but only with given by formula (13) probability.

This optimization algorithm is called Metropolis method, and is base for simulated annealing method. Basic difference between this methods is possibility of temperature change in (13). Now parameter t is still called temperature, despite that analogy between it and thermodynamics is not so obvious. Instead of Boltzman's constant there is defined a group of coefficients. In every algorithm iteration drawn are points in neigh borough of actual best solution. From this set of points algorithm choose new, not always best, point as an actual optimization solution. Thanks to possibility of temperature change, simulated annealing method can go out of local minimum area and find global solution for specified issue. Higher temperature allows to robust change of actual solution, so it has higher probability of founding global minimum area. After each iteration temperature is reduced according to selected *cooling strategy*. More detail description of basic *cooling strategies* can be found in [3]. Selection of proper strategy should be adapted to given issue. Simulated annealing method tuning consist of proper selection of start temperature and *cooling strategy*.

4. Simulation

This paper presents simulation studies for purpose to define optimal fractional order PID controller parameters - equation (12) - for test object with given transfer function in closed system with feedback. Scheme of such an system is shown on Figure 1.

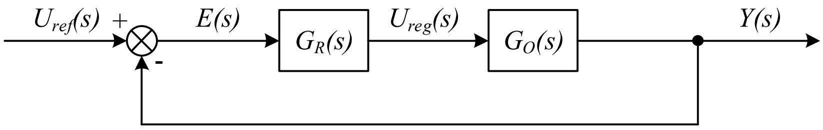

Fig. 1. Scheme of analyzed closed system with no noise: $U_{ref}(s)$ – referene signal, $E(s)$ – control error, $U_{reg}(s)$ – object steering signal, $G_R(s)$ – regulator transfer function, $Y(s)$ – output signal

As the controled object defined was second order oscillating object with transfer function defined as [6]:

$$G_O = \frac{K}{T^2 s^2 + 2\xi T s^1 + 1} \tag{14}$$

where: K – gain, T – self oscillations period, ξ – relative absorption coefficient.

In studies parameters of object G_O was given as: T = 0.1 [s], ξ = 0.2 and K = 1. Figure 2 presents step response of open system with no controller.

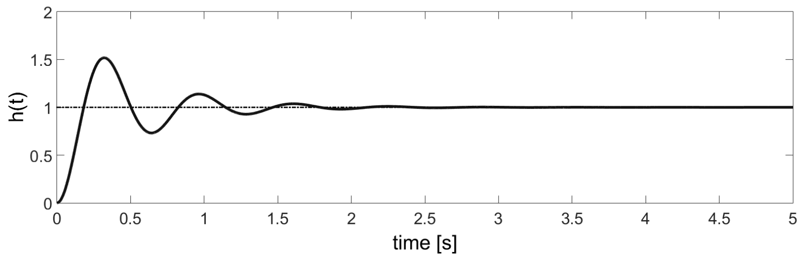

Fig. 2. Step response h(t) of object G_O in open system with no controller

On the basis of minimization of given control quality factor for each of objects selected were five parameters of fractional order PID controller. The simulated annealing method was used for minimization.

As an objective function selected was Integral Square Error (ISE) criterion defined as:

$$ISE = \int_0^{10ms} e^2(t)dt \tag{15}$$

where: $e(t)$ – control error.

Such a defined objective function lets to obtain system with small oscillations in step response and short time of settling time.

The calculating application was created in Matlab® & Simulnik® Environment with use of additional Toolboxes such as: Control System Toolbox® and freeware FOMCON library which has implemented a packet of functions calculating and simulating fractional order calculus systems [4]. The simulation was held for selected time interval of step response from 0 to 10 milliseconds and sampling period $T_S = 1$ [μs].

First experiment was to compare three different *cooling strategies* defined respectively as:

$$T_i = T_0 - i\frac{T_0 - T_N}{N} \tag{16.a}$$

$$T_i = T_0\left(\frac{T_N}{T_0}\right)^{i/N} \tag{16.b}$$

$$T_i = \frac{T_0 - T_N}{1 + \exp(0.3(i - N/2))} + T_N \tag{16.c}$$

where: T_i – temperature in i-th iteration, T_0 – start temperature, T_N – end temperature, N – iteration number.

Charts on Figure 3 present temperature and probability (from equation (13)) in every iteration for each of strategies respectively defined: strategy I – equation (16.a), strategy II – equation (16.b), strategy III – equation (16.c).

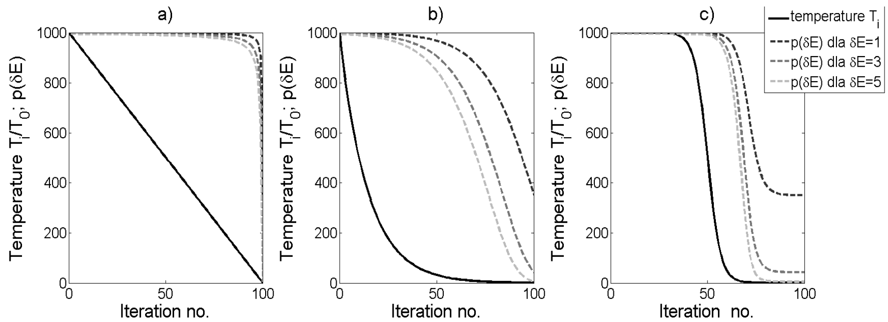

Fig. 3. Cooling strategies (solid line) used for simulated annealing optimization and probability of jumping to worse point (dashed line): a) strategy I, b) strategy II, c) strategy III

In compare of those three system cooling strategies, it is seen that first strategy has highest probability of jumping to the worse solutions than previous one. Chart on Figure 3b and Figure 3c differs in intensity of temperature changes, and hence in decrease of probability of jumping to the worse solutions. Disadvantages of these two cooling strategies is relatively high probability of jumping to the worse solutions at the final phase of optimization, especially in small differences of objective function value (small gradient). All three strategies were implemented in simulated model.

For compare purpose optimization were done from two different start points $\mathbf{X}_1$ and $\mathbf{X}_2$. Figure 4, Figure 5 and Figure 6 present actual point of optimization after each iteration and best solution from whole optimization.

a) b)

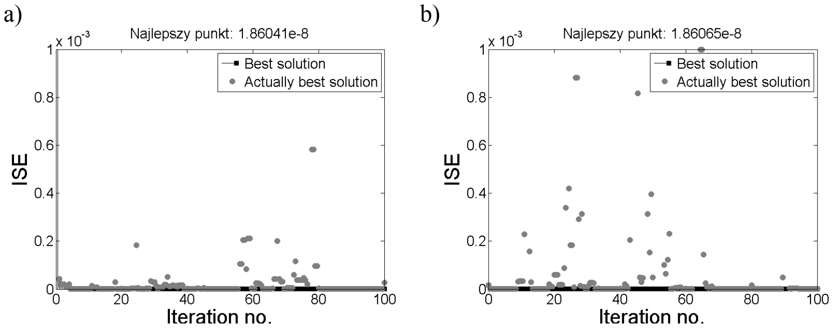

Fig. 4. The course of searching the optimal fractional order PID controller parameters with use of cooling strategy I: a) from point $\mathbf{X}_1$, b) from point $\mathbf{X}_2$

a) b)

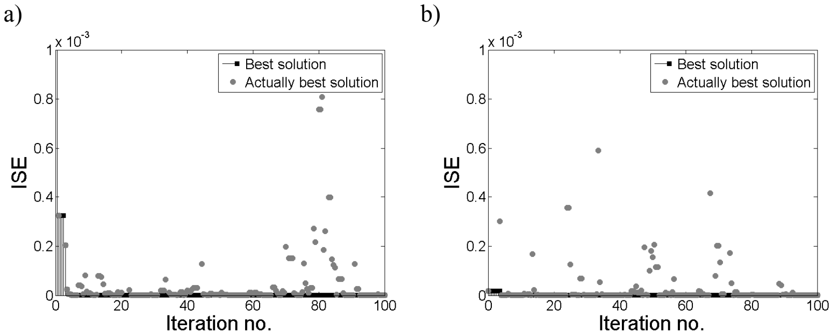

Fig. 5. The course of searching the optimal fractional order PID controller parameters with use of cooling strategy II: a) from point $\mathbf{X}_1$, b) from point $\mathbf{X}_2$

a) b)

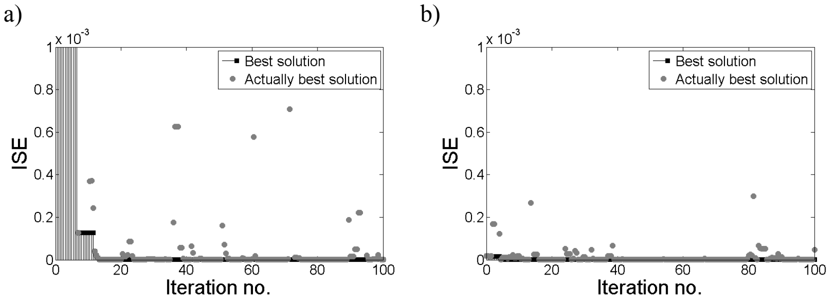

Fig. 6. The course of searching the optimal fractional order PID controller parameters with use of cooling strategy III: a) from point $\mathbf{X}_1$, b) from point $\mathbf{X}_2$

Figure 7 presents step response for control system with fractional order PID controller with parameters given by actually selected solution after first few iteration. The example concerns optimization with strategy I started from point $\mathbf{X}_1$. It is seen that after first iteration step response of system is worse than responses in after next iterations. This solution gives much longer stabilization time than solutions given in next iterations. Comparing step response for solution after seventh and ninth iteration, it can be seen that algorithm jump to worse solution.

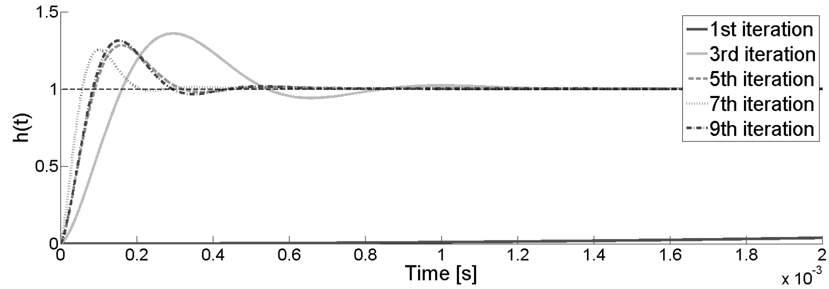

Fig. 7. Step responses of controller with parameter given from optimization solutions at first nine iteration

Table 3.1 presents result for experiment for different *cooling strategies* and different optimization start points. Results were selected as best objective function value from five optimization in every case.

Tabela 3.1. Optimization results for each attempt

No	Start point	Cooling strategy	K_P	K_I	K_D	μ	λ	ISE
1	$\mathbf{X}_1$	I	101,8	264,5	999,9	0,77	0,99	1,8604E-08
2	$\mathbf{X}_2$	I	0,02	324,4	999,9	0,47	0,99	1,8606E-08
3	$\mathbf{X}_1$	II	915,9	962,4	999,9	0,99	0,99	1,8607E-08
4	$\mathbf{X}_2$	II	135	748,5	999,8	0,1	0,99	1,8605E-08
5	$\mathbf{X}_1$	III	0,01	633,4	999,9	0,29	0,99	1,8605E-08
6	$\mathbf{X}_2$	III	291,8	433,5	998,5	0,03	0,99	1,8770E-08

Second experiment was to optimize regulator for automatic voltage regulator (AVR) with fractional order PID controller. As an AVR applied was DC/DC buck converter showed in Figure 8. Load voltage was given as an system feedback to controller.

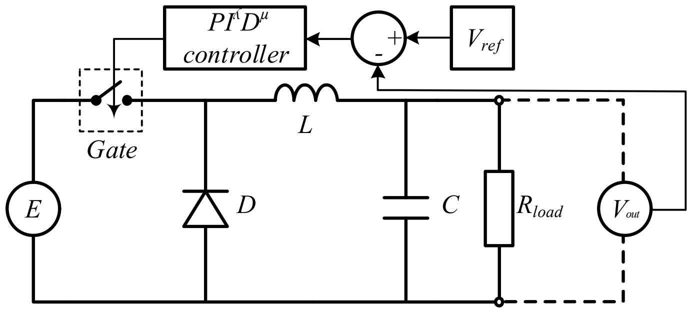

Fig. 8. DC/DC buck converter as the AVR

Figure 9 presents step response of system from Figure 8 for optimal fractional order controller and for comparison classic PID controller.

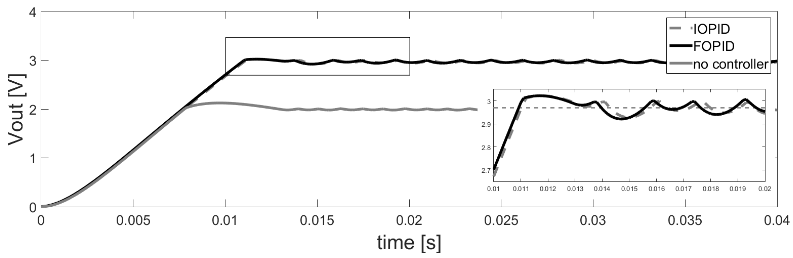

Fig. 9. Step response of AVR with fractional order PID controller (solid line)
and integral order PID controller (dashed)

5. Summary

The paper presents application for method of simulated annealing in fitting fractional order PID controller for object with given transfer function and automatic voltage regulator. Result shows that studied optimization method can be very effective tool for selecting parameters of controllers.

Analyzing algorithm path of searching, in the first experiment, the best solution for each attempt we can present the following conclusion:

Strategy I, despite that more frequently than other strategies jump to wore solution at the beginning, has been most efficient. This strategy is in accordance with expectations in the final stages reduces the likelihood of recourse to a new point worse than actual solution.

Solutions in the all strategies were highly different for every start point. Especially it is seen in strategy II and III. Differences between values of quality factor ISE were very small. It indicates a large number of local extremes in objective whit very similar values. Despite this the simulated annealing method gives good results in all cases.

Comparison of fractional and integral order controller shows that both controllers gives similar step response. Fractional controller has slightly shorter rise time.

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