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CONTROL AND INFORMATICS

# Single–server queueing system with limited queue, random volume customers and unlimited sectorized memory buffer

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**Abstract.** In the present paper, we analyze the model of a single–server queueing system with limited number of waiting positions, random volume customers and unlimited sectorized memory buffer. In such a system, the arriving customer is additionally characterized by a non–negative random volume vector whose indications usually represent the portions of unchanged information of a different type that are located in sectors of unlimited memory space dedicated for them during customer presence in the system. When the server ends the service of a customer, information immediately leaves the buffer, releasing resources of the proper sectors. We assume that in the investigated model, the service time of a customer is dependent on his volume vector characteristics. For such defined model, we obtain a general formula for steady–state joint distribution function of the total volume vector in terms of Laplace-Stieltjes transforms. We also present practical results for some special cases of the model together with formulae for steady–state initial moments of the analyzed random vector, in cases where the memory buffer is composed of at most two sectors. Some numerical computations illustrating obtained theoretical results are attached as well.

**Key words:** single–server queueing system; queueing systems with random volume customers; sectorized memory buffer; total volume vector; Laplace–Stieltjes transform.

## 1. INTRODUCTION

The classical queueing theory usually deals with models in which the arriving customers are assumed to be homogeneous. It means that their basic characteristics such as service time distribution are the same and they differ substantially only in arrival times. This assumption is taken into consideration in almost all analyses of well–known models of type $M/M/n/m$, $M/G/n/0$, $M/G/1/\infty$, $M/G/1/m$, $GI/M/n/\infty$ and queueing system with processor sharing $M/G/1/\infty - EPS$ [1–3]. On one hand such simplification makes analysis less complicated, but on the other hand the results of investigations cannot be often applied in real computer or telecommunication systems.

Indeed, nowadays in many modern technical systems that are designed for customers' servicing, we must treat customers as non–homogeneous. They may exemplary have different service time characteristics, different priorities, come from different sources or be characterized by other random requirements. These additional assumptions make research more complicated but let us successfully use introduced models in practice. Some aspects of customers' non–homogeneity have recently appeared in many works of researchers from different countries [4–9].

In many papers, authors assume that customers differ in their volumes (sizes). Such situation happens when they transport information that is integrally stored in memory buffer of a system until their service termination. This approach has caused appearance of the new discipline called queueing systems with random volume customers. It is still rather a novel but strongly developing direction in applied mathematics which has many various applications, especially in computer science.

Moving on to practice, in many real systems customer service time is dependent on his volume (size of the portion of information he delivers, measured in bytes). The area of investigation in this case is much wider than for the classical queueing models. Besides getting the characteristics of number of customers present in the system or waiting time characteristics (for models with unlimited queue), we also want to obtain characteristics of the total volume of customers (the sum of the volumes of all customers present in the system) and loss characteristics when memory buffer size is limited. Initially, such models were investigated by the tools of classical queueing theory [10, 11] but it turned out that obtained theoretical results were not compatible with simulation ones. It was because computations did not take into account an existing dependence between customer volume and his service time.

First papers investigating systems with random volume customers that introduced new methods of research and extended mathematical apparatus appeared in the last decades of the twentieth century [12, 13]. The popularity of this direction has been increasing in the last few years and there are a lot of publications on this topic, mainly due to the progress in computer science and possible applications of analyzed models or obtained results in real systems. Some interesting investigations

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can be found e.g. in [14–28] but we must emphasize that in many of cited papers service time of a customer is still often treated as independent with regard to his volume which does not let us use the results in many practical systems, and the others include only approximate analysis and do not deal with calculating exact total volume characteristics (even in steady state).

Considering the theory of queueing systems with random volume customers, we always must take into account two main important aspects. The first of them is a possible limitation of total volume and the second is the character of dependency between customer's volume and his service time (dependent or not). So we can analyze models with limited or unlimited memory buffer in which service time of a customer can be independent or dependent on his volume. Finally, we have four classes of models [2, 29, 30]. Models from first class (unlimited total volume and service time of a customer independent on his volume) are trivial and their analysis needs no extended methods (we can obtain results using classical ones). Models belonging to second class (limited memory buffer but still independent customer volume and his service time) are only a little bit more difficult because they need just small modifications in classical methods. Unfortunately, independence assumption in these two classes makes that models rarely can be used in practice as service time of a customer (e.g. packet) in real computer or telecommunication systems is usually dependent on his size (often proportional). The third class (unlimited memory buffer and dependence between customer service time and his volume) is very interesting because we may obtain in this case very practical characteristics of the total volume of customers present in the system. Here analysis demands introducing significant generalizations of methods known from classical queueing theory (see e.g. [31] in which servers are additionally non–identical). Moreover, results obtained for the models from the third class can be used to calculate approximate loss characteristics for analogous models but with limited memory buffer (the fourth class) [32]. The last fourth class is the most difficult (but the most practical) and exact results were obtained only for systems with no waiting positions (see e.g. [33–35]). Some of obtained results can be used in real computer or telecommunication system designing process (e.g. calculating required sizes of memory buffers) as packets of data seem to be good representation of random volume customers.

In addition, investigations have recently become concentrated on systems in which customers are characterized by some random volume vectors (their volume is understood as multidimensional). This assumption is connected with practical observation that in computer networks customers (packets) may transport information of a different type (packets are composed of parts storing some specific data – text parts, attachment parts, audio parts, video parts and so on). These parts are located in separate sectors of memory buffer until customer ends his service. The behavior of such systems is schematically presented in Fig. 1. This approach can be found in technical reports [36, 37] and first important analyses of systems with random volume customers and sectorized memory buffer can be found e.g. in [38–40].

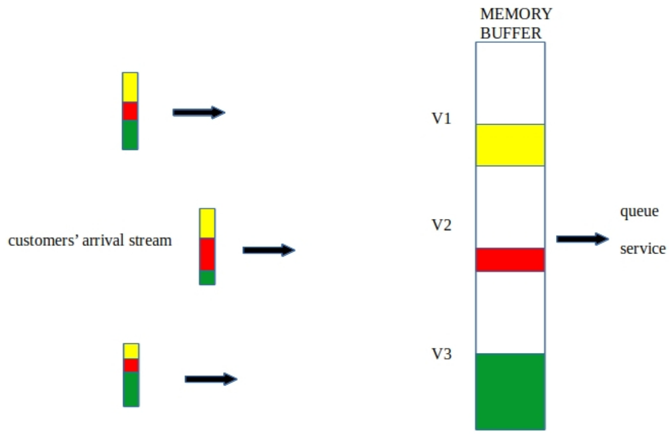


**Fig. 1.** Scheme of a queueing system with random volume customers and sectorized memory buffer

In our paper we also deal with analysis of chosen model of the mentioned above type. However, this time we investigate much more complicated model in which the queue of waiting customers is limited and service time of a customer is dependent on his volume vector. These practical (in line with reality) assumptions make research more complex from the mathematical point of view (compared to the cited above papers). E.g. in [40] we analyze either models without a queue or models in which the queue is unlimited. The analysis is a little bit less complicated in this case. But, on the other hand, results obtained in the present paper may have many various applications in real telecommunication or computer systems designing process. The model might be exemplary used for application–level servers and routers inside the computer network. This paper presents an exact analysis of single–server queueing system (of $M/G/1/m$–type) with random volume customers and sectorized unlimited memory buffer. The rest of the paper is organized as follows. In Section 2, we introduce necessary notations and show some mathematical background of our research. In Section 3, we present obtained formula for steady–state Laplace–Stieltjes transform of total volume vector for analyzed model which is the main result of this paper. Next Section 4 contains exemplary results for some special cases of the model. In Section 5, we present calculations for the limitary case ($M/G/1/\infty$ system). In Section 6, we discuss situation in which random volume vector is one-dimensional whereas in Section 7 – the case of two-dimensional vector. These two sections additionally contain formulae for steady–state initial moments of total volume vector and some numerical computations. The last Section 8 presents conclusions and final remarks.

## 2. THE MODEL AND BASIC NOTATIONS

In this section, we introduce some necessary notations that are used in analyzing of single–server queueing system with limited number of waiting positions and unlimited sectorized memory buffer.

We analyze a queueing system $M/G/1/m$, $m = 0, 1, \ldots,$ (in our notation $m$ does not include service position) in steady state (we assume that the utilization coefficient $\rho$ satisfies the condition: $\rho = a\beta_1 < \infty$), where $a$ is an arrival rate (time between neighboring moments of customers' arrival is exponentially distributed with parameter $a$) and $\beta_1$ is the mean value of customer's service time $\xi$. We denote by $B(t) = \mathbf{P}\{\xi < t\}$ its distribution function (DF). We assume that each customer in the system is characterized by $n$-dimensional random (volume) vector $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_n)$, $n = 1,2,\ldots$, where indications $\zeta_1,\ldots,\zeta_n$ are non–negative random variables (RVs). Let

$$L(\boldsymbol{x}) = L(x_1,\ldots,x_n) = \mathbf{P}\{\zeta_1 < x_1,\ldots,\zeta_n < x_n\} = \mathbf{P}\{\boldsymbol{\zeta} < \boldsymbol{x}\}$$

be the joint DF of RVs $\zeta_1,\ldots,\zeta_n$ (or DF of the random vector $\boldsymbol{\zeta}$).

We also assume that service time of the customer on service $\xi$ generally depends on vector $\boldsymbol{\zeta}$. So, the following $(n+1)$-dimensional DF is defined:

$$F(\boldsymbol{x},t) = \mathbf{P}\{\boldsymbol{\zeta} < \boldsymbol{x}, \xi < t\}.$$

Then, we obviously have $L(\boldsymbol{x}) = F(\boldsymbol{x},\infty)$, $B(t) = F(\vec{\infty},t)$, where $\vec{\infty} = \underbrace{(\infty,\ldots,\infty)}_{n}$.

Let $\sigma_i(t)$ be the sum of $i$th indications of all customers present in the system at time instant $t$, $i = 1,\ldots,n$. Now we introduce vector $\boldsymbol{\sigma}(t) = (\sigma_1(t),\ldots,\sigma_n(t))$, that is usually called total volume vector. In steady state we have $\sigma_i(t) \Rightarrow \sigma_i$, $i = 1,\ldots,n$, and $\boldsymbol{\sigma}(t) \Rightarrow \boldsymbol{\sigma} = (\sigma_1,\ldots,\sigma_n)$ in the sense of a weak convergence, where $\sigma_i$, $\boldsymbol{\sigma}$ are the proper steady–state (limit) random characteristics of the system.

Our aim is the determination of $n$-dimensional Laplace–Stieltjes transform (LST) $\delta(\boldsymbol{s}) = \delta(s_1,\ldots,s_n)$ of the random vector $\boldsymbol{\sigma}$:

$$\delta(\boldsymbol{s}) = \mathbf{E}e^{-(\boldsymbol{s},\boldsymbol{\sigma})} = \int_{\boldsymbol{0}}^{\vec{\infty}} e^{-(\boldsymbol{s},\boldsymbol{x})}\,\mathrm{d}D(\boldsymbol{x}),$$

where $D(\boldsymbol{x}) = D(x_1,\ldots,x_n) = \mathbf{P}\{\sigma_1 < x_1,\ldots,\sigma_n < x_n\} = \mathbf{P}\{\boldsymbol{\sigma} < \boldsymbol{x}\}$ is the joint DF of vector's $\boldsymbol{\sigma}$ indications and $(\boldsymbol{s},\boldsymbol{x}) = \sum_{i=1}^{n} s_i x_i$ denotes the scalar product of the vectors $\boldsymbol{s}$ and $\boldsymbol{x}$.

## 3. THE MAIN RESULT

In this section we present main result of our investigations connected with obtaining of general formula for LST $\delta(\boldsymbol{s})$.

Denote by $p_k(u)\,\mathrm{d}u$ the steady–state probability that there are $k$ customers in the system at arbitrary time instant and time $\xi^*$ passed from the beginning of service (of currently being served customer) to this time, where $\xi^* \in [u, u+\mathrm{d}u)$, $k = 1,\ldots,m+1$. Time $\xi^*$ is often called elapsed time of a customer. Let $\eta$ be the number of customers present in the system in steady state. Evidently, due to the total probability theorem, we obtain the following formula for the steady–state probabilities $p_k$:

$$p_k = \mathbf{P}\{\eta = k\} = \int_0^\infty p_k(u)\,\mathrm{d}u, \quad k = 1,\ldots,m+1,$$

whereas value of $p_0 = \mathbf{P}\{\eta = 0\}$ can be found from the normalization condition $\sum_{k=0}^{m+1} p_k = 1$.

As it follows from [1,2], we have (if $m > 0$)

$$p_k(u) = e^{-au}[1 - B(u)] \sum_{i=0}^{k-1} p_{k-i}(0)\frac{(au)^i}{i!}, \ k = 1,\ldots,m, \quad (1)$$

$$p_{m+1}(u) = [1 - B(u)] \sum_{k=1}^{m} p_k(0) - \sum_{k=1}^{m} p_k(u). \quad (2)$$

In the case of $m = 0$, we easily obtain

$$p_1(u) = ap_0[1 - B(u)]. \quad (2a)$$

Let

$$\alpha(\boldsymbol{s},q) = \mathbf{E}e^{-(\boldsymbol{s},\boldsymbol{\zeta})-q\xi} = \int_{\boldsymbol{x}=\boldsymbol{0}}^{\vec{\infty}} \int_{t=0}^{\infty} e^{-(\boldsymbol{s},\boldsymbol{x})-qt}\,\mathrm{d}F(\boldsymbol{x},t)$$

be LST of the function $F(\boldsymbol{x},t)$ and analogously

$$\varphi(\boldsymbol{s}) = \alpha(\boldsymbol{s},0) = \int_{\boldsymbol{0}}^{\vec{\infty}} e^{-(\boldsymbol{s},\boldsymbol{x})}\,\mathrm{d}L(\boldsymbol{x})$$

and

$$\beta(q) = \alpha(\boldsymbol{0},q) = \int_0^\infty e^{-qt}\,\mathrm{d}B(t)$$

be LSTs of the function $L(\boldsymbol{x})$ and $B(t)$, respectively.

**Theorem 1.** The function $\delta(\boldsymbol{s})$ for the system under consideration is determined by the following relations:

$$\delta(\boldsymbol{s}) = p_0 - (\varphi(\boldsymbol{s}))^m \alpha'_q(\boldsymbol{s},q)\big|_{q=0} \sum_{k=1}^{m} p_k(0) + \frac{1}{a}\sum_{k=1}^{m}\left[(\varphi(\boldsymbol{s}))^{k-1} - (\varphi(\boldsymbol{s}))^m\right] \times \sum_{i=0}^{k-1} p_{k-i}(0)\left[\varphi(\boldsymbol{s}) - \sum_{j=0}^{i} R_j(\boldsymbol{s},a)\right],$$

where

$$R_j(\boldsymbol{s},a) = \frac{a^j}{j!}\int_{\boldsymbol{x}=\boldsymbol{0}}^{\vec{\infty}} \int_{t=0}^{\infty} t^j e^{-(\boldsymbol{s},\boldsymbol{x})-at}\,\mathrm{d}F(\boldsymbol{x},t), \quad \text{if } m > 0;$$

$$\delta(\boldsymbol{s}) = p_0\left[1 - a\alpha'_q(\boldsymbol{s},q)\big|_{q=0}\right], \quad \text{if } m = 0. \quad (3)$$

**Proof.** Assume that $m > 0$. On the base of total probability theorem, in analogous way as it was done in [40] for the $M/G/1/\infty$ system with random volume customers and sectorized memory buffer, we can present DF $D(\boldsymbol{x})$ in the following form:

$$D(\boldsymbol{x}) = p_0 + \sum_{k=1}^{m+1} \int_0^\infty p_k(u) \left[ L_*^{(k-1)} * E_u(\mathbf{x}) \right] \, du,$$

where $L_*^{(k-1)}(\boldsymbol{x})$ denotes $(k-1)$-dimensional convolution of DF $L(\boldsymbol{x})$ and $E_u(\boldsymbol{x})$ – conditional joint DF of customer on service volume vector, under condition that his elapsed time equals $u$.

Obviously, on the base of properties of LST transform (applied to the left and right side of the above formula), we can present function $\delta(\boldsymbol{s})$ in the following form:

$$\delta(\boldsymbol{s}) = p_0 + \sum_{k=1}^{m+1} (\varphi(\boldsymbol{s}))^{k-1} \int_0^\infty p_k(u) e_u(\boldsymbol{s}) \, du,$$

where $e_u(\boldsymbol{s}) = [1-B(u)]^{-1} \int_{\mathbf{0}}^{\vec{\infty}} e^{-(\boldsymbol{s},\boldsymbol{x})} \int_{y=u}^{\infty} dF(\boldsymbol{x},y)$ is LST of DF $E_u(\boldsymbol{x})$. From (1) and (2) we obtain

$$\begin{aligned}
\delta(\boldsymbol{s}) &= p_0 + \sum_{k=1}^{m} (\varphi(\boldsymbol{s}))^{k-1} \int_0^\infty p_k(u) e_u(\boldsymbol{s}) \, du \\
&\quad + (\varphi(\boldsymbol{s}))^m \int_0^\infty \left\{ [1-B(u)] \sum_{k=1}^m p_k(0) - \sum_{k=1}^m p_k(u) \right\} e_u(\boldsymbol{s}) \, du \\
&= p_0 + \sum_{k=1}^m \left[ (\varphi(\boldsymbol{s}))^{k-1} - (\varphi(\boldsymbol{s}))^m \right] \int_0^\infty p_k(u) e_u(\boldsymbol{s}) \, du \\
&\quad + \ (\varphi(\boldsymbol{s}))^m \sum_{k=1}^m p_k(0) \int_0^\infty [1-B(u)] \, e_u(\boldsymbol{s}) \, du. \qquad (4)
\end{aligned}$$

If we calculate the last integral, taking into consideration the relation for $e_u(\boldsymbol{s})$, we obtain:

$$\begin{aligned}
\int_0^\infty [1-B(u)] \, e_u(\boldsymbol{s}) \, du &= \int_0^\infty du \int_{\mathbf{0}}^{\vec{\infty}} e^{-(\boldsymbol{s},\boldsymbol{x})} \int_u^\infty dF(\boldsymbol{x},t) \\
&= \int_{\mathbf{0}}^{\vec{\infty}} \int_0^\infty e^{-(\boldsymbol{s},\boldsymbol{x})} \, dF(\boldsymbol{x},t) \int_0^t du \\
&= \int_{\mathbf{0}}^{\vec{\infty}} \int_0^\infty t e^{-(\boldsymbol{s},\boldsymbol{x})} \, dF(\boldsymbol{x},t) \\
&= a^{-1} R_1(\boldsymbol{s},0) = -\alpha'_q(\boldsymbol{s},q)\big|_{q=0}. \qquad (5)
\end{aligned}$$

Let us calculate the integral

$$\begin{aligned}
\int_0^\infty & p_k(u) e_u(\boldsymbol{s}) \, du \\
&= \int_0^\infty e^{-au} \, du \sum_{i=0}^{k-1} p_{k-i}(0) \frac{(au)^i}{i!} \int_{\mathbf{0}}^{\vec{\infty}} e^{-(\boldsymbol{s},\boldsymbol{x})} \int_u^\infty dF(\boldsymbol{x},t) \\
&= \sum_{i=0}^{k-1} \frac{a^i p_{k-i}(0)}{i!} \int_{\mathbf{0}}^{\vec{\infty}} \int_0^\infty e^{-(\boldsymbol{s},\boldsymbol{x})} \, dF(\boldsymbol{x},t) \int_0^t u^i e^{-au} \, du.
\end{aligned}$$

It is known that

$$\int_0^t u^i e^{-au} \, du = \frac{i!}{a^{i+1}} \left[ 1 - e^{-at} \sum_{j=0}^i \frac{(at)^j}{j!} \right],$$

whereas we obtain:

$$\begin{aligned}
\int_0^\infty & p_k(u) e_u(\boldsymbol{s}) \, du \\
&= \sum_{i=0}^{k-1} \frac{p_{k-i}(0)}{a} \int_{\mathbf{0}}^{\vec{\infty}} \int_0^\infty e^{-(\boldsymbol{s},\boldsymbol{x})} \left[ 1 - e^{-at} \sum_{j=0}^i \frac{(at)^j}{j!} \right] dF(\boldsymbol{x},t) \\
&= \sum_{i=0}^{k-1} \frac{p_{k-i}(0)}{a} \left[ \int_{\mathbf{0}}^{\vec{\infty}} \int_0^\infty e^{-(\boldsymbol{s},\boldsymbol{x})} \, dF(\boldsymbol{x},t) \right. \\
&\qquad \left. - \sum_{j=0}^i \int_{\mathbf{0}}^{\vec{\infty}} \int_0^\infty \frac{(at)^j}{j!} e^{-(\boldsymbol{s},\boldsymbol{x})-at} \, dF(\boldsymbol{x},t) \right] \\
&= \frac{1}{a} \sum_{i=0}^{k-1} p_{k-i}(0) \left[ \varphi(\boldsymbol{s}) - \sum_{j=0}^i R_j(\boldsymbol{s},a) \right]. \qquad (6)
\end{aligned}$$

If we substitute (5) and (6) into (4), we obtain the first relation from the statement of the theorem.

If $m = 0$, DF $D(\boldsymbol{x})$ can be presented in a simpler form:

$$D(\boldsymbol{x}) = p_0 + \int_0^\infty p_1(u) E_u(\boldsymbol{x}) \, du.$$

If we use LST to the both sides of the above relation, we obtain:

$$\delta(\boldsymbol{s}) = p_0 + \int_0^\infty p_1(u) e_u(\boldsymbol{s}) \, du.$$

Taking into consideration relation for $e_u(\boldsymbol{s})$ and formulae (2a) and (5), we finally have:

$$\begin{aligned}
\delta(\boldsymbol{s}) &= p_0 + a p_0 \int_0^\infty du \int_{\mathbf{0}}^{\vec{\infty}} e^{-(\boldsymbol{s},\boldsymbol{x})} \int_u^\infty dF(\boldsymbol{x},t) \\
&= p_0 - a p_0 \alpha'_q(\boldsymbol{s},q)\big|_{q=0},
\end{aligned}$$

which proves the second relation from theorem. The same result can be obtained as the special case of the $M/G/n/0$ system with random volume customers and sectorized memory buffer, substituting $n = 1$ (see again [40]). □

Note that, to calculate vector's $\boldsymbol{\sigma}$ characteristics (if $m > 0$), we can use relations $p_1(0) = a(p_0 + p_1)$, $p_i(0) = ap_i$, $i = 2,\ldots,m$, and $p_{m+1} = 1-(1-p_0)/\rho$ [2]. Then, we can obtain the following, more convenient form of the formula from Theorem 1:

$$\begin{aligned}\delta(\boldsymbol{s}) &= p_0 - \frac{1-p_0}{\beta_1}(\varphi(\boldsymbol{s}))^m \alpha'_q(\boldsymbol{s},q)\big|_{q=0}\\ &+\sum_{k=1}^{m}\left[(\varphi(\boldsymbol{s}))^{k-1}-(\varphi(\boldsymbol{s}))^m\right]\left\{p_0\left[\varphi(\boldsymbol{s})-\sum_{j=0}^{k-1}R_j(\boldsymbol{s},a)\right]\right.\\ &\left.+\sum_{i=1}^{k}p_i\left[\varphi(\boldsymbol{s})-\sum_{j=0}^{k-i}R_j(\boldsymbol{s},a)\right]\right\}, \end{aligned} \tag{7}$$

where probabilities $p_k = \mathbf{P}\{\eta = k\}$ can be determined by the following algorithm [2]:

1. Calculate the quantities:
$$\beta_k(a) = \frac{1}{k!}\int_0^\infty (at)^k e^{-at}\,\mathrm{d}B(t), \quad k = 0,\ldots,m-1.$$
2. Calculate $\gamma_k(a)$ $(k = 0,\ldots,m-1)$ using following relations:
$$\gamma_0(a) = 1-\beta_0(a);$$
$$\gamma_k(a) = \gamma_{k-1}(a)-\beta_k(a), \quad k = 1,\ldots,m-1.$$
3. Calculate the quantities $S_k$ $(k = 0,\ldots,m-1)$:
$$S_0 = 1/\beta_0(a);$$
$$S_k = \frac{1}{\beta_0(a)}\sum_{i=0}^{k-1}\gamma_{k-i}(a)S_i, \quad k = 1,\ldots,m-1.$$
4. Calculate $p_0$:
$$p_0 = \left[1+\rho\sum_{k=0}^{m-1}S_k\right]^{-1}.$$
5. Calculate $p_k$ $(k = 1,\ldots,m+1)$ under following relations:
$$p_1 = p_0(S_0-1);$$
$$p_k = p_0 S_{k-1}, \quad k = 2,\ldots,m,\ m > 1;$$
$$p_{m+1} = 1-(1-p_0)/\rho.$$

Some approximate relations are also known for probabilities $p_k$ calculations (see e.g. [41]).

If $m = 0$, we obtain: $p_0 = \dfrac{1}{1+\rho}$, $p_1 = \dfrac{\rho}{1+\rho}$ and finally:

$$\delta(\boldsymbol{s}) = \frac{1}{1+\rho}\left[1-a\alpha'_q(\boldsymbol{s},q)\big|_{q=0}\right].$$

## 4. EXEMPLARY RESULTS FOR SOME SPECIAL CASES

In this section we present simple results for some special cases of the investigated model.

1. **System $M/G/1/1$.** Let $m = 1$. Then, we have from Theorem 1:
$$\begin{aligned}\delta(\boldsymbol{s}) &= p_0 - \varphi(\boldsymbol{s})\alpha'_q(\boldsymbol{s},q)\big|_{q=0}p_1(0)\\ &+ a^{-1}(1-\varphi(\boldsymbol{s}))\,p_1(0)\,[\varphi(\boldsymbol{s})-R_0(\boldsymbol{s},a)]. \end{aligned}$$
In this case we obviously have $R_0(\boldsymbol{s},a) = \alpha(\boldsymbol{s},a)$, $\beta_0(a) = \beta(a)$. It is also easy to prove that
$$p_0 + p_1 = \frac{p_0}{\beta(a)},$$
whereas, taking into consideration (7), we obtain the other form of the formula:
$$\begin{aligned}\delta(\boldsymbol{s}) &= p_0 - \frac{1-p_0}{\beta_1}\varphi(\boldsymbol{s})\alpha'_q(\boldsymbol{s},q)\big|_{q=0}\\ &+\frac{p_0}{\beta(a)}(1-\varphi(\boldsymbol{s}))(\varphi(\boldsymbol{s})-\alpha(\boldsymbol{s},a)). \end{aligned} \tag{8}$$
2. **System $M/G/1/2$.** For $m = 2$ we obtain analogously after some calculations:
$$\begin{aligned}\delta(\boldsymbol{s}) &= p_0 - \frac{1-p_0}{\beta_1}(\varphi(\boldsymbol{s}))^2\alpha'_q(\boldsymbol{s},q)\big|_{q=0}\\ &+(1-\varphi(\boldsymbol{s}))\left\{\frac{1}{\rho}(1-p_0)(\varphi(\boldsymbol{s}))^2+\frac{p_0}{\beta(a)}(\varphi(\boldsymbol{s})-\alpha(\boldsymbol{s},a))\right.\\ &-\frac{\beta(a)(1-p_0)-\rho p_0}{\rho\beta(a)}\varphi(\boldsymbol{s})\alpha(\boldsymbol{s},a)+\frac{p_0}{\beta(a)}\varphi(\boldsymbol{s})\\ &\left.\times[\varphi(\boldsymbol{s})-2\alpha(\boldsymbol{s},a)-R_1(\boldsymbol{s},a)]\right\}. \end{aligned} \tag{9}$$

## 5. THE CASE OF UNLIMITED QUEUE

In this section, basing on the proved theorem, we will obtain formula for $\delta(\boldsymbol{s})$ for single–server queueing system with random volume customers, unlimited queue and sectorized memory buffer.

Assume now that $m \to \infty$ and $\rho < 1$, so we analyze $M/G/1/\infty$ queueing system in a steady state, and let us calculate $\delta^{(\infty)}(\boldsymbol{s}) = \lim\limits_{m\to\infty}\delta(\boldsymbol{s})$. Then, from theorem 1 and the fact that $|\varphi(\boldsymbol{s})| < 1$ (so, obviously, $\lim\limits_{m\to\infty}(\varphi(\boldsymbol{s}))^m = 0$ and $\sum\limits_{i=0}^{\infty}(\varphi(\boldsymbol{s}))^i = (1-\varphi(\boldsymbol{s}))^{-1}$), we obtain:

$$\begin{aligned}\delta^{(\infty)}(\boldsymbol{s}) &= \lim_{m\to\infty}\delta(\boldsymbol{s}) = p_0^{(\infty)}\\ &+a^{-1}\sum_{k=1}^{\infty}(\varphi(\boldsymbol{s}))^{k-1}\sum_{i=0}^{k-1}p_{k-i}(0)\left[\varphi(\boldsymbol{s})-\sum_{j=0}^{i}R_j(\boldsymbol{s},a)\right], \end{aligned} \tag{10}$$

where $p_0^{(\infty)} = \lim\limits_{m\to\infty}p_0$.

Relation (10) can be written out in the following form (we denote here $p_i^{(\infty)}(0) = \lim_{m\to\infty} p_i(0)$):

$$\delta^{(\infty)}(\mathbf{s}) = p_0^{(\infty)} + a^{-1}\left[\sum_{k=1}^{\infty}(\varphi(\mathbf{s}))^k \sum_{i=1}^{k} p_i^{(\infty)}(0) - \sum_{k=1}^{\infty}(\varphi(\mathbf{s}))^{k-1}\sum_{i=1}^{k} p_i^{(\infty)}(0)\sum_{j=0}^{k-i} R_j(\mathbf{s},a)\right]. \quad (11)$$

The first sum in (11) is calculated as follows:

$$\sum_{k=1}^{\infty}(\varphi(\mathbf{s}))^k \sum_{i=1}^{k} p_i^{(\infty)}(0) = \sum_{i=1}^{\infty}(\varphi(\mathbf{s}))^i p_i^{(\infty)}(0)\sum_{k=i}^{\infty}(\varphi(\mathbf{s}))^{k-i} = (1-\varphi(\mathbf{s}))^{-1}\sum_{i=1}^{\infty} p_i^{(\infty)}(0)\,(\varphi(\mathbf{s}))^i .$$

If we denote $p_i^{(\infty)} = \lim_{m\to\infty} p_i$ and take into consideration that $p_1^{(\infty)}(0) = a\left(p_0^{(\infty)} + p_1^{(\infty)}\right)$ and $p_i^{(\infty)}(0) = a p_i^{(\infty)}$, $i = 2,\dots,m$, we easily obtain, when $m\to\infty$, after simple transformations that

$$(1-\varphi(\mathbf{s}))^{-1}\sum_{i=1}^{\infty} p_i^{(\infty)}(0)\,(\varphi(\mathbf{s}))^i = \frac{a}{1-\varphi(\mathbf{s})}\left[p_0^{(\infty)}\varphi(\mathbf{s}) + \sum_{i=1}^{\infty} p_i^{(\infty)}\,(\varphi(\mathbf{s}))^i\right] = \frac{a}{1-\varphi(\mathbf{s})}\left[p_0^{(\infty)}\varphi(\mathbf{s}) + P(\varphi(\mathbf{s})) - p_0^{(\infty)}\right],$$

where $P(z) = \dfrac{p_0^{(\infty)}(1-z)\beta(a-az)}{\beta(a-az)-z}$ is the generation function of steady–state customers number distribution for the system $M/G/1/\infty$. As a result, we obtain:

$$\sum_{k=1}^{\infty}(\varphi(\mathbf{s}))^k \sum_{i=1}^{k} p_i^{(\infty)}(0) = \frac{a p_0^{(\infty)}\varphi(\mathbf{s})}{\beta(a-a\varphi(\mathbf{s}))-\varphi(\mathbf{s})}. \quad (12)$$

The second sum in (11) can be presented as follows:

$$\sum_{k=1}^{\infty}(\varphi(\mathbf{s}))^{k-1}\sum_{i=1}^{k} p_i^{(\infty)}(0)\sum_{j=0}^{k-i} R_j(\mathbf{s},a) = (\varphi(\mathbf{s}))^{-1}\sum_{i=1}^{\infty} p_i^{(\infty)}(0)\,(\varphi(\mathbf{s}))^i \sum_{k=0}^{\infty}(\varphi(\mathbf{s}))^k \sum_{j=0}^{k} R_j(\mathbf{s},a). \quad (13)$$

Here we have:

$$\sum_{k=0}^{\infty}(\varphi(\mathbf{s}))^k\sum_{j=0}^{k} R_j(\mathbf{s},a) = \sum_{j=0}^{\infty} R_j(\mathbf{s},a)\,(\varphi(\mathbf{s}))^j \sum_{k=j}^{\infty}(\varphi(\mathbf{s}))^{k-j} = (1-\varphi(\mathbf{s}))^{-1}\sum_{j=0}^{\infty} R_j(\mathbf{s},a)\,(\varphi(\mathbf{s}))^j . \quad (14)$$

Taking into consideration relation (3), we obtain:

$$\sum_{j=0}^{\infty} R_j(\mathbf{s},a)\,(\varphi(\mathbf{s}))^j = \int_{\mathbf{0}}^{\vec{\infty}}\int_0^{\infty} e^{-(\mathbf{s},\mathbf{x})-at}\sum_{j=0}^{\infty}\frac{(at\varphi(\mathbf{s}))^j}{j!}\,\mathrm{d}F(\mathbf{x},t) = \int_{\mathbf{0}}^{\vec{\infty}}\int_0^{\infty} e^{-(\mathbf{s},\mathbf{x})-(a-a\varphi(\mathbf{s}))t}\,\mathrm{d}F(\mathbf{x},t) = \alpha(\mathbf{s},a-a\varphi(\mathbf{s})). \quad (15)$$

If we substitute (14) (taking into consideration (15)) to (13), we obtain (see also (12)):

$$\sum_{k=1}^{\infty}(\varphi(\mathbf{s}))^{k-1}\sum_{i=1}^{k} p_i^{(\infty)}(0)\sum_{j=0}^{k-i} R_j(\mathbf{s},a) = \frac{a p_0^{(\infty)}\alpha(\mathbf{s},a-a\varphi(\mathbf{s}))}{\beta(a-a\varphi(\mathbf{s}))-\varphi(\mathbf{s})}. \quad (16)$$

It can be easily shown that $p_0^{(\infty)} = \lim_{m\to\infty} p_0 = 1-\rho$. Indeed, we have relation

$$\lim_{m\to\infty} p_{m+1} = \lim_{m\to\infty}\left(1-\frac{1-p_0}{\rho}\right) = 1-\frac{1-p_0^{(\infty)}}{\rho}.$$

But, of course, series $\sum_{i=0}^{k} p_i$ is convergent what implies that $\lim_{m\to\infty} p_{m+1} = 0$, so we finally obtain mentioned above formula for $p_0^{(\infty)}$. If we now substitute (12) and (16) into (11), we finally obtain:

$$\delta^{(\infty)}(\mathbf{s}) = (1-\rho)\left[1+\frac{\varphi(\mathbf{s})-\alpha(\mathbf{s},a-a\varphi(\mathbf{s}))}{\beta(a-a\varphi(\mathbf{s}))-\varphi(\mathbf{s})}\right]. \quad (17)$$

The same result was obtained in [40].

## 6. SYSTEM $M/G/1/M$ WITH RANDOM VOLUME CUSTOMERS AND ONE-DIMENSIONAL MEMORY BUFFER

In this section we investigate the case $n = 1$, i.e. we assume that each customer is characterized by random one-dimensional volume $\zeta$. In the same way we define the following DF: $F(x,t) = \mathbf{P}\{\zeta < x,\ \xi < t\}$, $L(x) = \mathbf{P}\{\zeta < x\} = F(x,\infty)$, $B(t) = \mathbf{P}\{\xi < t\} = F(\infty,t)$, $\sigma(t)$ is a total volume of customers present in the system at time instant $t$ ($\sigma(t)\Rightarrow\sigma$, if $\rho<\infty$, where $\sigma$ is the steady-state total volume), $D(x) = \mathbf{P}\{\sigma < x\}$ is DF of RV $\sigma$, $\delta(s) = \mathbf{E}e^{-s\sigma} = \int_0^{\infty} e^{-sx}\,\mathrm{d}D(x)$ is LST of DF $D(x)$, $\alpha(s,q) = \int_0^{\infty}\int_0^{\infty} e^{-sx-qt}\,\mathrm{d}F(x,t)$ is LST of DF $F(x,t)$, $\varphi(s) = \alpha(s,0)$ is LST of DF $L(x)$ and $\beta(q) = \alpha(0,q)$ is LST of DF $B(t)$.

From Theorem 1 we obtain (in the case of $n = 1$ and $m > 0$):

$$\delta(s) = p_0 - (\varphi(s))^m \alpha'_q(s,q)\big|_{q=0} \sum_{k=1}^{m} p_k(0) + \frac{1}{a}\sum_{k=1}^{m}\left[(\varphi(s))^{k-1} - (\varphi(s))^m\right] \times \sum_{i=0}^{k-1} p_{k-i}(0)\left[\varphi(s) - \sum_{j=0}^{i} R_j(s,a)\right], \tag{18}$$

where

$$R_j(s,a) = \frac{a^j}{j!}\int_0^\infty\int_0^\infty t^j e^{-sx-at}\,\mathrm{d}F(x,t),$$

or, in other form,

$$\delta(s) = p_0 - \frac{1-p_0}{\beta_1}(\varphi(s))^m \alpha'_q(s,q)\big|_{q=0} + \sum_{k=1}^{m}\left[(\varphi(s))^{k-1} - (\varphi(s))^m\right]\left\{p_0\left[\varphi(s) - \sum_{j=0}^{k-1} R_j(s,a)\right] + \sum_{i=1}^{k} p_i\left[\varphi(s) - \sum_{j=0}^{k-i} R_j(s,a)\right]\right\}. \tag{19}$$

Let

$$\alpha_{ij} = \mathbf{E}\left(\zeta^i \xi^j\right) = (-1)^{i+j}\frac{\partial^{i+j}}{\partial s^i \partial q^j}\alpha(s,q)\big|_{s=q=0},$$

$i, j = 1, 2, \ldots$, be the mixed $(i+j)$-th moment of RVs $\zeta$ and $\xi$, $\beta_i$ denotes the $i$-th moment of RV $\xi$ and $\varphi_i$ – the $i$-th moment of RV $\zeta$.

The first and second moments of RV $\sigma$ are determined by the following relations:

$$\delta_1 = \mathbf{E}\sigma = -\delta'(0) = \frac{1-p_0}{\beta_1}(\alpha_{11} + m\varphi_1\beta_1) - \sum_{k=1}^{m}(m-k+1)\varphi_1\left\{p_0\left[1 - \sum_{j=0}^{k-1}\beta_j(a)\right] + \sum_{i=1}^{k} p_i\left[1 - \sum_{j=0}^{k-i}\beta_j(a)\right]\right\}; \tag{20}$$

$$\delta_2 = \mathbf{E}\sigma^2 = \delta''(0) = \frac{1-p_0}{\beta_1}\left[\alpha_{21} + 2m\varphi_1\alpha_{11} + m\varphi_2\beta_1 + m(m-1)\varphi_1^2\beta_1\right] - \sum_{k=1}^{m}\left\{(m-k+1)\varphi_2 + [m(m-1) - (k-1)(k-2)]\varphi_1^2\right\} \times \left\{p_0\left[1 - \sum_{j=0}^{k-1}\beta_j(a)\right] + \sum_{i=1}^{k} p_i\left[1 - \sum_{j=0}^{k-i}\beta_j(a)\right]\right\} - 2(m-k+1)\varphi_1\left\{p_0\left[\varphi_1 + \sum_{j=0}^{k-1} R'_j(0,a)\right] + \sum_{i=1}^{k} p_i\left[\varphi_1 + \sum_{j=0}^{k-i} R'_j(0,a)\right]\right\}, \tag{21}$$

where $R'_j(0,a) = \frac{\partial R_j(s,a)}{\partial s}\big|_{s=0}$.

Let us consider as an example a system $M/M/1/1$ with service time proportional to customer volume: $\xi = c\zeta$, $c > 0$, where RV $\zeta$ has an exponential distribution with parameter $f$. Then $\rho = \frac{ac}{f}$, $\varphi(s) = \frac{f}{s+f}$, $\beta(q) = \varphi(cq) = \frac{f}{f+cq}$, $\alpha(s,q) = \varphi(s+cq) = \frac{f}{s+f+cq}$, $\alpha'_q(s,q)|_{q=0} = -\frac{cf}{(s+f)^2}$, $R_0(s,a) = \frac{f}{s+f+ac}$, $p_k = p_0\rho^k$, $k = 1,2$, $p_0 = \frac{1-\rho}{1-\rho^3}$, if $\rho \neq 1$ and $p_0 = \frac{1}{3}$, if $\rho = 1$. If we substitute these functions into (19), taking into consideration the relations for $p_k$, we obtain:

$$\delta(s) = p_0 + \frac{f^2}{(f+s)^2}\left[\frac{(1-p_0)f}{f+s} + \frac{p_0\rho(1+\rho)s}{f+f\rho+s}\right]. \tag{22}$$

Inversion of Laplace transform $\delta(s)/s$ presents the exact form of $D(x)$ function:

$$D(x) = \left(1+\rho+\rho^2\right)^{-1} \times \left\{1 + e^{-f(1+\rho)x}(1+\rho^{-1}) + \rho(1+\rho) - (2\rho)^{-1}e^{-fx}(1+\rho)\left[2 - 2f\rho x + \rho^2(2 + 2fx + f^2x^2)\right]\right\},$$

when $\rho \neq 1$ and

$$D(x) = 1 + \frac{2}{3}e^{-2fx} - \frac{1}{3}e^{-fx}\left(4 + f^2x^2\right),$$

when $\rho = 1$.

First two moments are equal:

$$\delta_1 = \frac{\rho(2+3\rho)}{f(1+\rho+\rho^2)}; \quad \delta_2 = \frac{2\rho(3+2\rho(5+3\rho))}{f^2(1+\rho)(1+\rho+\rho^2)}.$$

In Tables 1, 2 we show numerical results of $\delta_1$ and $\delta_2$ for some various values of $\rho$ and $f$.

Obtained results are consistent with the intuitive understanding of the analyzed problem. If we increase the value of $\rho$ (without changing $f$), the values of $\delta_1$ and $\delta_2$ also increase whereas increasing of $f$ value (without c hanging of $\rho$) causes decreasing of these characteristics because the mean value of the volumes of the arriving customers is smaller then. It is obvious that these characteristics must be limited (because of limited number of waiting positions). In this case $\delta_1 \to \frac{3}{f}$ and $\delta_2 \to \frac{12}{f^2}$ (when we change only the value of $\rho \to \infty$) and $\delta_1, \delta_2 \to 0$ if $f \to \infty$ (when we change only value of $f$).

**Table 1**

Values of $\delta_1$ for $M/M/1/1$ queueing system with customer service time proportional to his volume

| $\delta_1$ | $f=0.1$ | $f=0.3$ | $f=0.5$ | $f=0.7$ | $f=0.9$ |
|---|---|---|---|---|---|
| $\rho=0.1$ | 2.0721 | 0.6907 | 0.4144 | 0.2960 | 0.2302 |
| $\rho=0.6$ | 11.6327 | 3.8776 | 2.3265 | 1.6618 | 1.2925 |
| $\rho=1.1$ | 17.6133 | 5.8711 | 3.5227 | 2.5162 | 1.9570 |
| $\rho=1.6$ | 21.0853 | 7.0284 | 4.2171 | 3.0122 | 2.3428 |
| $\rho=2.1$ | 23.2091 | 7.7364 | 4.6418 | 3.3156 | 2.5788 |
| $\rho=2.6$ | 24.5946 | 8.1982 | 4.9189 | 3.5135 | 2.7327 |

**Table 2**

Values of $\delta_2$ for $M/M/1/1$ queueing system with customer's service time proportional to his volume

| $\delta_2$ | $f=0.1$ | $f=0.3$ | $f=0.5$ | $f=0.7$ | $f=0.9$ |
|---|---|---|---|---|---|
| $\rho=0.1$ | 66.503 | 7.389 | 2.660 | 1.357 | 0.821 |
| $\rho=0.6$ | 427.041 | 47.449 | 17.082 | 8.715 | 5.272 |
| $\rho=1.1$ | 672.882 | 74.765 | 26.915 | 13.732 | 8.307 |
| $\rho=1.6$ | 819.559 | 91.062 | 32.782 | 16.726 | 10.118 |
| $\rho=2.1$ | 910.322 | 101.147 | 36.413 | 18.578 | 11.239 |
| $\rho=2.6$ | 969.841 | 107.760 | 38.794 | 19.793 | 11.973 |

**Remark 1.** It can be easily shown that the character of dependency between customer's service time and his volume has a substantial influence on total volume characteristics. Consider now, as the next example, system $M/M/1/1$ with service time independent of customer's volume and assume that customer's volume is also exponentially distributed with parameter $f$ and his service time is exponentially distributed with parameter $\mu$. Let us substitute $\mu = \frac{f}{c}$. Then, from the classical point of view, this system behaves in the same way as investigated earlier system with customer's service time proportional to his volume (e.g. distribution function $B(t)$ is identical for both systems). But here we have: $\alpha(s,q) = \dfrac{f^2}{(f+s)(f+cq)}$ and relation for $\delta(s)$ has the following form:

$$\delta(s) = \frac{f^2 + p_0 f s(2+\rho) + p_0 s^2}{(f+s)^2}.$$

The exact form of $D(x)$ function is presented below:

$$D(x) = 1 - \frac{e^{-fx}\rho(1+\rho+\rho f x)}{1+\rho+\rho^2},$$

when $\rho \neq 1$ and

$$D(x) = 1 - \frac{1}{3}e^{-fx}(2+fx),$$

when $\rho = 1$.

Notice that even formulae for first two moments are completely different in this case:

$$\delta_1 = \frac{\rho(1+2\rho)}{f(1+\rho+\rho^2)}; \; \delta_2 = \frac{2\rho(1+3\rho)}{f^2(1+\rho+\rho^2)}.$$

## 7. THE CASE OF TWO-DIMENSIONAL MEMORY BUFFER

This section presents results in the case when memory buffer is composed of two sectors ($n=2$). In this case we obtain (for $m>0$):

$$\begin{aligned}\delta(s_1,s_2) = p_0 - \frac{1-p_0}{\beta_1}(\varphi(s_1,s_2))^m \alpha'_q(s_1,s_2,q)\big|_{q=0} \\ + \sum_{k=1}^{m}\left[(\varphi(s_1,s_2))^{k-1} - (\varphi(s_1,s_2))^m\right]\Big\{p_0\Big[\varphi(s_1,s_2) \\ - \sum_{j=0}^{k-1} R_j(s_1,s_2,a)\Big] \\ + \sum_{i=1}^{k} p_i\Big[\varphi(s_1,s_2) - \sum_{j=0}^{k-i} R_j(s_1,s_2,a)\Big]\Big\},\end{aligned} \tag{23}$$

where

$$R_j(s_1,s_2,a) = \frac{a^j}{j!}\int_{x_1=0}^{\infty}\int_{x_2=0}^{\infty}\int_{t=0}^{\infty} t^j e^{-s_1x_1-s_2x_2-at}\,\mathrm{d}F(x_1,x_2,t).$$

Introduce the notation

$$\begin{aligned}\alpha_{ijk} = \mathbf{E}\left(\zeta_1^i\zeta_2^j\xi^k\right) = \int_0^\infty\int_0^\infty\int_0^\infty x_1^i x_2^j t^k\,\mathrm{d}F(x_1,x_2,t) \\ = (-1)^{i+j+k}\frac{\partial^{i+j+k}\alpha(s_1,s_2,q)}{\partial s_1^i\partial s_2^j\partial q^k}\Big|_{s_1=s_2=q=0}, \quad i,j,k = 0,1,\ldots.\end{aligned}$$

For example, $\mathbf{E}\zeta_1 = \varphi_1^{(1)} = \alpha_{100}$; $\mathbf{E}\zeta_2 = \varphi_1^{(2)} = \alpha_{010}$; $\mathbf{E}(\zeta_1\zeta_2) = \alpha_{110} = \varphi_{11}$; $\mathbf{E}\xi = \beta_1 = \alpha_{001}$; $\mathbf{E}\zeta_1^2 = \varphi_2^{(1)} = \alpha_{200}$; $\mathbf{E}\zeta_2^2 = \varphi_2^{(2)} = \alpha_{020}$; $\mathbf{E}\xi^2 = \beta_2 = \alpha_{002}$ etc.

Then, we obtain from (23):

$$\begin{aligned}\delta_1^{(1)} = \mathbf{E}\sigma_1 = -\frac{\partial\delta(s_1,0)}{\partial s_1}\Big|_{s_1=0} = \frac{1-p_0}{\beta_1}\left(\alpha_{101} + m\varphi_1^{(1)}\beta_1\right) \\ - \sum_{k=1}^{m}(m-k+1)\varphi_1^{(1)}\Big\{p_0\Big[1 - \sum_{j=0}^{k-1}\beta_j(a)\Big] \\ + \sum_{i=1}^{k} p_i\Big[1 - \sum_{j=0}^{k-i}\beta_j(a)\Big]\Big\};\end{aligned} \tag{24}$$

$$
\begin{aligned}
\delta_2^{(1)} &= \mathbf{E}\sigma_1^2 = \frac{\partial^2\delta(s_1,0)}{\partial s_1^2}\bigg|_{s_1=0} \\
&= \frac{1-p_0}{\beta_1}\Big[\alpha_{201} + 2m\varphi_1^{(1)}\alpha_{101} + m\varphi_2^{(1)}\beta_1 \\
&\quad + m(m-1)\left(\varphi_1^{(1)}\right)^2\beta_1\Big] \\
&\quad - \sum_{k=1}^{m}\Big\{(m-k+1)\varphi_2^{(1)} \\
&\quad + [m(m-1)-(k-1)(k-2)]\left(\varphi_1^{(1)}\right)^2\Big\} \\
&\quad \times\left\{p_0\left[1-\sum_{j=0}^{k-1}\beta_j(a)\right] + \sum_{i=1}^{k}p_i\left[1-\sum_{j=0}^{k-i}\beta_j(a)\right]\right\} \\
&\quad - 2(m-k+1)\varphi_1^{(1)}\left\{p_0\left[\varphi_1^{(1)} + \sum_{j=0}^{k-1}R'_j(0,a)\right]\right. \\
&\quad \left. + \sum_{i=1}^{k}p_i\left[\varphi_1^{(1)} + \sum_{j=0}^{k-i}R'_j(0,a)\right]\right\}. \qquad (25)
\end{aligned}
$$

It is clear that the relations for $\delta_1^{(2)} = \mathbf{E}\sigma_2$ and $\delta_2^{(2)} = \mathbf{E}\sigma_2^2$ take similar forms. For the mixed moment $\delta_{11} = \mathbf{E}(\sigma_1\sigma_2) = \frac{\partial^2(s_1,s_2)}{\partial s_1\partial s_2}\Big|_{s_1=s_2=0}$ we obtain:

$$
\begin{aligned}
\delta_{11} &= (1-p_0)\left\{\left[\varphi_{11} + m(m-1)\varphi_1^{(1)}\varphi_1^{(2)}\right]\right. \\
&\quad \left. + \frac{1}{\beta_1}\left(\alpha_{111} + \varphi_1^{(1)}\alpha_{011} + \varphi_1^{(2)}\alpha_{101}\right)\right\} \\
&\quad - \sum_{k=1}^{m}\Big\{\Big[(m-k+1)\varphi_{11} \\
&\quad + (m(m-1)-(k-1)(k-2))\varphi_1^{(1)}\varphi_1^{(2)}\Big] \\
&\quad \times\left[p_0\left(1-\sum_{j=0}^{k-1}\beta_j(a)\right) + \sum_{i=1}^{k}p_i\left(1-\sum_{j=0}^{k-i}\beta_j(a)\right)\right] \\
&\quad + (m-k+1)\varphi_1^{(1)}\left[p_0\left(\varphi_1^{(2)} + \sum_{j=0}^{k-1}R'_{js_2}(0,0,a)\right)\right. \\
&\quad \left. + \sum_{i=1}^{k}p_i\left(\varphi_1^{(2)} + \sum_{j=0}^{k-i}R'_{js_2}(0,0,a)\right)\right] \\
&\quad + (m-k+1)\varphi_1^{(2)}\left[p_0\left(\varphi_1^{(1)} + \sum_{j=0}^{k-1}R'_{js_1}(0,0,a)\right)\right. \\
&\quad \left. + \sum_{i=1}^{k}p_i\left(\varphi_1^{(1)} + \sum_{j=0}^{k-i}R'_{js_1}(0,0,a)\right)\right]\Big\}, \qquad (26)
\end{aligned}
$$

where

$$
R'_{js_1}(0,0,a) = -\frac{a^j}{j!}\int_0^\infty\int_0^\infty x_1 t^j \,\mathrm{d}F(x_1,\infty,t),
$$

$$
R'_{js_2}(0,0,a) = -\frac{a^j}{j!}\int_0^\infty\int_0^\infty x_2 t^j \,\mathrm{d}F(\infty,x_2,t).
$$

Consider, as an example, system $M/G/1/1$, in which service time of a customer is proportional to his length i.e. $\xi = c(\zeta_1+\zeta_2)$, where RV $\zeta_1$, $\zeta_2$ are independent, and assume additionally that indications $\zeta_1$ and $\zeta_2$ are exponentially distributed with parameters $f$ and $g$ ($f \neq g$), respectively. Then, after simple calculations we have: $\rho = ac\left(\frac{1}{f}+\frac{1}{g}\right)$, $\varphi(s_1,s_2) = \frac{fg}{(f+s_1)(g+s_2)}$, $\alpha(s_1,s_2,q) = \frac{fg}{(f+s_1+cq)(g+s_2+cq)}$, $\alpha'_q(s_1,s_2,q)\big|_{q=0} = \frac{-cfg(f+g+s_1+s_2)}{(f+s_1)^2(g+s_2)^2}$ and we finally have:

$$
\begin{aligned}
\delta(s_1,s_2) &= p_0 - \frac{(fg)^3(p_0-1)(f+g+s_1+s_2)}{(f+g)(f+s_1)^3(g+s_2)^3} \\
&\quad + \frac{p_0(fg)^2\rho(1+\rho)(gs_1+(f+s_1)s_2)}{(f+s_1)^2(f^2+gs_1+f(g(1+\rho)+s_1))} \\
&\quad \times\frac{f^2+g(g+s_1+s_2)+f(g(2+\rho)+s_1+s_2)}{(g+s_2)^2(g(g+s_2)+f(g(1+\rho)+s_2))}. \qquad (27)
\end{aligned}
$$

On the base of (27), we may obtain exact formula for distribution function $D(x_1,x_2)$ e.g. using *Mathematica* environment (we only must run **InverseLaplaceTransform** function to $\delta(s_1,s_2)/s_1s_2)$ [42]. Finally, we can calculate moments of vector $(\sigma_1,\sigma_2)$:

$$
\delta_1^{(1)} = \frac{3(1-p_0)}{f} + \frac{p_0-1}{f+g} - \frac{p_0\rho(1+\rho)\left[(f+g)^2+fg\rho\right]}{f(f+g+f\rho)(f+g+g\rho)}, \qquad (28)
$$

$$
\begin{aligned}
\delta_2^{(1)} &= \frac{2(3f+6g-p_0(5f+8g))}{f^2(f+g)} - \frac{4p_0}{f^2} \\
&\quad - \frac{2p_0(2f^2g+fg^2-g^3)}{f^2(f-g)^2(f+g+f\rho)} + \frac{2(f+g)^2p_0}{f(f-g)(f+g+g\rho)^2} \\
&\quad + \frac{2p_0(f^3+g^3)}{f^2(f-g)^2(f+g+g\rho)}. \qquad (29)
\end{aligned}
$$

The above formulae for the first indication of total volume vector can be also used to compute analogous characteristics for the second one. We only have to change $f$ into $g$ and vice versa.

Whereas mixed (1+1)th moment of $(\sigma_1, \sigma_2)$ is equal:

$$\delta_{11} = \frac{9(1-p_0)}{fg} + \frac{3(p_0-1)}{f+g}\left(\frac{1}{f}+\frac{1}{g}\right)$$
$$- \frac{p_0\rho(1+\rho)(f+g)\left((f+g)^2+fg\rho\right)}{\left(f^2+fg(1+\rho)\right)\left(g^2+fg(1+\rho)\right)}$$
$$\times\left[\frac{g}{g^2+fg(1+\rho)} - \frac{f}{f^2+fg(1+\rho)}\right]$$
$$- \frac{p_0\rho(1+\rho)\left[2(f+g)^2+3fg\rho\right]}{\left(f^2+fg(1+\rho)\right)\left(g^2+fg(1+\rho)\right)}. \tag{30}$$

In (28)–(30) we substitute $p_0 = \dfrac{1-\rho}{1-\rho^3}$, if $\rho \neq 1$ or $p_0 = \dfrac{1}{3}$ otherwise

## 8. CONCLUSIONS AND FINAL REMARKS

In the present paper, we have investigated the model of $M/G/1/m$ queueing system with random volume customers and sectorized memory buffer. We proved general formula for the Laplace–Stieltjes transform of steady-state distribution of total volume vector for this system and presented exact results for some special cases of the model. Moreover, we showed that well–known results for the $M/G/1/\infty$ queueing system may be obtained with the help of limitary calculations for the analyzed model. Besides, we discussed the cases of one and two-dimensional memory buffers together with presentation of numerical computations in practical situations when service time of a customer is proportional to his length. Finally, we paid attention to the fact that the character of dependency between customer service time and his volume vector has a substantial influence on total volume vector characteristics (even on his first moments). Our work can be used by engineers and scientists who are involved in the process of computer or telecommunication networks designing (e.g. calculating needed sizes of memory buffers).

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