Vibrations in Physical Systems Vol.26 (2014)

Vibrations of Thin Functionally Graded Plates with Tolerance-Periodic Microstructure

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Abstract

This paper presents a problem of vibrations of thin functionally graded plates. To describe this kind of plates three averaged models are proposed: a tolerance model, an asymptotic model and a combined asymptotic-tolerance model, cf. [10]. Calculational results obtained for a functionally graded plate band using the proposed models are compared to each other.

Keywords: thin functionally graded plates, tolerance-periodic microstructure, tolerance modelling

1. Introduction

There are considered thin plates with functionally graded macrostructure in planes parallel to the plate midplane. However, the microstructure is tolerance-periodic, cf. Figure 1.

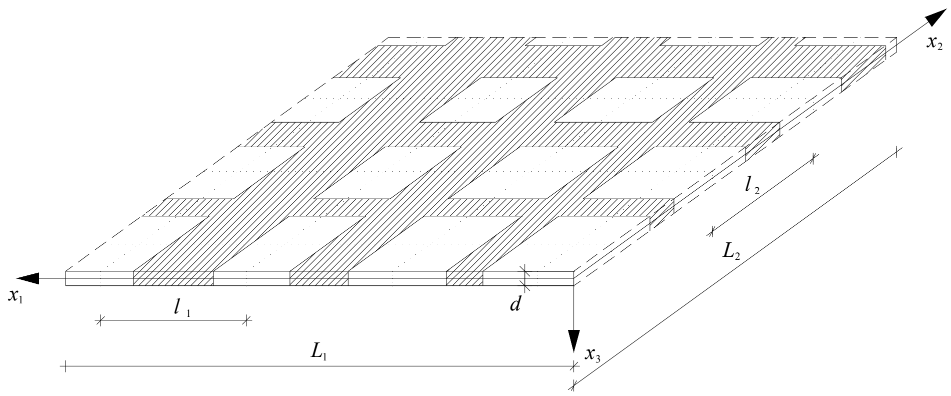

Figure 1. Fragment of a functionally graded plate

Plates of this kind consist of many small elements, where adjacent elements are nearly identical but the distant ones may be variable. Every element is treated as a thin plate with spans l_1 and l_2 along the x_1- and x_2-axis, respectively, cf. Figure 2.

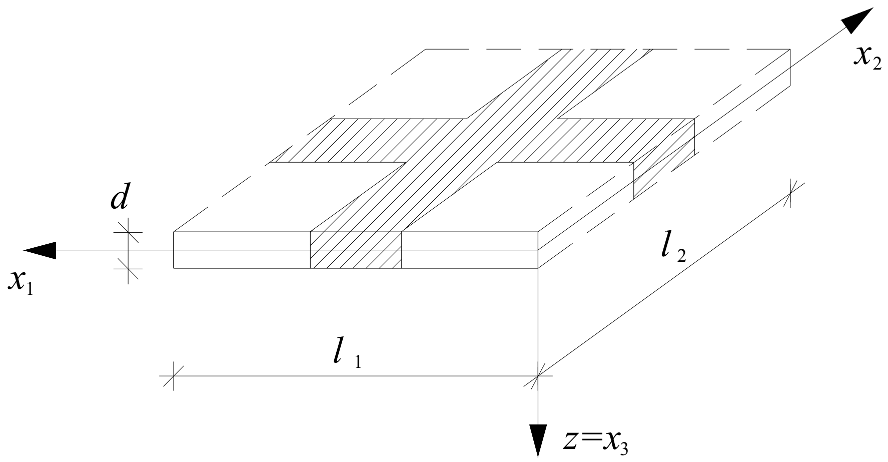

Figure 2. Element of a functionally graded plate

Structures made of functionally graded material are often analysed using averaging approaches for macroscopically homogeneous structures. Models, which are proposed, based on the asymptotic homogenization, where the effect of the microstructure size is neglected in the governing equations.

Using the tolerance averaging technique the effect of the microstructure size can be taken into account in governing equations for structures under consideration, cf. [9,10]. Some applications of this method to the modelling of various periodic structures are shown in a series of papers, [1,5]. The tolerance modelling was adopted to functionally graded structures like transversally tolerance-periodic plates, cf. [2,3,4,5,6] and for longitudinally functionally graded structures [8].

2. Modelling foundations

Denote by $Ox_1x_2x_3$ orthogonal Cartesian coordinate system and by t the time coordinate. Set $\mathbf{x}\equiv(x_1,x_2)$ and $z\equiv x_3$. The region of the undeformed plate is denote by $\Omega\equiv\{(\mathbf{x}, z)\text{: } -d(\mathbf{x})/2 \le z \le d(\mathbf{x})/2,\ \mathbf{x}\in\Pi\}$, where Π is the midplane and $d(\cdot)$ is the plate thickness. The "basic cell" on Ox_1x_2 is denoted by $\Omega\equiv[-l_1/2,\ l_1/2]\times[-l_2/2,\ l_2/2]$. The diameter of cell Ω, called *the parameter of microstructure*, is defined by $l\equiv[(l_1)^2+(l_2)^2]^{1/2}$ and satisfies the condition $d_{max} << l << (L_1, L_2)$. Thickness $d(\cdot)$ can be a tolerance-periodic function in $\mathbf{x}$.

Define tolerance-periodic functions in $\mathbf{x}$: a mass density μ, a rotational inertia ϑ and bending stiffnesses $d_{\alpha\beta\gamma\delta}$ in the form:

$$\begin{aligned} &\mu(\mathbf{x}) \equiv \int_{-d/2}^{d/2} \rho(\mathbf{x}, z)dz, \quad \vartheta(x) \equiv \int_{-d/2}^{d/2} \rho(\mathbf{x}, z)z^2 dz, \\ &d_{\alpha\beta\gamma\delta}(\mathbf{x}) \equiv \int_{-d/2}^{d/2} c_{\alpha\beta\gamma\delta}(\mathbf{x}, z)z^2 dz. \end{aligned} \tag{1}$$

From the Kirchhoff-type plates theory assumptions the equation for deflection $w(\mathbf{x}, t)$ of functionally graded plates with highly oscillating, tolerance-periodic, non-continuous coefficients is described by:

$$\partial_{\alpha\beta}(d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) + \mu\ddot{w} - \partial_{\alpha}(\vartheta\partial_{\alpha}\ddot{w}) = p. \tag{2}$$

3. Modelling concepts and assumptions

Averaged equations for functionally graded plates will be obtained using the tolerance averaging technique. The basic concepts of the modelling procedure are defined in books, cf. [9,10].

Let $\Omega(\mathbf{x})\equiv\mathbf{x}+\Omega$, $\Pi_{\Omega}=\{\mathbf{x}\in\Pi\colon \Omega(\mathbf{x})\subset\Pi\}$, be a cell at $\mathbf{x}\in \Pi_{\Omega}$. The averaging operator for an arbitrary integrable function f is defined by

$$< f > (\mathbf{x}) = \frac{1}{l_1 l_2}\int_{\Omega(\mathbf{x})} f(y_1, y_2)dy_1 dy_2, \quad \mathbf{x}\in\Pi_{\Omega}. \tag{3}$$

If a function f is tolerance-periodic in $\mathbf{x}$, then averaged value by (3) is a slowly-varying function in $\mathbf{x}$.

Following the aforementioned books let us denote a set of tolerance-periodic functions by $TP_{\delta}^{\alpha}(\Pi, \Omega)$, a set of slowly-varying functions by $SV_{\delta}^{\alpha}(\Pi, \Omega)$, a set of highly oscillating functions by $HO_{\delta}^{\alpha}(\Pi, \Omega)$, where $\alpha \geq 0$, δ is a tolerance parameter. Denote by $h(\cdot)$ a highly oscillating function, $h\in HO_{\delta}^{2}(\Pi, \Omega)$, continuous together with gradient $\partial^1 h$ and having a piecewise continuous and bounded gradient $\partial^2 h$. Function $h(\cdot)$ is called *the fluctuation shape function* of the 2-nd kind, if it depend on l as a parameter and satisfies conditions: $\partial^k h\in O(l^{\alpha-k})$ for $k=0,1,\ldots,\alpha$, $\partial^k h\equiv h$, and $<\mu h>(\mathbf{x})\approx 0$ for every $\mathbf{x}\in\Pi_{\Omega}$, $\mu > 0$, $\mu\in TP_{\delta}^{1}(\Pi, \Omega)$.

4. Governing equations

In this section will be presented equations for three models: the tolerance model, the asymptotic model and the combined asymptotic-tolerance model.

The tolerance modelling procedure is outlined here following the book [3].

The first assumption of this procedure is *the micro-macro decomposition* plate deflection w:

$$w(\mathbf{x},t) = U(\mathbf{x},t) + h^A(\mathbf{x})Q^A(\mathbf{x},t), \quad A = 1,\ldots,N, \quad \mathbf{x}\in\Pi. \tag{4}$$

Functions $U(\cdot,t)$ and $Q^A(\cdot,t)$ are kinematic unknowns, called *the macrodeflection* and *the fluctuation amplitudes*, respectively, $h^A(\cdot)$ are the known *fluctuation shape functions*.

The second assumption is *the tolerance averaging approximation*, i.e. terms of an order of $O(\delta)$ are negligibly small.

Using the above assumptions in equation (2), after some manipulations we obtain the system of equations:

$$
\begin{aligned}
&\partial_{\alpha\beta}(< d_{\alpha\beta\gamma\delta} > \partial_{\gamma\delta}U + < d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^B > Q^B) + < \mu > \ddot{U} - < \vartheta > \partial_{\alpha\alpha}\ddot{U} = < p >,\\
&< d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^A > \partial_{\alpha\beta}U + < d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B > Q^B +\\
&\qquad\qquad + (< \mu h^A h^B > + < \vartheta\partial_\alpha h^A \partial_\alpha h^B >)\ddot{Q}^B = < ph^A >,
\end{aligned}
\tag{5}
$$

where all coefficients are slowly-varying functions in $\mathbf{x}$. Equations (5) together the micro-macro decomposition (4) present *the tolerance model of thin functionally graded plates*. These equations take into account the effect of the microstructure size on dynamic problems of this kind of plates.

The asymptotic modelling procedure is shown in books [3,10].

The starting point of this procedure is equation (2). We introduce a parameter $\varepsilon\in(0,1]$, an interval $\Omega_\varepsilon\equiv[-\varepsilon l_1/2,\ \varepsilon l_1/2]\times[-\varepsilon l_2/2,\ \varepsilon l_2/2]$, ε-cell $\Omega_\varepsilon(\mathbf{x})\equiv\mathbf{x}+\Omega_\varepsilon$, $\mathbf{x}\in\Pi$. For function $\tilde{f}(\mathbf{x},\cdot)\in H^1(\Omega)$, □$\mathbf{x}\in\Pi$, we define $\tilde{f}_\varepsilon(\mathbf{x},\mathbf{y})\equiv\tilde{f}(\mathbf{x},\mathbf{y}/\varepsilon)$, $\tilde{f}_\varepsilon(\mathbf{x},\cdot)\in H^1(\Omega_\varepsilon)\subset H^1(\Omega)$, $\mathbf{y}\in\Omega_\varepsilon(\mathbf{x})$. Functions $h^A(\cdot)$, $h^A(\cdot)\in HO_\delta^2(\Pi,\Omega)$, A=1,…,N, have their periodic approximations $\tilde{h}^A(\mathbf{x},\cdot)$, given by $\tilde{h}_\varepsilon^A(\mathbf{x},\mathbf{y})\equiv\tilde{h}^A(\mathbf{x},\mathbf{y}/\varepsilon)$, $\mathbf{y}\in\Omega_\varepsilon(\mathbf{x})$.

The fundamental assumption of the asymptotic modelling in *the asymptotic decomposition* for the deflection $w(\mathbf{x},t)$,

$$
w_\varepsilon(\mathbf{x},\mathbf{y},t) = U(\mathbf{y},t) + \varepsilon^2\tilde{h}_\varepsilon^A(\mathbf{x},\mathbf{y})Q^A(\mathbf{y},t),
\tag{6}
$$

where $\mathbf{y}\in\Omega_\varepsilon(\mathbf{x})$, $t\in(t_0,\ t_1)$, functions w, U, Q^A (A=1,…,N) are continuous and bounded in Π with they derivatives.

Using the assumption (6) and the limit passage $\varepsilon\to 0$, after some manipulations we obtain the equations of *the asymptotic model* in the form:

$$
\begin{aligned}
&\partial_{\alpha\beta}((< d_{\alpha\beta\gamma\delta} > -\\
&- < d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^B >< d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B >^{-1} < d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A >)\partial_{\gamma\delta}U) +\\
&+ < \mu > \ddot{U} - < \vartheta > \partial_{\alpha\alpha}\ddot{U} = < p >,\\
&Q^B = - < d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B >^{-1} < d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^A > \partial_{\alpha\beta}U.
\end{aligned}
\tag{7}
$$

These equations have smooth, slowly-varying coefficients in the contrast to equation (2). The effect of the microstructure size is neglected.

The last model presented in this paper is *the combined asymptotic-tolerance model*, cf [10, 6].

At the beginning we apply the asymptotic modelling procedure. Because the macrodeflection U is the solution to equation $(7)_1$ and the fluctuation amplitudes Q^A are determined by relation $(7)_2$, we have the known following function

$$
w_0(\mathbf{x},t) = U(\mathbf{x},t) + h^A(\mathbf{x})Q^A(\mathbf{x},t).
\tag{8}
$$

The next step of this modelling procedure is to apply the tolerance modelling procedure. Using the known function $w_0(\cdot,t)$ and introducing the known fluctuation shape functions $g^K(\cdot)\in FS_\delta^2(\Pi,\Omega)$, K=1,…,N, we assume the plate deflection as

$$w(\mathbf{x},t) = w_0(\mathbf{x},t) + g^K(\mathbf{x})V^K(\mathbf{x},t), \tag{8}$$

where V^K are slowly-varying unknown functions in **x**.

Finally, after some manipulations, we arrive at the system of equations for *the asymptotic-tolerance model*, which can be written in the following form:

$$\begin{aligned}
&\partial_{\alpha\beta}((< d_{\alpha\beta\gamma\delta} > - \\
&- < d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^B >< d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B >^{-1} < d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A >)\partial_{\gamma\delta}U) + \\
&+ < \mu > \ddot{U} - < \vartheta > \partial_{\alpha\alpha}\ddot{U} = < p >, \\
&< d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g^K\partial_{\gamma\delta}g^J > V^J + (< \mu g^K g^J > + < \mu\partial_\alpha g^K \partial_\beta g^J > \ddot{V}^J = \\
&= < pg^K > - < d_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g^K\partial_{\gamma\delta}w_0 > .
\end{aligned} \tag{9}$$

This model makes it possible to analyse the effect of the microstructure size on vibrations of the plates under consideration.

5. Conclusions

In this paper three modelling procedures are applied to the known differential equation of Kirchhoff-type plates with functionally graded structure. These procedures make it possible to replace the governing equation with non-continuous, tolerance-periodic coefficients by the systems of differential equations with slowly-varying, continuous coefficients for each model.

All three models can be used to analyse various dynamic problems of functionally graded plates.

The tolerance model, which describes the effect of the microstructure size, makes it possible to investigate not only macrovibrations, but also microvibrations, which are related to the microstructure of the functionally graded plates.

Also the governing equations of the combined asymptotic-tolerance model take into account the effect of the microstructure size on vibrations of these plates.

The third presented model, the asymptotic model, neglecting the aforementioned effect, can describe only macrovibrations of the plates.

Acknowledgments

This contribution is supported by the National Science Centre of Poland under grant No. 2011/01/N/ST8/07758.

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