The compact decoupling for system of thermoelasticity in viscoporous media and exponential decay

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The problem of exponential decay for solutions of porous-thermoelasticity system, when time $t \to \infty$ is studied. For sufficiently small values of parameter of intensity of elasticity-porosity interactions the exponential decaying is established.

The idea of compact decoupling is applied for the system of equations. Exponential decay is proved first for the simpler decoupled system, then the property is derived for the original system.

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Classification: 35B35, 35L20, 74F05, 74L10.

Introduction

In this paper we continue studies,which we have begun in [6], on decaying of solutions as $t \to \infty$ for the thermoelasticity system of viscoporous media.

We consider the following system:

$$\begin{aligned}
&\partial_t^2 u = \Delta_e u + b\nabla\phi - M\nabla\theta \quad \text{in} \quad \Omega \times R_+,\\
&\partial_t^2 \phi = a\Delta\phi - b\mathrm{div}u - \gamma\phi - r\partial_t\phi + M_1\theta \\
&\qquad\qquad \text{in} \quad \Omega \times R_+,\\
&\partial_t\theta = d\Delta\theta - M\mathrm{div}\partial_t u - M_1\partial_t\phi \quad \text{in} \quad \Omega \times R_+,\\
&u = 0, \quad \phi = 0, \quad \theta = 0 \quad \text{on} \quad \partial\Omega \times R_+,\\
&u(0) = u^0, \partial_t u(0) = u^1, \phi(0) = \phi^0, \partial_t\phi(0) = \\
&\qquad = \phi^1, \theta(0) = \theta^0 \quad \text{in} \quad \Omega.
\end{aligned} \tag{1}$$

In the above we denoted: $\Omega \subset R^n$, $n = 2, 3$, is open bounded set, the regularity of $\partial\Omega$ will be precised later, $R_+ := (0, \infty)$, $\Delta e = \mu\Delta I + (\mu + \lambda)\nabla\mathrm{div}$ is Lame operator and coefficents a, b, d, r, M, M_1, r, $\mu > 0$ (more precise constraints on coefficents will be introduced later).

Interpretation of u, ϕ, θ and mechanical justification of (1) is given in [4] and was recalled in [6]. We recall that $u \in R^n$ is the displacement vector for media occupying domain Ω, ϕ denotes the change of volume fraction relative to equilibrium configuration, θ is the temperature.

The system (1) in one dimensional case was studied in [11] and it was proved that when $r > 0$, the exponential decay takes place, and when $r = 0$ this effect does not occur. It is known, that for thermoelasticity system (without ϕ) for $n = 1$ the energy decays exponentially (see literature references in [6]).

Let us denote $\epsilon_{ij}(u) := \frac{1}{2}(\partial_j u_i + \partial_i u_j)$,

$$\sigma_{ij}(u) := \lambda \sum_{l=1}^{n} \epsilon_{ll}(u)\delta_{ij} + 2\mu\epsilon_{ij}(u), \quad i, j \in \{1, \dots, n\},$$

$$\sigma(u) : \epsilon(v) := \sum_{i,j=1}^{n} \sigma_{ij}(u)\epsilon_{ij}(v).$$

We remind that $(\Delta_e u)_i = \sum_{j=1} \partial_j \sigma_{ij}(u), i \in \{1, \dots, n\}$.

Assumption 1.1 *We require $\lambda + \mu > 0$, $(\lambda + \mu)\gamma > b^2$ when $n = 2$ and $3\lambda + 2\mu > 0$, $(3\lambda + 2\mu)\gamma > 3b^2$ when $n = 3$.*

It was proved in [5, 6] that

Proposition 1.2 *If b, γ, ν, λ satisfy Assumption 1.1 then there exist constants $c_1, c_2 > 0$ such, that*

$$\sigma(u) : \epsilon(u) \geq c_1 \sum_{i,j=1}^{n} \epsilon_{ij}^2(u),$$

$$\sigma(u) : \epsilon(u) + 2b\phi\mathrm{div}u + \gamma\phi^2 \geq c_2 \left(\phi^2 + \sum_{i,j=1}^{n} \epsilon_{ij}^2(u) \right).$$

In this section we assume C^2 regularity of $\partial\Omega$.

We define spaces $V = H^1_0(\Omega)^n \times H^1(\Omega)$, $H = V \times L^2(\Omega) \times L^2(\Omega)^{n+1}$. It was proved in [5],[6] that

Proposition 1.3 *The bilinear form*

$$\left\langle \begin{pmatrix} u^1 \\ \phi^1 \end{pmatrix}, \begin{pmatrix} u^2 \\ \phi^2 \end{pmatrix} \right\rangle_V :=$$

$$:= \int_\Omega \left[\sigma(u^1) : \epsilon(u^2) + a\nabla\phi^1 \cdot \nabla\phi^2 + \gamma\phi^1\phi^2 + \right.$$

$$\left. + b\phi^1 \mathrm{div} v^2 + b\phi^2 \mathrm{div} u^1\right]$$

is the scalar product in V and V is the Hilbert space.

Let $\xi^i := \left(u^i, \phi^i, v^i, \psi^i, \theta^i\right)^T \in H$, $i = 1, 2$. The bilinear form

$$(\xi^1, \xi^2) := \left\langle \begin{pmatrix} u^1 \\ \phi^1 \end{pmatrix}, \begin{pmatrix} u^2 \\ \phi^2 \end{pmatrix} \right\rangle_V +$$

$$+ \int_\Omega \left[v^1 \cdot v^2 + \psi^1\psi^2 + \theta^1\theta^2\right]$$

is the inner product in H and H is the Hilbert space.

The norm given by this scalar product we denote $||\cdot||$.

We consider operator:

$$L := \begin{pmatrix} 0 & , & 0 & , & I & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 & , & I & , & 0 \\ \Delta_e & , & b\nabla & , & 0 & , & 0 & , & -M\nabla \\ -b\mathrm{div} & , & (a\Delta - \gamma I) & , & 0 & , & -rI & , & M_1 I \\ 0 & , & 0 & , & -M\mathrm{div} & , & -M_1 I & , & d\Delta \end{pmatrix},$$

with domain $X \subset H$, $X = \left(H^2(\Omega) \cap H^1_0(\Omega)\right)^n \times \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times H^1_0(\Omega)^n \times H^1_0(\Omega) \times \left(H^2(\Omega) \cap H^1_0(\Omega)\right)$.

We see that L : X → H. We can rewrite (1) as the ordinary differential equation in H:

$$\frac{d\eta}{dt} = L\eta \tag{2}$$

$$\eta(0) = \eta^0 \tag{3}$$

where $\eta \in H$ and $\eta^0 = (u^0, \phi^0, u^1, \phi^1, \theta^0)$.

In [6], the definition of strong solution of (1) when $\eta^0 \in X$ and weak solution when $\eta^0 \in H$ was given. It was proved in [6] (Theorem 2.7):

Theorem 1.4 *Let the coefficients satisfy Assumption 1.1, $\partial\Omega$ has regularity of class C^2.*

Then on H the operator L generates the c_0-semigroup of contractions $S(t)$, $t \geq 0$. Moreover when $(u(t), \phi(t), v(t), \psi(t), \theta(t))^T := S(t)\xi^0$ then $v(t) = \partial_t u(t)$, $\psi(t) = \partial_t\phi(t)$, $t > 0$, and $(u(t), \phi(t), \theta(t))$, $t \geq 0$ for $\xi^0 \in X$ is the unique strong solution of (1), and for $\xi^0 \in H$ is the unique weak solution of (1).

We recall from [12] that $S(t)\eta^0 \in H$ when $\eta^0 \in H$, $S(t)\eta^0 \in X$ when $\eta^0 \in X$, $S(t)\eta^0 \in D(L^k)$ when $\eta^0 \in D(L^k)$, $k \in N$, $t > 0$.

Definition 1.5 *We say that open, bounded set $\Omega \subset R^n$ satisfies Condition (C) if for every $s > 0$ the system:*

$$-\Delta v = sv \quad \text{in} \quad \Omega,$$
$$\mathrm{div} v = 0 \quad \text{in} \quad \Omega,$$
$$v = 0 \quad \text{on} \quad \partial\Omega,$$

has unique solution $v = 0 \in R^n$.

For strong and weak solutions their energy is given by the formula (see [6]):

$$E(t) := \frac{1}{2}||S(t)\eta^0||^2 \equiv$$

$$\equiv \frac{1}{2}\left(||u(t), \phi(t), \partial_t u(t), \partial_t\phi(t), \theta(t))\right||^2.$$

Because of contractivity property of semigroup S we have $E(t_2) \leq E(t_1)$ when $0 \leq t_1 \leq t_2$. In [6] it was proved:

Theorem 1.6 *If coefficients satisfy conditions from Theorem 1.4 and domain Ω satisfies Condition (C) then $\lim_{t\to\infty} E(t) = 0$.*

To prove uniform decaying the following assumption introduced in the paper [10] is necessary:

Assumption 1.7 *We require that there exists $T > 0$ and $C > 0$ such that for every solution of the system*

$$\begin{aligned} \partial_t^2\phi = \Delta_e\phi \quad &\text{in} \quad \Omega \times R_+, \\ \phi = 0 \quad &\text{on} \quad \partial\Omega \times R_+, \\ \phi(0) = \phi^0, \partial_t\phi(0) = \phi^1 \quad &\text{in} \quad \Omega. \end{aligned} \tag{4}$$

the following inequality holds

$$\begin{aligned} ||\phi^0||^2_{(L^2(\Omega))^n} + ||\phi^1||^2_{(H^{-1}(\Omega))^n} \leq \\ \leq C\int_0^T ||\mathrm{div}\phi(t)||^2_{H^{-1}(\Omega)}, \end{aligned} \tag{5}$$

It was proved that in the class of domains Ω in dimenstions $n = 2, 3$ satifying condition (C) the uniform decaying of energy of thermoelastic system (without porosity, see [2, 10]) holds if, and only if the inequality (5) is satisfied.

For dimension $n = 2$ description of geometric properties of domains Ω, for which the estimation (5) holds, was given in [10]. For dimension $n = 3$ analogous theorem was proved in [2] (see Theorem 3). In both cases the additional condition $\lambda + 2\mu > \mu$ was assumed. When μ, λ satisfy the condition from Assumption 1.1 and additionally $\lambda \in \left(-\frac{2}{3}\mu, -\frac{\mu}{2}\right) \cup R_+$ the above condition is satisfied.

Decoupled system corresponding to system (1)

We are going to introduce decoupled system, derived from (1), for which we are able to prove exponential decay of energy. This result will be then carried over to system (1). To obtain decoupled system we proceed in a similar way as for the classical system of thermoelasticity [8, 10, 13]. We drop the term $M_1\theta$ in the second equation of system (1) because θ as a solution of parabolic equation has good regularity. Then, keeping the leading terms in the third equation, we get:

$$\Delta\theta = \frac{M}{d}\mathrm{div}\partial_t u, \quad \text{in} \quad R_+ \times \Omega,$$
$$\theta = 0 \quad \text{on} \quad R_+ \times \partial\Omega$$

We consider operator $P = \nabla(\Delta^{-1})\mathrm{div}$, where $\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ and $\mathrm{div} : L^2(\Omega)^n \to H^{-1}(\Omega)$. It is known that P is the orthogonal projection operator in $\left(L^2(\Omega)\right)^n$ onto the subspace

$$\left\{\nabla\psi : \psi \in H_0^1(\Omega)\right\}.$$

This means that $\nabla\theta = \frac{M}{d}P\partial_t u$. By replacing $\nabla\theta$ in the first equation in (1) by $\frac{M}{d}P\partial_t u$ we obtain the following decoupled system corresponding to system (1) :

$$\begin{aligned} \partial_t^2\overline{u} &= \Delta_e\overline{u} + b\nabla\overline{\phi} - \frac{M^2}{d}P\partial_t\overline{u} \quad \text{in} \quad \Omega \times R_+, \\ \partial_t^2\overline{\phi} &= a\Delta\overline{\phi} - b\mathrm{div}\overline{u} - \gamma\overline{\phi} - \gamma\partial_t\overline{\phi} \quad \text{in} \quad \Omega \times R_+, \end{aligned} \tag{6}$$

$$\partial_t\overline{\theta} = d\Delta\overline{\theta} - M\mathrm{div}\partial_t\overline{u} - M_1\partial_t\overline{\phi} \quad \text{in} \quad \Omega \times R_+. \tag{7}$$

Initial and boundary conditons are the same as in the system (1).

Let us introduce $\overline{v} = \partial_t\overline{u}$, $\overline{\psi} = \partial_t\overline{\phi}$ and $\eta := (\overline{u}, \overline{\phi}, \overline{v}, \overline{\psi})^T$,

$\overline{H} := \left\{\eta : (\overline{u}, \overline{\phi}) \in V, \overline{v} \in \left(L^2(\Omega)\right)^n, \overline{\psi} \in L^2(\Omega)\right\}.$

The space $\overline{H}$ equipped with inner product $(\eta^1, \eta^2)_{\overline{H}} := \left((\eta^1, 0), (\eta^2, 0)\right)$ is the Hilbert space; we remind that $(\cdot, \cdot)$ denotes inner product in H. For $\begin{pmatrix} \eta^i \\ \theta^i \end{pmatrix} \in H$, $i = 1, 2$, $\left(\begin{pmatrix} \eta^1 \\ \theta 1 \end{pmatrix}, \begin{pmatrix} \eta^2 \\ \theta^2 \end{pmatrix}\right) = (\eta^1, \eta^2)_{\overline{H}} + \int_\Omega \eta^1\eta^2$. We denote the norm in $\overline{H}$ induced by inner product $(\cdot, \cdot)_{\overline{H}}$ by $||\cdot||_{\overline{H}}$.

We consider the subspace $\overline{X} \subset \overline{H}$, $\overline{X} := (H^2(\Omega) \cup H_0^1(\Omega))^n \times \left(H^2(\Omega) \cup H_0^1(\Omega)\right) \times \left(H_0^1(\Omega)\right)^n \times H_0^1(\Omega)$ and the following operator $\overline{A} : \overline{X} \to \overline{H}$:

$$\overline{A} := \begin{pmatrix} 0 & , & 0 & , & I & , & 0 \\ 0 & , & 0 & , & 0 & , & I \\ \Delta_e & , & b\nabla & , & -\frac{M^2}{d}P & , & 0 \\ -b\mathrm{div} & , & (a\Delta - \gamma I) & , & 0 & , & -rI \end{pmatrix}.$$

We notice that $H = \overline{H} \times L^2(\Omega)$, $X = \overline{X} \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right)$.

We write the system (6)-(7) in the form of linear equation in H:

$$\frac{d\overline{\xi}}{dt} = \overline{L\xi} \tag{8}$$

where $\overline{\xi} := \begin{pmatrix} \eta \\ \theta \end{pmatrix}$,

$$\overline{L} := \begin{pmatrix} \overline{A} & , & 0 \\ B & , & d\Delta \end{pmatrix},$$

where $B\eta := -M\mathrm{div}v - M_1\psi$.

We see that $B : \overline{X} \to L^2(\Omega)$. As the domain of operator $\overline{L}$ we take X. We have $\overline{L} : X \to H$ and it is easy to see that $\overline{L}$ is a closed operator.

In the further considerations we shall need the following proposition:

Theorem 2.1 *Operator $\overline{L}$ is the generator of c_0-semigroup in H, which we denote by $\overline{S}(t)$,$t \geq 0$.*

For the fixed value of coefficient $b > 0$ in the system (6), (7) we denote the corresponding c_0-semigroup with $\overline{S}_b(\cdot)$.

Theorem 2.2 *Let domain Ω satisfy the Condition (C) and guarantee that inequality* (5) *holds. Then exists $b_0 > 0$ such, that for every $b \in (0, b_0)$ the semigroup $\overline{S}_b(t)$ has the property of uniform decaying.*

The final theorem of this section establishes the relationship between semigroups of main problem and its decoupled version:

Theorem 2.3 *For every $\tau > 0$, operator $S(\cdot) - \overline{S}(\cdot) : H \to C([0, \tau], H)$ is compact.*

Main result

First we recall from [7] the following theorem.

Theorem 3.1 (Guo) Let $T(t)$ and $T_B(t)$ be c_0-semigroups on a Banach space Y with generators G and $G + B$ respectively. Assume the folowing hypotheses:

1. $\lim_{t\to\infty} ||T(t)y|| = 0$ *for every* $y \in Y$,
2. $||T_B(t)|| \leq Me^{-\omega t}$, $t \geq 0$, *where* $M, \omega > 0$ *are constants,*
3. $T_(t_0) - T_B(t_0)$ *is compact for some* $t_0 > 0$.

Then there exist constants $M_1, \omega_1 > 0$, such that $||T(t)|| \leq M_1 e^{-\omega_1 t}$, $t \geq 0$.

Let $S_b(\cdot) \equiv S(\cdot)$, for fixed value of parameter b, where the semigroup $S(\cdot)$ was defined in Theorem 1.6. The main result of this paper is the following:

Theorem 3.2 *Let the assumptions of Theorem 1.6 and Theorem 2.2 hold and $b \in (0, b_0)$, where b_0 was defined in Theorem 2.2. Then the semigroup $S_b(\cdot)$ has the property of uniform decaying.*

Proof. We derive the assertion from Theorem 3.1 applied to semigroups $S_b(t), \overline{S}_b(t)$, $b \in (0, b_0)$, where b_0 was defined in Theorem 2.2. From Theorem 1.6 we deduce that $S_b(t)$, $t \geq 0$ satisfies condition (i) of Theorem 3.1. From Theorem 2.2, $\overline{S}_b(t)$, $t \geq 0$, satisfies the condition (ii) of Theorem 3.1. From Theorem 2.3 we deduce that the difference $S_b(t) - \overline{S}_b(t)$ satisfies, for every $t > 0$, condition (iii) from Theorem 3.1. From the conclusion of Theorem 3.1 we obtain the assertion.

We indicate the possibility of slow decay for appropriate domains Ω. Let us recall that an orbit of the billiard in Ω consists of lines, which are reflected transvesally at the boundary. A two-periodic orbit consists of a line segment which insersects boundary perpendicularly.

We announce the following proposition:

Proposition 3.3 *Suppose that there exists a two-periodic orbit of billiard in Ω and let $\omega(t)$ be a positive function with $\lim_{t\to\infty} \omega(t) = 0$. Then there exists $\xi^0 \in H$ such that $\lim_{t\to\infty} \omega^{-1}(t)||S(t)\xi^0|| = \infty$.*

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