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ON THE HYPER-ORDER OF TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF CERTAIN HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate the growth of meromorphic solutions of the linear differential equation

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)} + \ldots + h_0(z)e^{P_0(z)}f = 0,$$

where $k \geq 2$ is an integer, $P_j(z)$ $(j = 0, 1, \ldots, k-1)$ are nonconstant polynomials and $h_j(z)$ are meromorphic functions. Under some conditions, we determine the hyper-order of these solutions. We also consider nonhomogeneous linear differential equations.

Keywords: linear differential equation, transcendental meromorphic function, order of growth, hyper-order.

Mathematics Subject Classification: 34M10, 30D35.

1. INTRODUCTION AND RESULTS

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [15,21]). Let $\sigma(f)$ denote the order of growth of a meromorphic function f. We recall the following definitions.

Definition 1.1 ([9, 16]). Let f be a meromorphic function. Then the hyper-order $\sigma_2(f)$ of f is defined by

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic of f (see [15, 21]).

Definition 1.2 ([9])**.** Let f be a meromorphic function. Then the hyper-exponent of convergence of zeros sequence of f is defined by

$$\lambda_2(f)=\limsup_{r\to+\infty}\frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N(r,\frac{1}{f})$ is the integrated counting function of zeros of f in $\{z:|z|\leq r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}_2(f)=\limsup_{r\to+\infty}\frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of f in $\{z:|z|\leq r\}$.

We define the logarithmic measure of a set $E\subset(1,+\infty)$ by $lm(E)=\int_1^{+\infty}\frac{\chi_E(t)}{t}dt$, where χ_E is the characteristic function of E.

For the second order linear differential equation

$$f''+h_1(z)e^{P(z)}f'+h_0(z)e^{Q(z)}f=0, \tag{1.1}$$

where $P(z)$ and $Q(z)$ are nonconstant polynomials, $h_1(z)$ and $h_0(z)\not\equiv 0$ are entire functions satisfying $\sigma(h_1)<\deg P$ and $\sigma(h_0)<\deg Q$, Gundersen showed in ([12, p. 419]) that if $\deg P\neq\deg Q$, then every nonconstant solution of equation (1.1) is of infinite order. If $\deg P=\deg Q$, then equation (1.1) may have nonconstant solutions of finite order. Indeed, $f(z)=e^z+2$ satisfies $f''+\frac{1}{2}e^zf'-\frac{1}{2}e^zf=0$. Kwon [16] studied the case where $\deg P=\deg Q$ and proved the following result.

Theorem 1.3 ([16])**.** *Let* $P(z)=a_nz^n+\ldots+a_1z+a_0$ *and* $Q(z)=b_nz^n+\ldots+b_1z+b_0$ *be nonconstant polynomials, where* a_i,b_i $(i=0,1,\ldots,n)$ *are complex numbers such that* $a_nb_n\neq 0$*. Let* $h_j(z)$ $(j=0,1)$ *be entire functions with* $\sigma(h_j)<n$*. Suppose that* $\arg a_n\neq\arg b_n$ *or* $a_n=cb_n$ $(0<c<1)$*. Then every nonconstant solution* f *of equation* (1.1) *is of infinite order and satisfies* $\sigma_2(f)\geq n$*.*

In [7], Chen improved the result of Theorem 1.3 for the linear differential equation (1.1) as follows.

Theorem 1.4 ([7])**.** *Let* $P(z)=a_nz^n+\ldots+a_1z+a_0$ *and* $Q(z)=b_nz^n+\ldots+b_1z+b_0$ *be nonconstant polynomials, where* a_i,b_i $(i=0,1,\ldots,n)$ *are complex numbers such that* $a_nb_n\neq 0$*. Let* $h_1(z),h_0(z)$ $(\not\equiv 0)$ *be entire functions with* $\sigma(h_j)<n$*. Suppose that* $\arg a_n\neq\arg b_n$ *or* $a_n=cb_n$ $(0<c<1)$*. Then every solution* $f(\not\equiv 0)$ *of* (1.1) *satisfies* $\sigma_2(f)=n$*.*

In [2], Belaïdi extended Theorem 1.3 for higher order linear differential equations with meromorphic coefficients as follows.

Theorem 1.5 ([2]). *Let* $k \geq 2$ *be an integer and* $P_j(z)=\sum_{i=0}^{n} a_{i,j} z^i$ $(j=0,1,\dots,k-1)$ *be nonconstant polynomials, where* $a_{0,j},\dots,a_{n,j}$ $(j=0,\dots,k-1)$ *are complex numbers such that* $a_{n,j}a_{n,0} \neq 0$ $(j=1,\dots,k-1)$. *Let* $h_j(z)$ $(\not\equiv 0)$ $(j=0,1,\dots,k-1)$ *be meromorphic functions. Suppose that* $\arg a_{n,j} \neq \arg a_{n,0}$ *or* $a_{n,j}=c_j a_{n,0}$ $(0<c_j<1)$ $(j=1,\dots,k-1)$ *and* $\sigma(h_j)<n$ $(j=0,1,\dots,k-1)$. *Then every meromorphic solution* f $(\not\equiv 0)$ *of the differential equation*

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)} f^{(k-1)} + \ldots + h_1(z)e^{P_1(z)} f + h_0(z)e^{P_0(z)} f = 0 \tag{1.2}$$

is of infinite order.

In 2008, Tu and Yi obtained the following result.

Theorem 1.6 ([18]). *Let* $k \geq 2$ *be an integer and* $P_j(z)=\sum_{i=0}^{n} a_{i,j} z^i$ $(j=0,1,\dots,k–1)$ *be polynomials with degree* $n \geq 1$, *where* $a_{n,j}$ $(j=0,1,\dots,k-1)$ *are complex numbers. Let* $h_j(z)$ $(j=0,1,\dots,k-1)$ *be entire functions with* $\sigma(h_j)<n$. *Suppose that there exist nonzero complex numbers* $a_{n,s}$ *and* $a_{n,l}$ *such that* $0<s<l\leq k-1$, $a_{n,s}=|a_{n,s}|e^{i\theta_s}$, $a_{n,l}=|a_{n,l}|e^{i\theta_l}$, θ_s, $\theta_l \in [0,2\pi)$, $\theta_s \neq \theta_l$, $h_s h_l \not\equiv 0$ *and for* $j \neq s,l$, $a_{n,j}$ *satisfies either* $a_{n,j}=d_j a_{n,s}$ $(0<d_j<1)$ *or* $a_{n,j}=d_j a_{n,l}$ $(0<d_j<1)$. *Then every transcendental solution* f *of equation* (1.2) *satisfies* $\sigma(f)=+\infty$. *Furthermore, if* f *is a polynomial solution of equation* (1.2), *then* $\deg f \leq s-1$; *if* $s=1$, *then every nonconstant solution* f *of equation* (1.2) *satisfies* $\sigma(f)=+\infty$.

Recently, Xiao and Chen considered higher order linear differential equations and proved the following result.

Theorem 1.7 ([20]). *Let* $k \geq 2$ *be an integer,* $A_j(z)$ $(j=0,1,\dots,k-1)$ *be entire functions with* $\sigma(A_j)<1$ *and* a_j $(j=0,1,\dots,k-1)$ *be complex numbers. If* $A_j \not\equiv 0$, *then* $a_j \neq 0$. *Suppose that there exists* $\{a_{i_1},a_{i_2},\dots,a_{i_m}\} \subset \{a_1,a_2,\dots,a_{k-1}\}$ *such that* $\arg a_{i_j} (j=1,2,\dots,m)$ *are distinct and for every nonzero* $a_l \in \{a_1,a_2,\dots,a_{k-1}\} \setminus \{a_{i_1},a_{i_2},\dots,a_{i_m}\}$, *there exists some* $a_{i_j} \in \{a_{i_1},a_{i_2},\dots,a_{i_m}\}$ *such that* $a_l = c_l^{(i_j)} a_{i_j}$ $(0<c_l^{(i_j)}<1)$. *Then every transcendental solution of equation*

$$f^{(k)} + A_{k-1}(z)e^{a_{k-1}z} f^{(k-1)} + \ldots + A_1(z)e^{a_1 z} f + A_0(z)e^{a_0 z} f = 0 \tag{1.3}$$

is of infinite order. Furthermore, if $a_0 = a_{i_{j_0}}$ *or* $a_0 = c_0^{(i_{j_0})} a_{i_{j_0}}$ $(0<c_0^{(i_{j_0})} \neq c_s^{(i_{j_0})}<1)$, *where* $s \in \{1,\dots,k-1\}$ *and* $a_{i_{j_0}} \in \{a_{i_1},a_{i_2},\dots,a_{i_m}\}$, *then every solution* $f(\not\equiv 0)$ *of equation* (1.3) *is of infinite order.*

In 2008, Belaïdi and Abbas [4] considered equations of the form (1.2), where $h_j(z)$ $(j=0,\dots,k-1)$ are entire functions. Recently, Habib and Belaïdi [13] studied higher order linear differential equations with meromorphic functions. In this paper, we continue the research in this type of problems. The main purpose of this paper is to extend and improve the above results to equations of the form (1.2) with meromorphic coefficients. We also consider the nonhomogeneous case. We will prove the following results.

Theorem 1.8. *Let* $k \geq 2$ *be an integer and* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \ldots, k-1)$ *be nonconstant polynomials with degree* $n \geq 1$, *where* $a_{0,j}, a_{1,j}, \ldots, a_{n,j}$ $(j = 0, \ldots, k-1)$ *are complex numbers. Let* $h_j(z)$ $(j = 0, 1, \ldots, k-1)$ *be meromorphic functions with* $\sigma(h_j) < n$. *Suppose that there exists* $s \in \{1, \ldots, k-1\}$ *such that* $h_s \not\equiv 0, a_{n,j} = c_j a_{n,s}$ $(0 < c_j < 1)$ $(j \neq s)$. *Then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicity of equation* (1.2) *is of infinite order and satisfies* $\sigma_2(f) = n$. *Furthermore, if* $h_0 \not\equiv 0$ *and* $\max\{c_1, \ldots, c_{s-1}\} < c_0$, *then every meromorphic solution* $f (\not\equiv 0)$ *whose poles are of uniformly bounded multiplicity of equation* (1.2) *is of infinite order and satisfies* $\sigma_2(f) = n$.

Example 1.9. Consider the linear differential equation

$$f''' - \Big(\frac{3z+2}{z+1}\Big) e^z f'' + \Big(\frac{2z+1}{z+1}\Big) e^{2z} f' - \Big(1 + \frac{3}{z}\Big) e^z f = 0.$$

Obiviously, the conditions of Theorem 1.8 are satisfied. So, every transcendental meromorphic solution f of this equation whose poles are of uniformly bounded multiplicity is of infinite order and satisfies $\sigma_2(f) = 1$. Remark that $f(z) = ze^{e^z}$ is a solution of this equation with $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$.

Theorem 1.10. *Let* $k \geq 2$ *be an integer,* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, \ldots, k-1)$ *be polynomials with degree* $n \geq 1$, *where* $a_{0,j}, \ldots, a_{n,j}$ $(j = 0, \ldots, k-1)$ *are complex numbers. Let* $h_j(z)$ $(j = 0, \ldots, k-1)$ *be meromorphic functions with* $\sigma(h_j) < n$. *Suppose that there exist* s, $d \in \{1, \ldots, k-1\}$ *such that* $h_s h_d \not\equiv 0, a_{n,s} = |a_{n,s}| e^{i\theta_s}$, $a_{n,d} = |a_{n,d}| e^{i\theta_d}$,$\theta_s, \theta_d \in [0, 2\pi), \theta_s \neq \theta_d$ *and for* $j \in \{0, \ldots, k-1\} \diagdown \{d, s\}$, $a_{n,j}$ *satisfies either* $a_{n,j} = c_j a_{n,s}$ *or* $a_{n,j} = c_j a_{n,d}$ $(0 < c_j < 1)$. *Then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicity of equation* (1.2) *is of infinite order and satisfies* $\sigma_2(f) = n$.

Theorem 1.11. *Let* $k \geq 2$ *be an integer and* $P_j(z) = \sum_{i=0}^{n} a_{i,j} z^i$ $(j = 0, 1, \ldots, k-1)$ *be polynomials with degree* $n \geq 1$, *where* $a_{0,j}, \ldots, a_{n,j}$ $(j = 0, \ldots, k-1)$ *are complex numbers. Let* $h_j(z)$ $(j = 0, 1, \ldots, k-1)$ *be meromorphic functions with* $\sigma(h_j) < n$. *If* $h_j \not\equiv 0$, *then* $a_{n,j} \neq 0$. *Suppose that there exists* $\{a_{n,i_1}, a_{n,i_2}, \ldots, a_{n,i_m}\} \subset \big\{a_{n,1}, a_{n,2}, \ldots, a_{n,k-1}\big\}$ *such that* $\arg a_{n,i_j}$ $(j = 1, 2, \ldots, m)$ *are distinct and for every nonzero*

$$a_{n,l} \in \big\{a_{n,1}, a_{n,2}, \ldots, a_{n,k-1}\big\} \diagdown \{a_{n,i_1}, a_{n,i_2}, \ldots, a_{n,i_m}\},$$

there exists some $a_{n,i_j} \in \{a_{n,i_1}, a_{n,i_2}, \ldots, a_{n,i_m}\}$ *such that* $a_{n,l} = c_l^{(i_j)} a_{n,i_j}$ $(0 < c_l^{(i_j)} < 1)$. *Then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicity of equation* (1.2) *is of infinite order and satisfies* $\sigma_2(f) = n$. *Furthermore, if* $a_{n,0} = a_{n,i_{j_0}}$ *or* $a_{n,0} = c_0^{(i_{j_0})} a_{n,i_{j_0}}$ $(0 < c_0^{(i_{j_0})} \neq c_s^{(i_{j_0})} < 1)$, *where* $s \in \{1, \ldots, k-1\}$ *and* $a_{n,i_{j_0}} \in \{a_{n,i_1}, a_{n,i_2}, \ldots, a_{n,i_m}\}$, *then every meromorphic solution* $f (\not\equiv 0)$ *whose poles are of uniformly bounded multiplicity of equation* (1.2) *is of infinite order and satisfies* $\sigma_2(f) = n$.

Theorem 1.12. *Let $k \geq 2$ be an integer, $P_j(z)$, $h_j(z)$ and $a_{n,j}$ $(j = 0, 1, \ldots, k-1)$ satisfy hypotheses of Theorem 1.8 or Theorem 1.10 or Theorem 1.11. Let $F \not\equiv 0$ be a meromorphic function of order $\sigma(f) < n$. Then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicity of the linear differential equation*

$$f^{(k)} + h_{k-1}(z)e^{P_{k-1}(z)}f^{(k-1)} + \ldots + h_1(z)e^{P_1(z)}f + h_0(z)e^{P_0(z)}f = F \tag{1.4}$$

is of infinite order and satisfies $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) = n$ with at most one exceptional solution f_0 of finite order.

Remark 1.13. It is well-known that a linear differential equation with holomorphic coefficients must have holomorphic solutions. But the characteristic of solutions is more complicated for a linear differential equation with meromorphic coefficients. For some works related to existence of meromorphic solutions of linear differential equations, see [10, 17, 22].

2. PRELIMINARY LEMMAS

Lemma 2.1 ([1])**.** *Let $P_j(z)$ $(j = 0, 1, \ldots, k)$ be polynomials with $\deg P_0(z) = n$ $(n \geq 1)$ and $\deg P_j(z) \leq n$ $(j = 0, 1, \ldots, k)$. Let $A_j(z)$ $(j = 0, \ldots, k)$ be meromorphic functions with finite order and $\max\{\sigma(A_j) : j = 0, 1, \ldots, k\} < n$ such that $A_0(z) \not\equiv 0$. We denote*

$$F(z) = A_k(z)e^{P_k(z)} + A_{k-1}(z)e^{P_{k-1}(z)} + \ldots + A_1(z)e^{P_1(z)} + A_0(z)e^{P_0(z)}.$$

If $\deg(P_0(z) - P_j(z)) = n$ for all $j = 1, \ldots, k$, then f is a nontrivial meromorphic function with finite order and satisfies $\sigma(F) = n$.

Lemma 2.2 ([11])**.** *Let $f(z)$ be a transcendental meromorphic function and let $\alpha > 1$ and $\varepsilon > 0$ be given constants. Then there exist a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure and a constant $B > 0$ that depends only on α and (i, j) (i, j positive integers with $i > j$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(i)}(z)}{f^{(j)}(z)}\right| \leq B\left[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right]^{i-j}.$$

Lemma 2.3 ([19])**.** *Let $g(z)$ be a transcendental entire function and $\nu_g(r)$ be the central index of g. For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. Then there exist a constant δ_r (> 0) and a set E_2 of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have*

$$\frac{g^{(n)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1 + o(1)) \ (n \geq 1 \text{ is an integer}).$$

Lemma 2.4 ([11, p. 89])**.** *Let $f(z)$ be a transcendental meromorphic function of finite order σ. Let $\Gamma = \{(k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m)\}$ denote a set of distinct pairs* *of integers satisfying* $k_i > j_i \geq 0$ $(i = 1, 2, \ldots, m)$ *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E_3 \subset [1, +\infty)$ *having finite logarithmic measure such that for all* z *satisfying* $|z| = r \notin [0, 1] \cup E_3$ *and* $(k, j) \in \Gamma$, *we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma - 1 + \varepsilon)}.$$

Lemma 2.5. *Let* $f(z) = g(z)/d(z)$ *be a meromorphic function with* $\sigma(f) = \sigma \leq +\infty$, *where* $g(z)$ *and* $d(z)$ *are entire functions satisfying one of the following conditions:*

(i) g *being transcendental and* d *being polynomial,*
(ii) g, d *all being transcendental and* $\lambda(d) = \sigma(d) = \beta < \sigma(g) = \sigma$.

For each sufficiently large $|z| = r$, *let* $z_r = re^{i\theta_r}$ *be a point satisfying* $|g(z_r)| = M(r, g)$ *and let* $\nu_g(r)$ *be the central index of* g. *Then there exist a constant* δ_r (> 0), *a sequence* $\{r_m\}_{m \in \mathbb{N}}$, $r_m \to +\infty$ *and a set* E_4 *of finite logarithmic measure such that the estimation*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r_m)}{z} \right)^n (1 + o(1)) \ (n \geq 1 \textit{ is an integer})$$

holds for all z *satisfying* $|z| = r_m \notin E_4$, $r_m \to +\infty$ *and* $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

Proof. By mathematical induction, we obtain

$$f^{(n)} = \frac{g^{(n)}}{d} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{d} \sum_{(j_1 \cdots j_n)} C_{jj_1 \ldots j_n} \left(\frac{d'}{d} \right)^{j_1} \cdots \left(\frac{d^{(n)}}{d} \right)^{j_n}, \tag{2.1}$$

where $C_{jj_1 \ldots j_n}$ are constants and $j + j_1 + 2j_2 + \ldots + nj_n = n$. Hence

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + \sum_{j=0}^{n-1} \frac{g^{(j)}}{g} \sum_{(j_1 \cdots j_n)} C_{jj_1 \ldots j_n} \left(\frac{d'}{d} \right)^{j_1} \cdots \left(\frac{d^{(n)}}{d} \right)^{j_n}. \tag{2.2}$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.3, there exist a constant δ_r (> 0) and a set E_2 of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{\nu_g(r)}{z} \right)^j (1 + o(1)) \ (j = 1, 2, \ldots, n), \tag{2.3}$$

where $\nu_g(r)$ is the central index of g. Substituting (2.3) into (2.2) yields

$$\begin{aligned} \frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z} \right)^n \Bigg[(1 + o(1)) + \sum_{j=0}^{n-1} \left(\frac{\nu_g(r)}{z} \right)^{j-n} (1 + o(1)) \\ \times \sum_{(j_1 \cdots j_n)} C_{jj_1 \ldots j_n} \left(\frac{d'}{d} \right)^{j_1} \cdots \left(\frac{d^{(n)}}{d} \right)^{j_n} \Bigg]. \end{aligned} \tag{2.4}$$

We can choose a constant ρ such that $\beta < \rho < \sigma$. By Lemma 2.4, for any given ε $(0 < 2\varepsilon < \rho - \beta)$, we have

$$\left|\frac{d^{(s)}(z)}{d(z)}\right| \leq r^{s(\beta-1+\varepsilon)} \ (s = 1, 2, \ldots, n), \tag{2.5}$$

where $|z| = r \notin [0,1] \cup E_3, E_3 \subset (1, +\infty)$ with $lm(E_3) < +\infty$. From this and $j_1 + 2j_2 + \ldots + nj_n = n - j$, we have

$$|z|^{n-j} \left| \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \right| \leq |z|^{(n-j)(\beta+\varepsilon)} \tag{2.6}$$

for $|z| = r \notin [0,1] \cup E_3$. By the Wiman-Valiron theory [17, p. 51], we have

$$\sigma(g) = \limsup_{r \to +\infty} \frac{\log \nu_g(r)}{\log r} = \sigma.$$

Then, by the definition of the limit superior, there exists a sequence $\{r'_m\}$ $(r'_m \to +\infty)$ satisfying

$$\lim_{r'_m \to +\infty} \frac{\log \nu_g(r'_m)}{\log r'_m} = \sigma. \tag{2.7}$$

Setting the logarithmic measure of $E_4 = [0,1] \cup E_2 \cup E_3, lm(E_4) = \delta < +\infty$. We have $[r'_m, (\delta+1)r'_m] \setminus E_4 \neq \varnothing$. Indeed, if $[r'_m, (\delta+1)r'_m] \setminus E_4 = \varnothing$, then for all $m \in \mathbb{N}, [r'_m, (\delta+1)r'_m] \subset E_4$. It follows that $\bigcup_{m \in \mathbb{N}} [r'_m, (\delta+1)r'_m] \subset E_4$ and

$$lm(\bigcup_{m \in \mathbb{N}} [r'_m, (\delta+1)r'_m]) = \sum_{m=0}^{\infty} \int_{r'_m}^{(\delta+1)r'_m} \frac{dt}{t} = \sum_{m=0}^{\infty} \log(\delta+1) = \infty \leq lm(E_4) = \delta$$

which is a contraction. So, there exists a point $r_m \in [r'_m, (\delta+1)r'_m] \setminus E_4$. Since

$$\frac{\log \nu_g(r_m)}{\log r_m} \geq \frac{\log \nu_g(r'_m)}{\log[(\delta+1)r'_m]} = \frac{\log \nu_g(r'_m)}{(\log r'_m)[1 + \frac{\log(\delta+1)}{\log r'_m}]}, \tag{2.8}$$

then we have

$$\lim_{r_m \to +\infty} \frac{\log \nu_g(r_m)}{\log r_m} = \sigma. \tag{2.9}$$

Hence for sufficiently large m, we obtain

$$\nu_g(r_m) \geq r_m^{\sigma-\varepsilon} \geq r_m^{\rho-\varepsilon}, \tag{2.10}$$

where $\sigma - \varepsilon$ can be replaced by a large enough number M if $\sigma = +\infty$. This and (2.5) lead to

$$\left| \left(\frac{\nu_g(r)}{z}\right)^{j-n} \left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n} \right| \leq r_m^{(n-j)(\beta-\rho+2\varepsilon)} \to 0, \ r_m \to +\infty, \tag{2.11}$$

where $|z| = r_m \notin E_4$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. From (2.4) and (2.11), we obtain our result. □

Lemma 2.6. *Let $f(z) = g(z)/d(z)$ be a meromorphic function with $\sigma(f) = \sigma \le +\infty$, where $g(z)$ and $d(z)$ are entire functions satisfying one of the following conditions:*

(i) *g being transcendental and d being polynomial,*
(ii) *g, d all being transcendental and $\lambda(d) = \sigma(d) = \rho < \sigma(g) = \sigma$.*

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. Then there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_5 of finite logarithmic measure such that the estimation

$$\left|\frac{f(z)}{f^{(n)}(z)}\right| \le r_m^{2n} \text{ } (n \ge 1 \textit{ is an integer})$$

holds for all z satisfying $|z| = r_m \notin E_5, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$.

Proof. Let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.5, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_5 of finite logarithmic measure such that the estimation

$$\frac{f^{(n)}(z)}{f(z)} = \Big(\frac{\nu_g(r_m)}{z}\Big)^n (1 + o(1)) \text{ } (n \ge 1 \text{ is an integer}) \tag{2.12}$$

holds for all z satisfying $|z| = r_m \notin E_5, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. On the other hand, we obtain for any given $\varepsilon > 0$ and sufficiently large m

$$\nu_g(r_m) \ge r_m^{\sigma-\varepsilon}, \tag{2.13}$$

where $\sigma - \varepsilon$ can be replaced by a large enough number M if $\sigma = +\infty$. Hence we have

$$\left|\frac{f(z)}{f^{(n)}(z)}\right| \le r_m^{2n}. \tag{2.14}$$

□

Lemma 2.7 ([14])**.** *Let $P(z) = (\alpha + i\beta)z^n + \dots$ (α, β are real numbers, $|\alpha| + |\beta| \ne 0$) be a polynomial with degree $n \ge 1$ and $A(z)$ be a meromorphic function with $\sigma(A) < n$. Set $f(z) = A(z)e^{P(z)}$ $(z = re^{i\theta})$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $E_6 \subset [1, +\infty)$ having finite logarithmic measure such that for any $\theta \in [0, 2\pi) \setminus H$ $(H = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\})$ and for $|z| = r \notin [0, 1] \cup E_6$, $r \to +\infty$, we have*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta)r^n\} \le \left|f(re^{i\theta})\right| \le \exp\{(1 + \varepsilon)\delta(P, \theta)r^n\},$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(P, \theta)r^n\} \le \left|f(re^{i\theta})\right| \le \exp\{(1 - \varepsilon)\delta(P, \theta)r^n\}.$$

Lemma 2.8 ([12]). *Let* $\varphi : [0,+\infty) \to \mathbb{R}$ *and* $\psi : [0,+\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r) \leq \psi(r)$ *for all* $r \notin E_7 \cup [0,1]$, *where* $E_7 \subset (1,+\infty)$ *is a set of finite logarithmic measure. Let* $\alpha > 1$ *be a given constant. Then there exists an* $r_0 = r_0(\alpha) > 0$ *such that* $\varphi(r) \leq \psi(\alpha r)$ *for all* $r > r_0$.

Lemma 2.9 ([8]). *Let* $k \geq 2$ *be an integer and let* $A_j(z)$ $(j = 0,1,\ldots,k-1)$ *be meromorphic functions of finite order. Set* $\rho = \max\{\sigma(A_j) : j = 0,1,\ldots,k-1\}$. *If* f *is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of the equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0,$$

then $\sigma_2(f) \leq \rho$.

Lemma 2.10 ([5]). *Let* $f(z) = g(z)/d(z)$ *be a meromorphic function with* $\sigma(f) = \sigma \leq +\infty$, *where* $g(z)$ *and* $d(z)$ *are entire functions satisfying one of the following conditions:*

(i) *g being transcendental and d being polynomial,*
(ii) *g, d all being transcendental and* $\lambda(d) = \sigma(d) = \beta < \sigma(g) = \sigma$.

Let $\nu_g(r)$ *be the central index of* g. *Then there exist a sequence* $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ *and a set* E_8 *of finite logarithmic measure such that the estimation*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r_m)}{z}\right)^n (1+o(1)) \ (n \geq 1 \textit{ is an integer})$$

holds for all z *satisfying* $|z| = r_m \notin E_8$, $r_m \to +\infty$ *and* $|g(z)| = M(r_m, g)$.

Lemma 2.11 ([6]). *Let* $g(z)$ *be a transcendental meromorphic function of order* $\sigma(g) = \sigma < +\infty$. *Then for any given* $\varepsilon > 0$, *there exists a set* $E_9 \subset (1,+\infty)$ *that has finite logarithmic measure such that*

$$|g(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\}$$

holds for $|z| = r \notin [0,1] \cup E_9$, $r \to +\infty$.

Lemma 2.12 ([9]). *Let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *be an entire function of infinite order with the hyper-order* $\sigma_2(f) = \sigma$ *and let* $\nu_f(r)$ *be the central index of* f. *Then*

$$\limsup_{r\to+\infty} \frac{\log\log \nu_f(r)}{\log r} = \sigma.$$

Lemma 2.13. *Let* $k \geq 2$ *be an integer,* $A_0(z),\ldots,A_{k-1}(z)$ *and* $F(\not\equiv 0)$ *be meromorphic functions of finite order and let* $\sigma = \max\{\sigma(F), \sigma(A_j) : j = 0,\ldots,k-1\}$. *If* f *is an infinite order meromorphic solution whose poles are of uniformly bounded multiplicity of equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F, \tag{2.15}$$

then $\sigma_2(f) \leq \sigma$.

Proof. Assume that f is an infinite order meromorphic solution whose poles are of uniformly bounded multiplicity of equation (2.15). By (2.15), we have

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \ldots + |A_1(z)| \left|\frac{f'}{f}\right| + \left|\frac{F}{f}\right| + |A_0(z)|. \tag{2.16}$$

By (2.15), it follows that the poles of f can only occur at the poles of A_j $(j = 0, \ldots, k-1)$ and F. Note that the poles of f are of uniformly bounded multiplicity. Hence $\lambda(1/f) \leq \sigma$. By the Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \sigma(d) = \lambda(1/f) \leq \sigma < \sigma(f) = \sigma(g) = +\infty$$

and $\sigma_2(f) = \sigma_2(g)$. By Lemma 2.10, there exist a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_8 of finite logarithmic measure such that the estimation

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r_m)}{z}\right)^j (1 + o(1)) \; (j = 1, \ldots, k) \tag{2.17}$$

holds for all z satisfying $|z| = r_m \notin E_8, r_m \to +\infty$ and $|g(z)| = M(r_m, g)$. By Lemma 2.11, for any given $\varepsilon > 0$, there exists a set $E_9 \subset (1, +\infty)$ that has finite logarithmic measure, such that

$$|F(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\}, \quad |d(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\} \tag{2.18}$$

and

$$|A_j(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\} \; (j = 0, \ldots, k-1) \tag{2.19}$$

hold for $|z| = r \notin [0,1] \cup E_9, r \to +\infty$. Since $M(r, g) \geq 1$ for r sufficiently large, it follows from (2.18) that

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|\,|d(z)|}{|g(z)|} = \frac{|F(z)|\,|d(z)|}{M(r, g)} \leq \exp\left\{2r^{\sigma+\varepsilon}\right\} \tag{2.20}$$

for $|z| = r \notin [0,1] \cup E_9, r \to +\infty$. Substituting (2.17), (2.19) and (2.20) into (2.16), we obtain

$$(\nu_g(r_m))^k\, |1 + o(1)| \leq (k+1) r_m^k (\nu_g(r_m))^{k-1}\, |1 + o(1)| \exp\left\{2r_m^{\sigma+\varepsilon}\right\} \tag{2.21}$$

for all z satisfying $|z| = r_m \notin [0,1] \cup E_8 \cup E_9, r \to +\infty$ and $|g(z)| = M(r_m, g)$. Thus, by (2.21), Lemma 2.8 and Lemma 2.12, we have

$$\sigma_2(g) = \limsup_{r_m \to +\infty} \frac{\log\log \nu_f(r_m)}{\log r_m} \leq \sigma + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\sigma_2(f) \leq \sigma$. □

Lemma 2.14. ([3]) *Let* $A_0, A_1, \ldots, A_{k-1}, F (\not\equiv 0)$ *be finite order meromorphic functions. If* f *is an infinite order meromorphic solution of equation* (2.15)*, then* $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$.

3. PROOF OF THEOREM 1.8

First we prove that every transcendental meromorphic solution f of equation (1.2) is of order $\sigma(f) \geq n$. Assume that f is a transcendental meromorphic solution f of equation (1.2) of order $\sigma(f) < n$. We can write equation (1.2) in the form

$$\sum_{j=0}^{k-1} h_j(z) f^{(j)} e^{P_j(z)} = -f^{(k)}, \tag{3.1}$$

where $h_j f^{(j)}$ ($j = 0, 1, \ldots, k-1$) are meromorphic functions of finite order with $\sigma(h_j f^{(j)}) < n$. We have $h_s f^{(s)} \not\equiv 0$. Indeed, if $h_s f^{(s)} \equiv 0$, it follows that $f^{(s)} \equiv 0$. Then f has to be a polynomial of degree less than s. This is a contradiction. Since $a_{n,j} = \alpha_j a_{n,s}$ $(0 < \alpha_j < 1)$ $(j \neq s)$, we get $\deg(P_s(z) - P_j(z)) = n$ $(j \neq s)$. Thus by (3.1) and Lemma 2.1, we have $\sigma(-f^{(k)}) = n$ and this is a contradiction. Hence every transcendental meromorphic solution f of equation (1.2) is of order $\sigma(f) \geq n$.

Assume f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.2). By Lemma 2.2, there exist a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq B[T(2r, f)]^{j+1} \ (0 \leq i < j \leq k). \tag{3.2}$$

By (1.2), it follows that the poles of f can only occur at the poles of $h_j(z)$ ($j = 0, \ldots, k-1$). Note that the poles of f are of uniformly bounded multiplicity. Hence

$$\lambda(1/f) \leq \max\{\sigma(h_j) : j = 0, \ldots, k-1\} < n.$$

By the Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \sigma(d) = \lambda(1/f) < n \leq \sigma(f) = \sigma(g).$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.6, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_5 of finite logarithmic measure such that the estimation

$$\left| \frac{f(z)}{f^{(i)}(z)} \right| \leq r_m^{2i} \ (i \geq 1 \text{ is an integer}) \tag{3.3}$$

holds for all z satisfying $|z| = r_m \notin E_5, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, where $H_1 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0\}$, we have

$$\delta(P_s, \theta) > 0 \text{ or } \delta(P_s, \theta) < 0.$$

Case 1. $\delta(P_s,\theta) > 0$. Put $\alpha = \max\{c_j : j \neq s)\}$. Then $0 < \alpha < 1$. By Lemma 2.7, for any given ε $(0 < 2\varepsilon < \frac{1-\alpha}{1+\alpha})$, there exists a set $E_6 \subset [1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, we have

$$\left|h_s(z)e^{P_s(z)}\right| \geq \exp\{(1-\varepsilon)\delta(P_s,\theta)r^n\} \tag{3.4}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\alpha\delta(P_s,\theta)r^n\} \quad (j \neq s). \tag{3.5}$$

We can rewrite (1.2) as

$$\begin{aligned} h_s(z)e^{P_s(z)} = {} & \frac{f^{(k)}}{f^{(s)}} + h_{k-1}(z)e^{P_{k-1}(z)}\frac{f^{(k-1)}}{f^{(s)}} \\ & + h_{s+1}(z)e^{P_{s+1}(z)}\frac{f^{(s+1)}}{f^{(s)}} + h_{s-1}(z)e^{P_{s-1}(z)}\frac{f^{(s-1)}}{f}\frac{f}{f^{(s)}} \\ & + \ldots + h_1(z)e^{P_1(z)}\frac{f'}{f}\frac{f}{f^{(s)}} + h_0(z)e^{P_0(z)}\frac{f}{f^{(s)}}. \end{aligned} \tag{3.6}$$

Substituting (3.2)–(3.5) into (3.6), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, we obtain

$$\begin{aligned} & \exp\{(1-\varepsilon)\delta(P_s,\theta)r_m^n\} \\ & \leq M_1 r_m^{2s} \exp\{(1+\varepsilon)\alpha\delta(P_s,\theta)r_m^n\}\,[T(2r_m,f)]^{k+1}, \end{aligned} \tag{3.7}$$

where M_1 (> 0) is a constant. Hence by using Lemma 2.8 and (3.7), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

Case 2. $\delta(P_s,\theta) < 0$. Set $\beta = \min\{c_j : j \neq s\} > 0$. By Lemma 2.7, for any given ε $(0 < 2\varepsilon < 1)$, there exists a set $E_6 \subset [0,2\pi)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, we have

$$\left|h_s(z)e^{P_s(z)}\right| \leq \exp\{(1-\varepsilon)\delta(P_s,\theta)r^n\} \tag{3.8}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1-\varepsilon)\beta\delta(P_s,\theta)r^n\}\ (j \neq s). \tag{3.9}$$

By (1.2), we get

$$\begin{aligned} -1 = {} & h_{k-1}(z)e^{P_{k-1}(z)}\frac{f^{(k-1)}}{f}\frac{f}{f^{(k)}} + \ldots + h_s(z)e^{P_s(z)}\frac{f^{(s)}}{f}\frac{f}{f^{(k)}} \\ & + \ldots + h_1(z)e^{P_1(z)}\frac{f'}{f}\frac{f}{f^{(k)}} + h_0(z)e^{P_0(z)}\frac{f}{f^{(k)}}. \end{aligned} \tag{3.10}$$

Substituting (3.2), (3.3), (3.8) and (3.9) into (3.10), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, we obtain

$$1 \leq M_2 r_m^{2k} \exp\{(1-\varepsilon)\beta\delta(P_s,\theta)r_m^n\}\,[T(2r_m,f)]^{k+1}, \tag{3.11}$$

where M_2 (> 0) is a constant. Hence by using Lemma 2.8 and (3.11), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

Suppose now that $h_0 \not\equiv 0$ and $\max\{c_1, \dots, c_{s-1}\} < c_0$. If f is a rational solution of (1.2), then by $h_0 \not\equiv 0$ and $\max\{c_1, \dots c_{s-1}\} < c_0$, the hypotheses of Theorem 1.8 and

$$\begin{aligned} f = -\bigg(&\frac{1}{h_0(z)} e^{-P_0(z)} f^{(k)} + \frac{h_{k-1}(z)}{h_0(z)} e^{P_{k-1}(z) - P_0(z)} f^{(k-1)} \\ &+ \dots + \frac{h_1(z)}{h_0(z)} e^{P_1(z) - P_0(z)} f'\bigg), \end{aligned} \tag{3.12}$$

we obtain a contradiction since the left side of equation (3.12) is a rational function but the right side is a transcendental meromorphic function.

Now we prove that equation (1.2) cannot have a nonzero polynomial solution. Set $\gamma = \max\{c_1, \dots, c_{s-1}\} < c_0$ and let f be a nonzero polynomial solution of equation (1.2) with $\deg f = q$. We take a ray $\arg z = \theta \in [0, 2\pi) \setminus H_1$ such that $\delta(P_s, \theta) > 0$. By Lemma 2.7, for any given ε ($0 < 2\varepsilon < \min\left\{\frac{1-\alpha}{1+\alpha}, \frac{c_0-\gamma}{c_0+\gamma}\right\}$), there exists a set $E_6 \subset [0, 2\pi)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta$, we have (3.4), (3.5),

$$\left|h_0(z) e^{P_0(z)}\right| \leq \exp\{(1+\varepsilon) c_0 \delta(P_s, \theta) r^n\}, \tag{3.13}$$

and

$$\left|h_j(z) e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon) \gamma \delta(P_s, \theta) r^n\} \ (j = 1, \dots, s-1). \tag{3.14}$$

If $q \geq s$, then by (1.2), (3.4) and (3.5), for all z with $|z| = r \notin [0, 1] \cup E_6, r \to +\infty$ and $\arg z = \theta$, we obtain

$$\begin{aligned} M_3 r^{q-s} \exp\{(1-\varepsilon)\delta(P_s, \theta) r^n\} &\leq \left|h_s(z) e^{P_s(z)}\right| \left|f^{(s)}(z)\right| \\ &\leq \sum_{j \neq s} \left|h_j(z) e^{P_j(z)}\right| \left|f^{(j)}(z)\right| \\ &\leq M_4 r^q \exp\{(1+\varepsilon)\alpha\delta(P_s, \theta) r^n\}, \end{aligned} \tag{3.15}$$

where M_3, M_4 (> 0) are constants. Hence (3.15) is a contradiction.

If $q < s$, then by (1.2), (3.13) and (3.14), for all z with $|z| = r \notin [0, 1] \cup E_6, r \to +\infty$ and $\arg z = \theta$, we obtain

$$\begin{aligned} M_5 r^{s-1} \exp\{(1-\varepsilon) c_0 \delta(P_s, \theta) r^n\} &\leq \left|h_0(z) e^{P_0(z)}\right| |f(z)| \\ &\leq \sum_{j=1}^{s-1} \left|h_j(z) e^{P_j(z)}\right| \left|f^{(j)}(z)\right| \\ &\leq M_6 r^{s-2} \exp\{(1+\varepsilon)\gamma\delta(P_s, \theta) r^n\}, \end{aligned} \tag{3.16}$$

where M_5, M_6 (> 0) are constants. By (3.16), we get a contradiction. Therefore, if $h_0 \not\equiv 0$ and $\max\{c_1, \dots, c_{s-1}\} < c_0$, then every meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.2) is of infinite order and satisfies $\sigma_2(f) = n$.

4. PROOF OF THEOREM 1.10

First we prove that every transcendental meromorphic solution f of equation (1.2) is of order $\sigma(f) \geq n$. Assume that f is a transcendental meromorphic solution f of equation (1.2) of order $\sigma(f) < n$. We can write equation (1.2) in the form (3.1), where $h_j f^{(j)}$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions of finite order with $h_s f^{(s)} \not\equiv 0, h_d f^{(d)} \not\equiv 0$ and $\sigma(h_j f^{(j)}) < n$ $(j = 0, 1, \ldots, k-1)$. Since $\theta_s \neq \theta_d$, it follows that $\deg(P_s(z) - P_j(z)) = n$ $(j \neq s)$. Thus by (3.1) and Lemma 2.1, we have $\sigma(-f^{(k)}) = n$ and this is a contradiction. Hence every transcendental meromorphic solution f of equation (1.2) is of order $\sigma(f) \geq n$.

Assume f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.2). By Lemma 2.2, there exist a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have (3.2). By (1.2), it follows that the poles of f can only occur at the poles of h_j $(j = 0, \ldots, k-1)$. Note that the poles of f are of uniformly bounded multiplicity. Hence

$$\lambda(1/f) \leq \max\{\sigma(h_j) : j = 0, \ldots, k-1\} < n.$$

By the Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \sigma(d) = \lambda(1/f) < n \leq \sigma(f) = \sigma(g).$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.6, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_5 of finite logarithmic measure such that the estimation (3.3) holds for all z satisfying $|z| = r_m \notin E_5, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. Set

$$H_2 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = 0 \text{ or } \delta(P_d, \theta) = 0\}$$

and

$$H_3 = \{\theta \in [0, 2\pi) : \delta(P_s, \theta) = \delta(P_d, \theta)\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we have

$$\delta(P_s, \theta) \neq 0,\ \delta(P_d, \theta) \neq 0 \text{ and } \delta(P_s, \theta) > \delta(P_d, \theta) \text{ or } \delta(P_s, \theta) < \delta(P_d, \theta).$$

Set $\delta_1 = \delta(P_s, \theta)$ and $\delta_2 = \delta(P_d, \theta)$.

Case 1. $\delta_1 > \delta_2$. Here we also divide our proof in three subcases.

Subcase 1.1. $\delta_1 > \delta_2 > 0$. Set $\delta_3 = \max\{\delta(P_j, \theta) : j \neq s\}$. Then $0 < \delta_3 < \delta_1$. Thus by Lemma 2.7, for any given ε $(0 < 2\varepsilon < \frac{\delta_1 - \delta_3}{\delta_1 + \delta_3})$, there exists a set $E_6 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_2 \cup H_3)$, we have

$$\left|h_s(z)e^{P_s(z)}\right| \geq \exp\{(1-\varepsilon)\delta_1 r^n\} \tag{4.1}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\delta_3 r^n\}\ (j \neq s). \tag{4.2}$$

Substituting (3.2), (3.3), (4.1) and (4.2) into (3.6), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_2 \cup H_3)$, we obtain

$$\exp\{(1-\varepsilon)\delta_1 r_m^n\} \leq M_1 r_m^{2s} \exp\{(1+\varepsilon)\delta_3 r_m^n\} [T(2r_m, f)]^{k+1}, \tag{4.3}$$

where $M_1 (> 0)$ is a constant. Hence by using Lemma 2.8 and (4.3), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

Subcase 1.2. $\delta_1 > 0 > \delta_2$. Set $\gamma = \max\{c_j : j \neq s, d\}$. By Lemma 2.7, for any given ε $(0 < 2\varepsilon < \frac{1-\gamma}{1+\gamma})$, there exists a set $E_6 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_2 \cup H_3)$, we have (4.1) and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\gamma\delta_1 r^n\}\ (j \neq s). \tag{4.4}$$

Substituting (3.2), (3.3), (4.1) and (4.4) into (3.6), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_2 \cup H_3)$, we obtain

$$\exp\{(1-\varepsilon)\delta_1 r_m^n\} \leq M_2 r_m^{2s} \exp\{(1+\varepsilon)\gamma\delta_1 r_m^n\} [T(2r_m, f)]^{k+1}, \tag{4.5}$$

where M_2 (> 0) is a constant. Hence by using Lemma 2.8 and (4.5), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

Subcase 1.3. $0 > \delta_1 > \delta_2$. Set $\lambda = \min\{c_j : j \neq s, d)\}$. By Lemma 2.7, for any given ε $(0 < 2\varepsilon < 1)$, there exists a set $E_6 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_2 \cup H_3)$, we have (3.8) and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1-\varepsilon)\lambda\delta_1 r^n\}\ (j \neq s). \tag{4.6}$$

Substituting (3.2), (3.3), (3.8) and (4.6) into (3.10), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_2 \cup H_3)$, we obtain

$$1 \leq M_3 r_m^{2k} \exp\{(1-\varepsilon)\lambda\delta_1 r_m^n\} [T(2r_m, f)]^{k+1}, \tag{4.7}$$

where M_3 (> 0) is a constant. Hence by using Lemma 2.8 and (4.7), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

Case 2. $\delta_1 < \delta_2$**.** Using the same reasoning as in Case 1, we can also obtain $\sigma(f) = +\infty$ and $\sigma_2(f) = n$.

5. PROOF OF THEOREM 1.11

First we prove that every transcendental meromorphic solution f of equation (1.2) is of order $\sigma(f) \geq n$. Assume that f is a transcendental meromorphic solution f of equation (1.2) of order $\sigma(f) < n$. We can rewrite equation (1.2) in the form (3.1), where $h_j f^{(j)}$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions of order $\sigma(h_j f^{(j)}) < n$ $(j = 0, 1, \ldots, k-1)$. We have $h_{i_s} f^{(i_s)} \not\equiv 0$ $(s = 1, \ldots, m)$. Indeed, if $h_{i_s} f^{(i_s)} \equiv 0$, it follows that $f^{(i_s)} \equiv 0$. Then f has to be a polynomial of degree less than i_s. This is a contradiction. We also have $\deg(P_{i_s}(z) - P_j(z)) = n$ $(j \neq i_s)$. Thus by (3.1) and Lemma 2.1, we obtain $\sigma(-f^{(k)}) = n$ and this is a contradiction. Hence every transcendental meromorphic solution f of equation (1.2) is of order $\sigma(f) \geq n$.

Assume f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.2). By Lemma 2.2, there exist a constant $B > 0$ and a set $E_1 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have (3.2). By (1.2), it follows that the poles of f can only occur at the poles of $h_j(z)$ $(j = 0, \ldots, k-1)$. Note that the poles of f are of uniformly bounded multiplicity. Hence

$$\lambda(1/f) \leq \max\{\sigma(h_j) : j = 0, \ldots, k-1\} < n.$$

By the Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \sigma(d) = \lambda(1/f) < n \leq \sigma(f) = \sigma(g).$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.6, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m\in\mathbb{N}}$, $r_m \to +\infty$ and a set E_5 of finite logarithmic measure such that the estimation (3.3) holds for all z satisfying $|z| = r_m \notin E_5, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. Set

$$H_4 = \bigcup_{j=0}^{k-1} \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0\}$$

and

$$H_5 = \bigcup_{1\leq s<d\leq m} \{\theta \in [0, 2\pi) : \delta(P_{i_s}, \theta) = \delta(P_{i_d}, \theta)\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we have $\delta(P_j, \theta) \neq 0$ $(j = 0, \ldots, k-1)$, $\delta(P_{i_s}, \theta) \neq \delta(P_{i_d}, \theta)$ $(1 \leq s < d \leq m)$. Since a_{n,i_j} $(j = 1, \ldots, m)$ are distinct complex numbers, then there exists only one $t \in \{1, \ldots, m\}$ such that

$$\delta_t = \delta(P_{i_t}, \theta) = \max\left\{\delta(P_{i_j}, \theta) : j = 1, \ldots, m\right\}.$$

For any given $\theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we have

$$\delta(P_{i_t}, \theta) > 0 \text{ or } \delta(P_{i_t}, \theta) < 0.$$

Case 1. $\delta_t > 0$. For $l \in \{0, \ldots, k-1\} \setminus \{i_1, \ldots, i_m\}$, we have

$$a_{n,l} = c_{n,l}^{(i_t)} a_{n,i_t} \text{ or } a_{n,l} = c_{n,l}^{(i_j)} a_{n,ij} \ (j \neq t).$$

Hence for $l \in \{0,\ldots,k-1\} \setminus \{i_1,\ldots,i_m\}$, we have $\delta(P_l,\theta) < \delta_t$. Set $\delta = \max\{\delta(P_j,\theta) : j \neq i_t\}$. Thus $\delta < \delta_t$.

Subcase 1.1. $\delta > 0$. Thus by Lemma 2.7, for any given ε $(0 < 2\varepsilon < \frac{\delta_t-\delta}{\delta_t+\delta})$, there exists a set $E_6 \subset (1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we have

$$\left|h_{i_t}(z)e^{P_{i_t}(z)}\right| \geq \exp\{(1-\varepsilon)\delta_{i_t} r^n\} \tag{5.1}$$

and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1+\varepsilon)\delta r^n\} \ (j \neq i_t). \tag{5.2}$$

We can rewrite (1.2) as

$$\begin{aligned} h_{i_t}(z)e^{P_{i_t}(z)} = {} & \frac{f^{(k)}}{f^{(i_t)}} + h_{k-1}(z)e^{P_{k-1}(z)}\frac{f^{(k-1)}}{f^{(i_t)}} \\ & + h_{i_t+1}(z)e^{P_{i_t+1}(z)}\frac{f^{(i_t+1)}}{f^{(i_t)}} + h_{i_t-1}(z)e^{P_{i_t-1}(z)}\frac{f^{(i_t-1)}}{f}\frac{f}{f^{(i_t)}} \\ & + \ldots + h_1(z)e^{P_1(z)}\frac{f'}{f}\frac{f}{f^{(i_t)}} + h_0(z)e^{P_0(z)}\frac{f}{f^{(i_t)}}. \end{aligned} \tag{5.3}$$

Substituting (3.2), (3.3), (5.1) and (5.2) into (5.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we obtain

$$\exp\{(1-\varepsilon)\delta_{i_t} r_m^n\} \leq M_1 r_m^{2i_t} \exp\{(1+\varepsilon)\delta r_m^n\} [T(2r_m,f)]^{k+1}, \tag{5.4}$$

where $M_1 (> 0)$ is a constant. Hence by using Lemma 2.8 and (5.4), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

Subcase 1.2. $\delta < 0$. By Lemma 2.7, for any given ε $(0 < 2\varepsilon < 1)$, there exists a set $E_6 \subset [0,2\pi)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we have (5.1) and

$$\left|h_j(z)e^{P_j(z)}\right| \leq \exp\{(1-\varepsilon)\delta(P_j,\theta)r^n\} < 1 \ (j \neq i_t). \tag{5.5}$$

Substituting (3.2),(3.3), (5.1) and (5.5) into (3.6), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we obtain

$$\exp\{(1-\varepsilon)\delta_{i_t} r_m^n\} \leq M_2 r_m^{2i_t} [T(2r_m,f)]^{k+1}, \tag{5.6}$$

where M_2 (> 0) is a constant. Hence by using Lemma 2.8 and (5.6), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

Case 2. $\delta_{i_t} < 0$. Set

$$c = \min\left\{c_{n,l}^{(i_j)} : l \in \{0,\ldots,k-1\} \setminus \{i_1,\ldots,i_m\} \text{ and } j = (1,\ldots,m)\right\}.$$

By Lemma 2.7, for any given ε $(0 < 2\varepsilon < 1)$, there exists a set $E_6 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_6$, $r \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we have

$$\left| h_j(z) e^{P_j(z)} \right| \leq \exp\{(1-\varepsilon) c \delta_{i_t} r^n\} \ (j = 0, \dots, k-1). \tag{5.7}$$

Substituting (3.2), (3.3) and (5.7) into (3.10), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6$, $r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_4 \cup H_5)$, we obtain

$$1 \leq M_3 r_m^{2k} \exp\left\{(1-\varepsilon) c \delta_{i_t} r_m^n\right\} \left[T(2r_m, f)\right]^{k+1}, \tag{5.8}$$

where M_3 (> 0) is a constant. Hence by using Lemma 2.8 and (5.8), we obtain $\sigma(f) = +\infty$ and $\sigma_2(f) \geq n$. From this and Lemma 2.9, we have $\sigma_2(f) = n$.

6. PROOF OF THEOREM 1.12

First, we show that (1.4) can possess at most one exceptional transcendental meromorphic solution f_0 of finite order. In fact, if f^* is another transcendental meromorphic solution of finite order of equation (1.3), then $f_0 - f^*$ is of finite order. But $f_0 - f^*$ is a transcendental meromorphic solution of the corresponding homogeneous equation of (1.4). This contradicts Theorem 1.8, Theorem 1.10 and Theorem 1.11. We assume that f is an infinite order meromorphic solution of (1.4) whose poles are of uniformly bounded multiplicity. By Lemma 2.13 and Lemma 2.14, we have $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f) \leq n$.

Now we prove that $\sigma_2(f) \geq n$. By Lemma 2.2, there exist a constant $B > 0$ and a set $E_1 \subset [1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_1$, we have (3.2). Set

$$\sigma = \max\left\{\sigma(F),\ \sigma(h_j) : j = 0, \dots, k-1\right\}.$$

By (1.4), it follows that the poles of f can only occur at the poles of $h_j(z)$ $(j = 0, \dots, k-1)$ and F. Note that the poles of f are of uniformly bounded multiplicity. Hence $\lambda(1/f) \leq \sigma$. By the Hadamard factorization theorem, we know that f can be written as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions with

$$\lambda(d) = \sigma(d) = \lambda(1/f) \leq \sigma < \sigma(f) = \sigma(g) = +\infty.$$

For each sufficiently large $|z| = r$, let $z_r = re^{i\theta_r}$ be a point satisfying $|g(z_r)| = M(r, g)$. By Lemma 2.6, there exist a constant δ_r (> 0), a sequence $\{r_m\}_{m \in \mathbb{N}}$, $r_m \to +\infty$ and a set E_5 of finite logarithmic measure such that the estimation (3.3) holds for all z satisfying $|z| = r_m \notin E_5, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$. Since $|g(z)|$ is continuous in $|z| = r$, then there exists a constant $\lambda_r (> 0)$ such that for all z satisfying $|z| = r$ sufficiently large and $\arg z = \theta \in [\theta_r - \lambda_r, \theta_r + \lambda_r]$, we have

$$\frac{1}{2} |g(z_r)| < |g(z)| < \frac{3}{2} |g(z_r)|. \tag{6.1}$$

On the other hand, by Lemma 2.11, for a given ε $(0 < \varepsilon < n - \sigma)$, there exists a set $E_9 \subset (1, +\infty)$ that has finite logarithmic measure such that

$$|F(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\} \text{ and } |d(z)| \leq \exp\left\{r^{\sigma+\varepsilon}\right\} \tag{6.2}$$

hold for $|z| = r \notin [0,1] \cup E_9, r \to +\infty$. Since $M(r,g) \geq 1$ for sufficiently large r, it follows from (6.2) that

$$\left|\frac{F(z)}{f(z)}\right| = \frac{|F(z)|\,|d(z)|}{|g(z)|} = \frac{|F(z)|\,|d(z)|}{M(r,g)} \leq \exp\left\{2r^{\sigma+\varepsilon}\right\} \tag{6.3}$$

for $|z| = r \notin [0,1] \cup E_9, r \to +\infty$. Set $\gamma = \min\{\delta_r, \lambda_r\}$.

(i) Suppose that $P_j(z), h_j(z)$ and $a_{n,j}$ $(j = 0, 1, \ldots, k-1)$ satisfy hypotheses of Theorem 1.8. For any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H_1$, where H_1 is defined in the proof of Theorem 1.8, we have

$$\delta(P_s, \theta) > 0 \text{ or } \delta(P_s, \theta) < 0.$$

Case 1. $\delta(P_s, \theta) > 0$**.** From (1.4), (3.2)–(3.5) and (6.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H_1$, we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(P_s,\theta)r_m^n\} \\ &\leq M_1 r_m^{2s} \exp\{(1+\varepsilon)\alpha\delta(P_s,\theta)r_m^n\}\,[T(2r_m, f)]^{k+1}, \end{aligned} \tag{6.4}$$

where M_1 (> 0) is a constant. From (6.4) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

Case 2. $\delta(P_s, \theta) < 0$**.** From (1.4), (3.2), (3.3), (3.9), (3.10) and (6.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus H_1$, we obtain

$$1 \leq M_2 r_m^{2k} \exp\{(1-\varepsilon)\beta\delta(P_s,\theta)r_m^n\}\,[T(2r_m, f)]^{k+1}, \tag{6.5}$$

where M_2 (> 0) is a constant. From (6.5) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

(ii) Suppose that $P_j(z)$, $h_j(z)$ and $a_{n,j}$ $(j = 0, 1, \ldots, k-1)$ satisfy the hypotheses of Theorem 1.10. For any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_2 \cup H_3)$, we have

$$\delta(P_s, \theta) \neq 0,\ \delta(P_d, \theta) \neq 0 \text{ and } \delta(P_s, \theta) > \delta(P_d, \theta) \text{ or } \delta(P_s, \theta) < \delta(P_d, \theta),$$

where H_2 and H_3 are defined in the proof of Theorem 1.10. Set $\delta_1 = \delta(P_s, \theta)$ and $\delta_2 = \delta(P_d, \theta)$.

Case 1. $\delta_1 > \delta_2$**.** Here we also divide our proof in three subcases:

Subcase 1.1. $\delta_1 > \delta_2 > 0$. From (1.4), (3.2), (3.3), (4.1), (4.2) and (6.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_2 \cup H_3)$, we obtain

$$\exp\{(1-\varepsilon)\delta_1 r_m^n\} \leq M_3 r_m^{2s} \exp\{(1+\varepsilon)\delta_3 r_m^n\}\,[T(2r_m, f)]^{k+1}, \tag{6.6}$$

where $M_3(>0)$ is a constant. From (6.6) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

Subcase 1.2. $\delta_1 > 0 > \delta_2$. From (1.4), (3.2), (3.3), (4.1), (4.4) and (6.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_2 \cup H_3)$, we obtain

$$\exp\left\{(1-\varepsilon)\delta_1 r_m^n\right\} \leq M_4 r_m^{2s} \exp\left\{(1+\varepsilon)\gamma\delta_1 r_m^n\right\} [T(2r_m, f)]^{k+1}, \tag{6.7}$$

where M_4 (>0) is a constant. From (6.7) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

Subcase 1.3. $0 > \delta_1 > \delta_2$. From (1.4), (3.2), (3.3), (3.9), (4.6) and (6.3), for all z satisfying $|z| = r_m \in [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_2 \cup H_3)$, we obtain

$$1 \leq M_5 r_m^{2k} \exp\left\{(1-\varepsilon)\lambda\delta_1 r_m^n\right\} T(2r_m, f)^{k+1}, \tag{6.8}$$

where M_3 (>0) is a constant. From (6.8) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

Case 2. $\delta_1 < \delta_2$**.** Using the same reasoning as in Case 1, we can also obtain $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

(iii) Suppose that $P_j(z), h_j(z)$ and $a_{n,j}$ $(j = 0, 1, \ldots, k-1)$ satisfy the hypotheses of Theorem 1.11. For any given $\theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_4 \cup H_5)$, we have

$$\delta_t = \delta(P_{i_t}, \theta) > 0 \text{ or } \delta(P_{i_t}, \theta) < 0,$$

where H_4, H_5 and δ_t are defined in the proof of Theorem 1.11.

Case 1. $\delta_t > 0$.

Subcase 1.1. $\delta > 0$, where $\delta = \max\left\{\delta(P_j, \theta) : j \neq i_t\right\}$. From (1.4), (3.2), (3.3), (5.1), (5.2) and (6.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_4 \cup H_5)$, we obtain

$$\exp\{(1-\varepsilon)\delta_{i_t} r_m^n\} \leq M_6 r_m^{2i_t} \exp\left\{(1+\varepsilon)\alpha r_m^n\right\} [T(2r_m, f)]^{k+1}, \tag{6.9}$$

where $M_6(>0)$ is a constant. From (6.9) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

Subcase 1.2 $\delta < 0$. From (1.4) ,(3.2),(3.3), (5.1), (5.5) and (6.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_4 \cup H_5)$, we obtain

$$\exp\left\{(1-\varepsilon)\delta_{i_t} r_m^n\right\} \leq M_7 r_m^{2i_t} [T(2r_m, f)]^{k+1}, \tag{6.10}$$

where $M_7(>0)$ is a constant. From (6.10) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

Case 2. $\delta_{i_t} < 0$**.** From (1.4), (3.2), (3.3), (5.7) and (6.3), for all z satisfying $|z| = r_m \notin [0,1] \cup E_1 \cup E_5 \cup E_6 \cup E_9, r_m \to +\infty$ and $\arg z = \theta \in [\theta_r - \gamma, \theta_r + \gamma] \setminus (H_4 \cup H_5)$, we obtain

$$1 \leq M_8 r_m^{2k} \exp\left\{(1-\varepsilon)c\delta_{i_t} r_m^n\right\} [T(2r_m, f)]^{k+1}, \tag{6.11}$$

where M_8 (> 0) is a constant. From (6.11) and Lemma 2.8, we get $\sigma_2(f) \geq n$. This and the fact that $\sigma_2(f) \leq n$ yield $\sigma_2(f) = n$.

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