Zbigniew Wesołowski[1]

[1] Institute of Computer and Information Systems Faculty of Cybernetic,
Military University of Technology, Kaliskiego Str. 2, 00-908 Warsaw, Poland

Estimating Reliability Characteristics for Homogeneous and Heterogeneous Systems

Abstract. The paper presents a method for estimating reliability characteristics dedicated for two important kinds of systems, i.e. homogeneous systems and heterogeneous systems. We propose an algorithm for evaluating these characteristics on the basis of observations. The paper gives examples of applications of the method for evaluating the reliability of a distributed system.

Keywords: reliability of systems, reliability characteristics, stochastic simulation

1. Introduction

Reliability theory is a field of research focused at understanding the important factors that have significant influence on the reliability of systems and other real existing structures [4, 5, 9]. System reliability is interpreted as its ability for the performance of tasks in defined terms and in determined time intervals.

One of the important issues of reliability theory and engineering is defining the reliability characteristics of systems [3, 4, 5, 7, 9, 10]. They are used not only for the evaluation of the reliability of existing systems, but are also applied for estimating the reliability of designed systems [4, 5, 7]. Since the issue of the estimation of the reliability characteristics of systems on the basis of observations is often difficult, in order to facilitate the solution of this issue it was proposed to introduce two classes of systems. They are homogeneous systems and heterogeneous systems [7, 15]. It is said that the system is homogeneous, if the random variables have the same probability distribution. If distributions of at least two of these random variables are different, it is said that the system is heterogeneous. Introducing new classes of systems is, above all, caused by problems of examining, whether stochastic processes are stationary or not. Examining the stationarity in the strict sense of the stochastic process requires examination of whether the multidimensional probability density function of this process is constant in time. As, so far, reliable methods of such examination have not been developed; one should recognize that statistical inference of stationarity in the strict sense of the stochastic process on the basis of observations of its value is not possible.

Examining the stationarity in the wide-sense of the stochastic process requires the inspection of the consistency in the time of the expected value and checking the dependence of the autocovariance function only on the difference of moments, rather than from situating these moments on the axis of the time. Due to the fact that so far, reliable tests for examining many autocovariance coefficients have not been proposed, one should recognize that statistical inference about stationarity in the wide sense of the stochastic process on the basis of observations of its value is difficult.

In the paper reliability characteristics both for homogeneous systems and heterogeneous systems have been described. The estimation of these characteristics is relatively easy. This can be accomplished using classical methods of mathematical statistics and the classical mean square approximation method.

Let us use the following notation: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}_+ = (0, \infty) \subset \mathbb{R}$, $\mathbb{R}_{[0,1]} = [0,1] \subset \mathbb{R}$, $\mathbb{N}$ - the set of natural numbers, $T = [0, \infty) \subset \mathbb{R}$ - a set of time parameters, $W(\alpha, \beta)$ - the Weibull distribution with the parameters α ($\alpha > 0$) and β ($\beta > 0$) [12], $\mathbf{y} \in \mathbb{R}^{m\times 1}$ - a column vector, $\mathbf{y} = \mathbf{vec}(\mathbf{y}_1, \dots, \mathbf{y}_n) \in \mathbb{R}^{nm\times 1}$, $\mathbf{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{nm\times 1}$ - the vectorization operator, where: $\mathbf{y}_i \in \mathbb{R}^{m\times 1}$, $i = 1, \dots, n$. Let $\Delta = (\Omega, \Xi, \mathrm{P})$ be a probability space on which all random variables are defined.

2. Model of the System Use Process

Let us assume that the system is bi-state in terms of reliability [4, 5, 7, 9, 10]. Let $B = \{0,1\}$ be a set of reliability states of the system, where: digit one means the working state (or the up-state), and digit zero means the failure state (or the down-state) of the system. The model of the system use process defines the ordered tuple,

$$M = (R, \{D_b; b \in B\}), \qquad (1)$$

where: R is a model of the time evolution of reliability states of the system, D_b is a model of changes of time intervals length of the system staying in the state $b \in B$.
Model R (1) defines the ordered tuple,

$$R = (T, B, \Delta, \mathscr{b}),$$

where $\mathscr{b}: \Omega \times T \to B$ is a stochastic process being a model of the process of the system reliability state changes over time.
Model D_b (1) defines the ordered triple,

$$D_b = \left(\mathbb{R}_+, \Delta, \left\{\mathscr{y}_b^{(n)}; n \in \mathbb{N}\right\}\right), \qquad b \in B, \qquad (2)$$

where: $\mathscr{y}_b^{(n)}: \Omega \to \mathbb{R}_+$ is a random variable being a stochastic model of the process of changes of time intervals length of the system staying, for the n, $n \in \mathbb{N}$, times, in the state $b \in B$.

3. Reliability Examinations of the System

The aim of the reliability examinations of the system is gathering statistical samples of lengths of time intervals of the system staying in reliability states from the set B. As the research of the reliability of systems in real terms of using them is difficult and frequently impossible [7, 14], statistical samples are usually collected on the basis of data generated by computer programs that simulate the use process of the studied system [7, 9, 14].
Simulation reliability examinations are carried out according to the plan,

$$(m_Q, m_N), \qquad (3)$$

which means that the examination will be repeated m_Q, $m_Q \in \mathbb{N}$, times; the examinations are terminated at the end of the duration, occurring after m_N, $m_N \in \mathbb{N}$, times, the failure state of the system. Let $Q = \{1, \dots, m_Q\}$ be a set of numbers of the simulation experiments. Let $N = \{1, \dots, m_N\}$ be a set of numbers of appearances of states from the set B.

A random vector

$$\boldsymbol{\mathcal{y}}_b = \left(\mathcal{y}_b^{(1)}, \mathcal{y}_b^{(2)}, \dots, \mathcal{y}_b^{(m_N)}\right)^T, \qquad (4)$$

is called the stochastic model of the process of changes time intervals length of the system staying in subsequent appearances of the state $b \in B$, where $\mathcal{y}_b^{(n)}$ (2), for $n \in N$.

Let $\mathbf{Y}_b = \left(\mathbf{y}_b^{(1)}, \mathbf{y}_b^{(2)}, \dots, \mathbf{y}_b^{(m_N)}\right) \in \mathbb{R}_+^{m_Q \times m_N}$ be a sample being the result of the simulation reliability examination of the studied system [14], where $\mathbf{y}_b^{(n)} = \left(y_b^{(n)}(1), \dots, y_b^{(n)}(m_Q)\right)^T$ is a vector which consists of realizations of the random variable $\mathcal{y}_b^{(n)} \in \boldsymbol{\mathcal{y}}_b$, i.e. $\mathcal{y}_b^{(n)}(\omega_q) = y_b^{(n)}(q)$, for: $q \in Q, n \in N, b \in B$.

Definition 1. Let $\mathrm{F}_b^{(n)}(t) = \mathrm{P}\left(\mathcal{y}_b^{(n)} \le t\right)$ be a distribution function of the random variable $\mathcal{y}_b^{(n)} \in \boldsymbol{\mathcal{y}}_b$, $n \in N$. It is said that the random vector $\boldsymbol{\mathcal{y}}_b = \left(\mathcal{y}_b^{(1)}, \dots, \mathcal{y}_b^{(m_N)}\right)^T$ is homogeneous if $\mathrm{F}_b^{(1)}(t) = \mathrm{F}_b^{(2)}(t) = \dots = \mathrm{F}_b^{(m_N)}(t)$. If $\mathrm{F}_b^{(i)}(t) \ne \mathrm{F}_b^{(j)}(t)$ for at least one pair $(i, j) \in N \times N$; $i \ne j$; $i, j = 1, \dots, m_N$, it is said that the random vector $\boldsymbol{\mathcal{y}}_b$ is heterogeneous.

□

Let $\mathbf{y}_b = \mathbf{vec}\left(\mathbf{y}_b^{(1)}, \dots, \mathbf{y}_b^{(m_N)}\right) = \left(y_b(1), y_b(2), \dots, y_b(m_M)\right) \in \mathbb{R}^{m_M \times 1}$, $m_M = m_Q m_N$, be a vector compound of observations from the samples $\mathbf{y}_b^{(1)}, \dots, \mathbf{y}_b^{(m_N)}$. Let $M = \{1, \dots, m_M\}$ be a set of numbers of components of the sample $\mathbf{y}_b$.

If the random vector $\boldsymbol{\mathcal{y}}_b = \left(\mathcal{y}_b^{(1)}, \mathcal{y}_b^{(2)}, \dots, \mathcal{y}_b^{(m_N)}\right)^T$ is homogeneous, its components $\mathcal{y}_b^{(1)}, \mathcal{y}_b^{(2)}, \dots, \mathcal{y}_b^{(m_N)}$ may be treated as independent copies of the same random variable,

$$\mathcal{y}_b : \Omega \to \mathbb{R}_+. \qquad (5)$$

In this case, the probabilistic space induced by the random vector $\boldsymbol{\mathcal{y}}_b = \left(\mathcal{y}_b^{(1)}, \dots, \mathcal{y}_b^{(m_N)}\right)^T$ is labeled $\Delta^{m_N} = (\Omega, \Xi, \mathrm{P})^{m_N}$.

A random vector $\boldsymbol{\mathcal{y}} = \mathbf{vec}(\boldsymbol{\mathcal{y}}_0, \boldsymbol{\mathcal{y}}_1)$ is called the stochastic model of the use process of the system.

Definition 2. It is said that the model $\boldsymbol{\mathcal{y}} = \mathbf{vec}(\boldsymbol{\mathcal{y}}_0, \boldsymbol{\mathcal{y}}_1)$ is homogeneous if its components $\boldsymbol{\mathcal{y}}_0$ and $\boldsymbol{\mathcal{y}}_1$ are homogeneous. If at least one of these components is heterogeneous, it is said that the model $\boldsymbol{\mathcal{y}}$ is heterogeneous.

□

Definition 3. It is said that the system is homogeneous if the stochastic model of its use process, i.e. the model $\boldsymbol{\mathcal{y}}$, is homogeneous. Otherwise, it is said that the system is heterogeneous.

□

4. Reliability Characteristics of Systems

In reliability theory different reliability characteristics are applicable depending on whether it is possible to acknowledge the system as homogeneous or whether the system is considered as heterogeneous.

4.1. Reliability Characteristics of Homogeneous Systems

Let us consider a homogeneous system. A stochastic model of the use process of such a system is the random vector $\boldsymbol{\mathcal{Y}} = \mathbf{vec}(\mathcal{Y}_0, \mathcal{Y}_1)$, where $\mathcal{Y}_b$ (5), for $b \in B$. Let $\mathrm{F}_b(t) = \mathrm{P}(\mathcal{Y}_b \leq t)$ be a distribution function and let $\mathrm{f}_b(t) = d\mathrm{F}_b(t)/dt$ be a probability density function of the random variable $\mathcal{Y}_b \in \boldsymbol{\mathcal{Y}}$.

For homogeneous systems the most widely used reliability characteristics are:

- the expected value of the working time of the system;
- the variance of the working time of the system;
- the reliability function of the system;
- the hazard rate function of the system;
- the expected value of the failure time of the system;
- the variance of the failure time of the system;
- the failure function of the system;
- the restoration intensity function of the system;
- the coefficient of the system availability.

The expected value of the working time of the system. An expected value $\mu_1 = \mathrm{E}(\mathcal{Y}_1)$ of the random variable $\mathcal{Y}_1$ is called the expected value of the working time of the system. This reliability characteristic is the average length of the time interval of the system staying in the working state.

The variance of the working time of the system. A variance $\sigma_1^2 = \mathrm{E}[(\mathcal{Y}_1 - \mu_1)^2]$ of the random variable $\mathcal{Y}_1$ is called the variance of the working time of the system. This reliability characteristic is the observed mean square deviation of the working time of the system from its expected value. The variance σ_1^2 is a measure of the volatility of the time interval length to maintain the system in the working state.

The reliability function of the system. The survival function $\mathrm{R}_1(t) = 1 - \mathrm{F}_1(t)$ of the random variable $\mathcal{Y}_1$ is called the reliability function of the system. This function expresses the probability of the system staying in the working state at least up to the moment *t*.

The hazard rate function of the system. The function $\lambda_1(t) = \mathrm{f}_1(t)/\mathrm{R}_1(t)$ is called the hazard rate function of the system. This function expresses the conditional density of the system being damaged at the moment *t* provided that up to this moment the system has remained in the working state.

The expected value of the failure time of the system. An expected value $\mu_0 = \mathrm{E}(\mathcal{Y}_0)$ of the random variable $\mathcal{Y}_0$ is called the expected value of the failure time of the system. This reliability characteristic is the average length of the time interval of the system staying in the failure state.

The variance of the failure time of the system. A variance $\sigma_0^2 = \mathrm{E}[(\mathcal{Y}_0 - \mu_0)^2]$ of the random variable $\mathcal{Y}_0$ is called the variance of the failure time of the system. This reliability characteristic is the observed mean square deviation of the failure time of the system from its expected value. The variance σ_0^2 is a measure of the volatility of the time interval length to maintain the system in the state of failure.

The failure function of the system. The survival function $\mathrm{R}_0(t) = 1 - \mathrm{F}_0(t)$ of the random variable y_0 is called the failure function of the system. This function expresses the probability of staying the system in the failure state at least up to the moment *t*.

The restoration intensity function of the system. A function $\lambda_0(t) = \mathrm{f}_0(t)/\mathrm{R}_0(t)$ is called the restoration intensity function of the system. This function express the conditional density of the system being renewed at the moment *t* provided that up to this moment the system has remained in the failure state.

The coefficient of the system availability. A random variable $\kappa = \mu_1/(\mu_0 + \mu_1)$ is called the coefficient of the system availability. This coefficient expresses the probability of the system staying in the working state at the start of its use.

It is worth noting that if the random variables y_b have the distributions $W(\alpha_b, \beta_b)$, $b \in B$, then the reliability characteristics of the system take, for *t*>0, the following forms,

$$\mathrm{R}_b(t) = \mathrm{e}^{-\left(\frac{t}{\beta_b}\right)^{\alpha_b}}, \tag{6}$$

$$\lambda_b(t) = \frac{\alpha_b \left(\frac{t}{\beta_b}\right)^{\alpha_b}}{t}, \tag{7}$$

$$\mu_b = \beta_b \Gamma\left(1 + \frac{1}{\alpha_b}\right), \tag{8}$$

$$\sigma_b^2 = \beta_b^2 \left[\Gamma\left(\frac{2 + \alpha_b}{\alpha_b}\right) - \Gamma\left(1 + \frac{1}{\alpha_b}\right)^2\right], \tag{9}$$

$$\kappa = \frac{\mu_1}{\mu_0 + \mu_1}. \tag{10}$$

4.2. Reliability Characteristics of Heterogeneous Systems

Let us consider a heterogeneous system. A stochastic model of the use process of such a system is the random vector $\boldsymbol{y} = \mathbf{vec}(\boldsymbol{y}_0, \boldsymbol{y}_1)$, where $\boldsymbol{y}_b$ (4), $b \in B$.

For heterogeneous systems, the most widely used reliability functions are:

- the trend of the expected value of the working time of the system;
- the trend of the variance of the working time of the system;
- the trend of the expected value of the failure time of the system;
- the trend of the variance of the failure time of the system;
- the trend of the coefficient of the system availability.

The trend of the expected value of the working time of the system. The trend $\phi_1 : \mathbb{N} \to \mathbb{R}_+$ of the sequence $\left(\mu_1^{(n)}\right)_{n\in\mathbb{N}}$ of expected values $\mu_1^{(n)} = \mathrm{E}\left[y_1^{(n)}\right]$ of random variables $y_1^{(n)}$, $n \in \mathbb{N}$, is called the trend of the expected value of the working time of the system. An instantaneous value $\phi_{1_n} \in \mathbb{R}_+$ of this trend expresses the evaluation of the expected value of the system for the n, $n \in \mathbb{N}$, times staying in the working state.

The trend of the variance of the working time of the system. The trend $\varphi_1 : \mathbb{N} \to \mathbb{R}_+$ of the sequence $\left(\sigma_1^{(n)^2}\right)_{n\in\mathbb{N}}$ of variances $\sigma_1^{(n)^2} = \mathrm{E}\left[\left(y_1^{(n)} - \mu_1^{(n)}\right)^2\right]$

of random variables $\mathcal{y}_1^{(n)}$, $n \in \mathbb{N}$, is called the trend of the variance of the working time of the system. An instantaneous value $\varphi_{1_n} \in \mathbb{R}_+$ of this trend expresses the evaluation of the variance of the system for the n, $n \in \mathbb{N}$, times staying in the working state.

The trend of the expected value of the failure time of the system. The trend $\phi_0 \colon \mathbb{N} \to \mathbb{R}_+$ of the sequence $\left(\mu_0^{(n)}\right)_{n\in\mathbb{N}}$ of expected values $\mu_0^{(n)} = \mathrm{E}\left(\mathcal{y}_0^{(n)}\right)$ of random variables $\mathcal{y}_0^{(n)}$, $n \in \mathbb{N}$, is called the trend of the expected value of the failure time of the system. An instantaneous value $\phi_{0_n} \in \mathbb{R}_+$ of this trend expresses the evaluation of the expected value of the system for the n, $n \in \mathbb{N}$, times staying in the failure state.

The trend of the variance of the failure time of the system. The trend $\varphi_0 \colon \mathbb{N} \to \mathbb{R}_+$ of the sequence $\left(\sigma_0^{(n)^2}\right)_{n\in\mathbb{N}}$ of variances $\sigma_0^{(n)^2} = \mathrm{E}\left[\left(\mathcal{y}_0^{(n)} - \mu_0^{(n)}\right)^2\right]$ of random variables $\mathcal{y}_0^{(n)}$, $n \in \mathbb{N}$, is called the trend of the variance of the failure time of the system. An instantaneous value $\varphi_{0_n} \in \mathbb{R}_+$ of this trend expresses the evaluation of the variance of the system for the n, $n \in \mathbb{N}$, times staying in the failure state.

The trend of the coefficient of the system availability. The trend $\omega : \mathbb{N} \to \mathbb{R}_{[0,1]}$ of the sequence $(\kappa_n)_{n\in\mathbb{N}}$, $\kappa_n = \mu_1^{(n)}/\left(\mu_0^{(n)} + \mu_1^{(n)}\right)$, for $n \in \mathbb{N}$, is called the trend of the coefficient of the system availability. An instantaneous value $\omega_n \in \mathbb{R}_{[0,1]}$ of this trend expresses the evaluation of the coefficient of availability of the system upon completion of it appearing for the n, $n \in \mathbb{N}$, times the state of its failure, assuming that the system at the beginning of its use was in the working state.

4.3. An Algorithm for Reliability Characteristics Evaluation

The following algorithm determines the procedure for evaluating the reliability characteristics of homogeneous and heterogeneous systems based on the observations $\mathbf{Y}_b$, $b \in B$.

Algorithm 1.

(1) examine the randomness of the samples $\mathbf{y}_b^{(n)} \in \mathbf{Y}_b$, $\mathbf{Y}_b$, $n \in N$, $b \in B$, using the Wald-Wolfowitz test [13]. If there are no reasons for rejecting null hypotheses stating the randomness of these samples, then go to step (2), otherwise terminate the algorithm;

(2) examine the homogeneity of the distributions of the random variables $\mathcal{y}_b^{(n)} \in \boldsymbol{\mathcal{y}}_b$, $n \in N$, with the help of the Kruskal-Wallis test [11, 13] based on the samples $\mathbf{y}_b^{(n)} \in \mathbf{Y}_b$, $\mathbf{Y}_b$, $n \in N$. If there are no reasons for rejecting null hypotheses stating the homogeneity of these random variables, then go to step (3), otherwise go to step (5), where $b \in B$;

(3) create a sample $\mathbf{y}_b = \mathbf{vec}\left(\mathbf{y}_b^{(1)}, \mathbf{y}_b^{(2)}, \dots, \mathbf{y}_b^{(m_N)}\right) \in \mathbb{R}_+^{m_Q m_N \times 1}$. Let us assume that this sample is compound of the realizations of the random variable $\mathcal{y}_b$ (5) about the distribution $W(\alpha_b, \beta_b)$. Let $W\left(\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)\right)$ be a the distribution estimated by the maximum likelihood method on the basis of sample $\mathbf{y}_b$. Conduct an examination of the goodness-of-fit of the observed

distribution of the sample $\mathbf{y}_b$ with the distribution $W\left(\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)\right)$ using the Pearson χ^2 test [8, 11, 13]. If there are no reasons for rejecting null hypotheses stating the goodness-of-fit of these distributions, for $b \in B$, then go to point (4), otherwise terminate the algorithm.

(4) based on the samples $\mathbf{y}_b$, $b \in B$, to evaluate of the system reliability characteristics determined by formulae (6)-(10). Terminate the algorithm.

(5) let us assume that the sample $\mathbf{y}_b^{(n)} \in \mathbf{Y}_b$ is compound of the realizations of the random variable $\mathcal{y}_b^{(n)} \in \boldsymbol{\mathcal{y}}_b$, $\boldsymbol{\mathcal{y}}_b$ (4), having the distribution $W\left(\alpha_b^{(n)}, \beta_b^{(n)}\right)$. Let $W\left(\hat{\alpha}_b^{(n)}\left(\mathbf{y}_b^{(n)}\right), \hat{\beta}_b^{(n)}\left(\mathbf{y}_b^{(n)}\right)\right)$ be a distribution estimated by the maximum likelihood method on the basis of the sample $\mathbf{y}_b^{(n)}$. Conduct an examination of the goodness-of-fit of the observed distribution of the sample $\mathbf{y}_b^{(n)}$ with the distribution $W\left(\hat{\alpha}_b^{(n)}\left(\mathbf{y}_b^{(n)}\right), \hat{\beta}_b^{(n)}\left(\mathbf{y}_b^{(n)}\right)\right)$ using the Pearson χ^2 test. If there are no reasons for rejecting null hypotheses stating the goodness-of-fit of these distributions, for: $n \in N$, $b \in B$, then go to point (6), otherwise terminate the algorithm.

(6) based on the samples $\mathbf{y}_b^{(n)}$, $n \in N$, $b \in B$, to evaluate the system reliability characteristics enumerated in section 4.2; the approximation of the trends ϕ_b, φ_b, $b \in B$, and ω, carry out with the least squares method [1, 2, 6, 8, 11, 13]. □

Example 1. Let us consider the issue of the estimation of the reliability characteristics of a distributed system with the client-server architecture, the technical structure of which is shown in fig. 1a. The elements of this structure are: clients h_1, h_2, the server h_3, network routers q_1, q_2 and one-way network links $c_1, \ldots, c_8$.

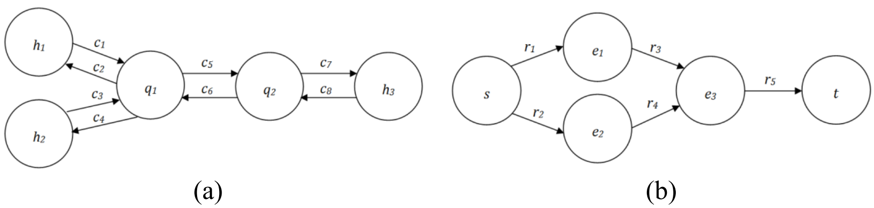


Figure 1. Schemes of the distributed system structures: (a) a block diagram of the technical structure; (b) a block diagram of the reliability structure

Let us suppose that the clients h_1, h_2 and the server h_3 are repairable with non-zero repair time, the network routers q_1, q_2 and the network links $c_1, \ldots, c_8$ are reliable. We consider the system to be able if the server and at least one of the clients are able. A model of the reliability structure [15] of the considered system is a graph (fig. 1b) in the form,

$$G = (V, R, \mathbf{A}),$$

where:

- $V = \{s, E, t\}$ is the set of vertices of the graph, $E = \{e_1, \ldots, e_{m_E}\}$ is the set of vertices, called operational elements of the system reliability structure (or briefly: elements), where: $m_E = 3$, $e_k \equiv h_k$, for $k = 1, \ldots, m_E$,
- $R = \{r_1, \ldots, r_{m_R}\}$ is the set of edges of the graph, where: $m_R = 5$, $r_1 = (s, e_1)$, $r_2 = (s, e_2)$, $r_3 = (e_1, e_3)$, $r_4 = (e_2, e_3)$, $r_5 = (e_3, t)$;
- $\mathbf{A} = \mathrm{A}^s \times \mathrm{A}_1 \times \mathrm{A}_2 \times \ldots \times \mathrm{A}_{m_E} \times \mathrm{A}^t$ is the set of reliability states of all elements from the set V, where: $\mathrm{A}^s = \{1\}$ is the set of reliability states of the source s, $\mathrm{A}^t = \{1\}$ is the set of reliability states of the destination t, $\mathrm{A}_k = \{0,1\}$ is the set of reliability states of the element $e_k \in E$, for $k = 1, \ldots, m_E$, where: digit 1 means the working state, and digit 0 means the failure state of any vertex from the set V.

Let $\mathbf{Y}_b = \left(\mathbf{y}_b^{(1)}, \mathbf{y}_b^{(2)}, \ldots, \mathbf{y}_b^{(m_N)}\right) \in \mathbb{R}_+^{m_Q \times m_N}$, $b \in B$, be the results of the simulation reliability examinations [7, 14, 15] of the distributed system made under the plan (3), where: $m_Q = 242$, $m_N = 20$. Components of the sample $\mathbf{y}_b^{(n)} \in \mathbf{Y}_b$ are interpreted as realizations of the random variable $\mathcal{y}_b^{(n)} \in \boldsymbol{\mathcal{y}}_b$, $\boldsymbol{\mathcal{y}}_b$ (4), for $n \in N$.

An evaluation of the reliability characteristics of the studied system will be conducted according to algorithm 1.

An examination of the randomness of the samples. For examining the randomness of the samples $\mathbf{y}_b^{(n)}$, $n \in N$, $b \in B$, the Wald-Wolfowitz test was applied. The results of this test indicate that, at the level of significance α=0.05, there were no reasons to reject null hypotheses stating the randomness of these samples.

An examination of the homogeneity of distributions of several random variables. Let us consider the issue of examining the homogeneity of distributions of the random variables $\mathcal{y}_b^{(n)} \in \boldsymbol{\mathcal{y}}_b$, $n \in N$. This issue can be formulated as the hypotheses verification problem of the form,

$$H_0\colon \mathrm{F}_b^{(1)}(t) = \mathrm{F}_b^{(2)}(t) = \cdots = \mathrm{F}_b^{(m_N)}(t), \tag{11}$$

$$H_1\colon \mathrm{F}_b^{(i)}(t) \neq \mathrm{F}_b^{(j)}(t), \textit{for at least one pair } (i,j);\, i \neq j;\, i,j = 1, \ldots, m_N. \tag{12}$$

For solving the problem (11)-(12) it is possible to apply the Kruskal-Wallis test. Table 1 shows the results of this test, where: $\hat{\tau}(\mathbf{Y}_b)$ is the evaluation of the test statistics from the sample $\mathbf{Y}_b$, $\hat{\alpha}^*[\hat{\tau}(\mathbf{Y}_b)]$ is the evaluation of the p-value α^* for the fixed evaluation of the test statistics, for $b \in B$.

Table 1. The results of the Kruskal-Wallis test

$b \in B$	$\hat{\tau}(\mathbf{Y}_b)$	$\hat{\alpha}^*[\hat{\tau}(\mathbf{Y}_b)]$
0	263.224	0.14975
1	211.788	0.918175

From table 1 it results that, at the level of significance α=0.05, there is no reason to reject the hypothesis H_0 (6), for samples $\mathbf{Y}_b$, $b \in B$. This means that the system can be regarded as homogeneous.

An examination of the goodness-of-fit of observed distributions with theoretical distributions. From table 1 it follows that the components of the sample $\mathbf{y}_b = \mathbf{vec}\left(\mathbf{y}_b^{(1)}, \mathbf{y}_b^{(2)}, \dots, \mathbf{y}_b^{(m_N)}\right) \in \mathbb{R}_+^{m_M \times 1}$, $m_M = m_Q m_N$, can be interpreted as realizations of the random variable $\mathcal{Y}_b$ (5). Let us suppose that $\mathcal{Y}_b \sim W(\alpha_b, \beta_b)$. It implies that the reliability characteristics of the considered system are expressed by formulas (6)-(10).

Table 2 shows the evaluations of the parameters of distributions $W\left(\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)\right)$ from the samples $\mathbf{y}_b$, $b \in B$.

Table 2. The evaluations of the parameters of distributions $W\left(\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)\right)$

$b \in B$	$\hat{\alpha}_b(\mathbf{y}_b)$	$\hat{\beta}_b(\mathbf{y}_b)$
0	0.96019	9.53265
1	1.0221	453.857

Let us consider the issue of examining the goodness-of-fit of observed distributions of samples $\mathbf{y}_b$ with distributions $W\left(\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)\right)$, for $b \in B$. This issue can be formulated as the hypotheses verification problem of the form,

$$H_0\colon \tilde{\mathrm{F}}_b(t;\mathbf{y}_b) = \hat{\mathrm{F}}_b(t;\mathbf{y}_b), \textit{for all } t \in T, \tag{13}$$
$$H_1\colon \tilde{\mathrm{F}}_b(t;\mathbf{y}_b) \neq \hat{\mathrm{F}}_b(t;\mathbf{y}_b), \textit{for at least one } t \in T, \tag{14}$$

where: $\tilde{\mathrm{F}}_b(t;\mathbf{y}_b)$ is a distribution function of observed distribution of the sample $\mathbf{y}_b$, $\hat{\mathrm{F}}_b(t;\mathbf{y}_b)$ is a distribution function of the distribution $W\left(\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)\right)$.
For solving the problem (13)-(14) the Pearson χ^2 test was applied. Table 3 shows the results of this test, where: $\hat{\tau}(\mathbf{y}_b)$ is the evaluation of the test statistics from the sample $\mathbf{y}_b$, for $b \in B$.

Table 3. The results of the Pearson χ^2 test

$b \in B$	$\hat{\tau}(\mathbf{y}_b)$	$\hat{\alpha}^*[\hat{\tau}(\mathbf{y}_b)]$
0	55.876	0.591392
1	48.7603	0.826674

Table 3 shows that, at the level of significance α=0.05, there is no reason to reject the hypotheses H_0 (13), for samples $\mathbf{y}_b$, $b \in B$.

Evaluating the distributed system reliability characteristics. Table 4 shows the evaluations $\hat{\mu}_b(\mathbf{y}_b)$, $\hat{\sigma}_b^2(\mathbf{y}_b)$ and $\hat{\kappa}(\mathbf{y}_b)$ of the reliability coefficients, appropriately, μ_b (8), σ_b^2 (9) and κ (10), evaluated on the basis of the parameters of distributions $W\left(\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)\right)$ given in table 2.

Table 4. The evaluations of the reliability coefficients (10)-(12) of the system

$b \in B$	$\hat{\mu}_b(\mathbf{y}_b)$	$\hat{\sigma}_b^2(\mathbf{y}_b)$	$\hat{\kappa}(\mathbf{y}_b)$
0	9.70655	102.242	0.978876
1	449.788	193669.0	

Figure 2 shows plots of the evaluations $\hat{\mathrm{R}}_b(t;\mathbf{y}_b)$ and $\hat{\lambda}_b(t;\mathbf{y}_b)$ of the reliability functions, appropriately, $\mathrm{R}_b(t)$ (6) and $\lambda_b(t)$ (7), with the parameters $\hat{\alpha}_b(\mathbf{y}_b), \hat{\beta}_b(\mathbf{y}_b)$, for $b \in B$, given in table 2.

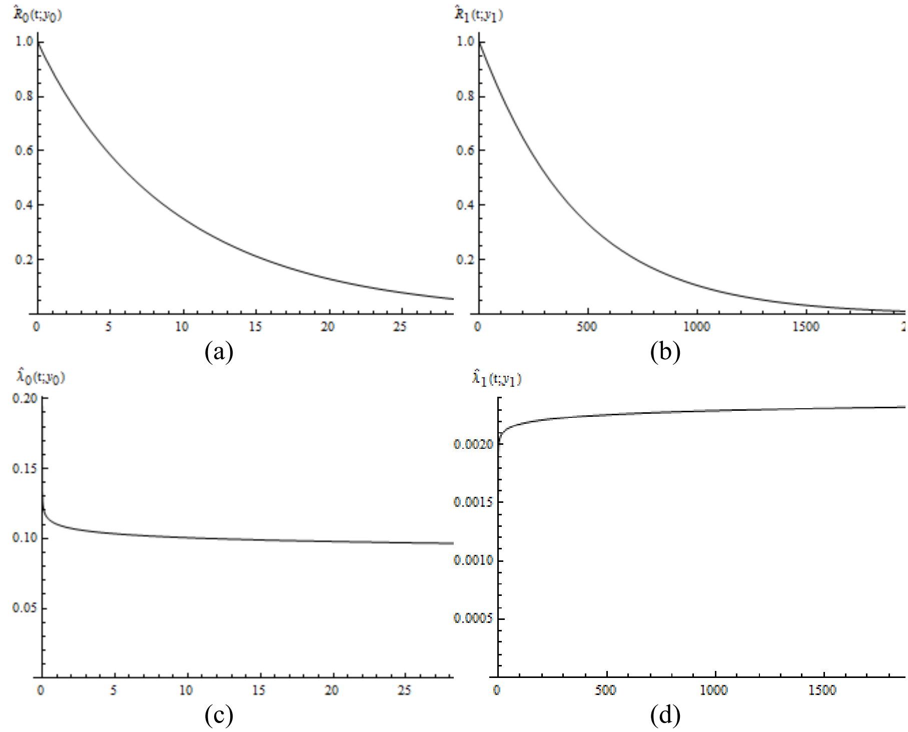


Figure 2. Plots of the evaluations of the distributed system reliability functions

□

Example 2. Let us consider the issue of the estimation of the reliability characteristics of the distributed system from example 1. Let $\mathbf{Y}_b = \left(\mathbf{y}_b^{(1)}, \mathbf{y}_b^{(2)}, \dots, \mathbf{y}_b^{(m_N)}\right) \in \mathbb{R}_+^{m_Q \times m_N}$, $b \in B$, be the results of the simulation reliability examinations of the system made under the plan (3), where: $m_Q = 546$, $m_N = 200$. Components of the sample $\mathbf{y}_b^{(n)} \in \mathbf{Y}_b$ are interpreted as realizations of the random variable $\mathcal{y}_b^{(n)} \in \boldsymbol{\mathcal{y}}_b$ (4), for $n \in N$.

An evaluation of the reliability characteristics of the studied system will be conducted according to algorithm 1.

An examination of the randomness of the samples. For examining the randomness of the samples $\mathbf{y}_b^{(n)}$, $n \in N$, $b \in B$, the Wald-Wolfowitz test was applied. The results of this test indicate that, at the level of significance α=0.05, there were no reasons to reject null hypotheses stating the randomness of these samples.

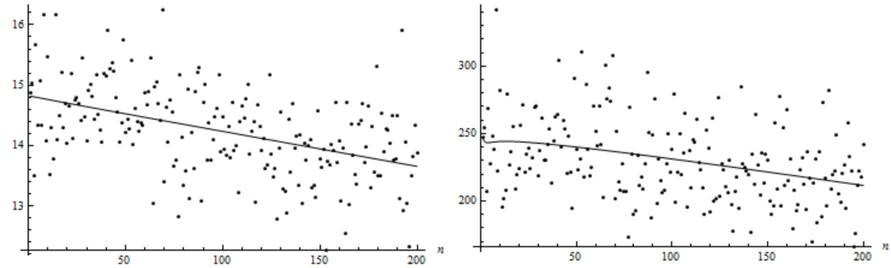
An examination of the homogeneity of distributions of several random variables. Table 5 shows the results of the Kruskal-Wallis test used for examining the homogeneity of the random variables $y_b^{(n)} \in \boldsymbol{y}_b$, $n \in N$, where: $\hat{\tau}(\mathbf{Y}_b)$ is the evaluation of the test statistics from the sample $\mathbf{Y}_b$, for $b \in B$.

Table 5. The results of the Kruskal-Wallis test

$b \in B$	$\hat{\tau}(\mathbf{Y}_b)$	$\hat{\alpha}^*[\hat{\tau}(\mathbf{Y}_b)]$
0	256.497	0.0036921
1	263.636	0.00144465

From table 5 it results that, at the level of significance α=0.05, one should reject the null hypothesis which states the homogeneity of random vectors $\boldsymbol{y}_b$, for $b \in B$. This means that the considered system can be regarded as heterogeneous.

Figures 3-5 show plots of the evaluations $\widehat{\phi}_b$, $\widehat{\varphi}_b$, for $b \in B$, and $\widehat{\omega}$, of the functions, appropriately, ϕ_b, φ_b, for $b \in B$, and ω, enumerated in section 4.2.



(a) (b)

Figure 3. Plots of the evaluations of the distributed system reliability functions: (a) the evaluations $\hat{\mu}_0^{(n)}\left(\mathbf{y}_0^{(n)}\right)$ (*points*) from the samples $\mathbf{y}_0^{(n)}$ of the expected values $\mu_0^{(n)}$ of the random variables $y_0^{(n)} \in \boldsymbol{y}_0$, for $n \in N$, and a plot of the evaluation $\widehat{\phi}_0$ (*the solid line*) of the function ϕ_0; (b) the evaluations $\hat{\sigma}_0^{(n)^2}\left(\mathbf{y}_0^{(n)}\right)$ (*points*) from the samples $\mathbf{y}_0^{(n)}$ of variances $\sigma_0^{(n)^2}$ of the random variables $y_0^{(n)} \in \boldsymbol{y}_0$, for $n \in N$, and a plot of the evaluation $\widehat{\varphi}_0$ (*the solid line*) of the function φ_0

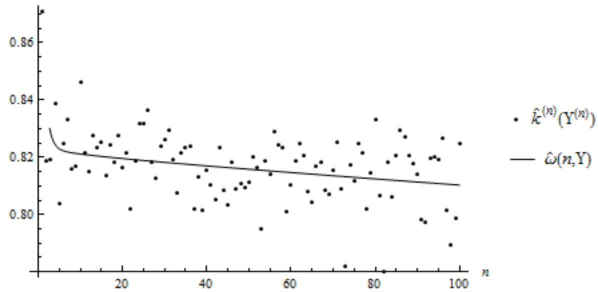
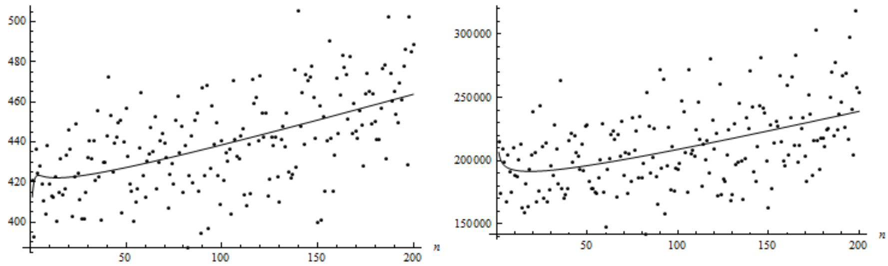


Figure 4. The evaluations $\hat{\kappa}^{(n)}\left(\mathbf{Y}^{(n)}\right)$ (*points*) of instantaneous availability coefficient of the distributed system $\kappa^{(n)}$ from the sample $\mathbf{Y}^{(n)} = \left(\mathbf{y}_0^{(n)}, \mathbf{y}_1^{(n)}\right)$, for $n \in N$, and a plot of the evaluation $\widehat{\omega}$ (*the solid line*) of the reliability function ω



(a) (b)

Figure 5. Plots of the evaluations of the distributed system reliability functions: (a) the evaluations $\hat{\mu}_1^{(n)}\left(\mathbf{y}_1^{(n)}\right)$ (*points*) from the samples $\mathbf{y}_1^{(n)}$ of the expected values $\mu_1^{(n)}$ of the random variables $\mathcal{Y}_1^{(n)} \in \boldsymbol{\mathcal{Y}}_1$, for $n \in N$, and a plot of the evaluation $\widehat{\phi}_1$ (*the solid line*) of the function ϕ_1; (b) the evaluations $\hat{\sigma}_1^{(n)^2}\left(\mathbf{y}_1^{(n)}\right)$ (*points*) from the samples $\mathbf{y}_1^{(n)}$ of variances $\sigma_1^{(n)^2}$ of the random variables $\mathcal{Y}_1^{(n)} \in \boldsymbol{\mathcal{Y}}_1$, for $n \in N$, and a plot of the evaluation $\widehat{\varphi}_1$ (*the solid line*) of the function φ_1

5. Summary

This paper proposes a relatively easy method for estimating the reliability characteristics of homogeneous systems and heterogeneous systems. The method uses a variety of classical methods of mathematical statistics, including: the method for examining the randomness of the samples, the method for examining the homogeneity of several distribution functions, and the method for examining the goodness-of-fit of observed distributions with theoretical distributions.
Directions for further work should be related to the estimation of the reliability characteristics of heterogeneous systems based on the results of stochastic simulation of the use process of that kind of systems.

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