Aleksey S. VICTOROV

Russian Academy of Science
The Institute of Environmental Geoscience, Russia
e-mail: direct@geoenv.ru

METHODS OF THE MATHEMATICAL MORPHOLOGY OF LANDSCAPE

key words: mathematical modeling, landscape metrics, diffuse exogenous processes, landscape pattern, the theory of random processes

Landscape morphology analysis is still a topical problem of the up-to-date landscape science. Many researchers, including V.A. Nikolaev, L.I. Ivashutina, Yu.G. Simonov, V.M. Fridland, B.V. Vinogradov, N.V. Fadeeva, K.H.Ritters, R.V.O'Neill, K. McGarigal, B.J. Marks, and others (Ivashutina, Nikolaev, 1971; Simonov, 1970; Fridland , 1972; Victorov, 1986; Riitters, O'Neill, Hunsaker a. o., 1995; Leitao A.B. et al. 2006; Moser, Jaeger, Tappeiner et al. 2007; McGarigal, Cushman, Neel, et al., 2002, Pshenitchnikov, 2003, and others), dealt with the quantitative analysis of landscape patterns.

Various tasks such as analysis of RSD and landscape patterns, exogenous process prediction and risk assessment involve landscape morphology analysis.

The researches devoted to the quantitative analysis of geometric features of morphological patterns have resulted now in the separate scientific branch – mathematical morphology of landscape (Victorov, 1998, 1992).

The Mathematical Morphology of Landscape is a branch of the landscape science dealing with quantitative laws for the Earth surface mosaics, which are formed by natural units (bogs, aeolian ridges, etc.), and mathematical methods of their analysis. Hence, the named branch is aimed on the quantitative aspects of the landscape morphology basing on its mathematical analysis. We can figuratively call it "the landscape geometry".

One of the most important results of the mathematical morphology of landscape is a concept of mathematical models of landscape patterns. A mathematical model of a landscape pattern is a set of mathematical dependences reflecting most essential

geometric characteristics of a landscape. The mathematical models are a core of the mathematical morphology of landscape. They appear to be an indivisible integral base for solving various tasks.

The mathematical models of morphological patterns formed under a single process in uniform geographical conditions play a special role of basic units. They are called canonical mathematical models of landscape patterns. The canonical models for number genetic types of territory are already developed now, including plains with karst processes, fluvial erosion and alluvial plains, and some others (Victorov, 1998). Some new canonical models appeared recently, such as mathematical models for ridged and ridged cellular eolian plains (Victorov, Zaitzev, 2000). The theory of random process appeared to be the most successful base for developing the canonical models of morphological patterns.

One can oppose the canonical mathematical models to the mathematical models of complicated morphological patterns, i.e. the landscape patterns formed under of a number of processes or (and) geomorphologic and geological heterogeneity. According to the combination of factors the mathematical models for complicated morphological patterns of the types I, II and III can be determined (Victorov, 2006).

The mathematical model of the landscape pattern for plains under soil subsidence processes is an example of canonical models.

Plains affected by soil subsidence are the plains developing under soil subsidence process. They are spread at different nature conditions including the south of the West Siberia, Kopet-Dag piedmont plain, Orsha-Mogilev plateau within the East European plain, Caspian lowland and others (fig. 1).

Landscape pattern of the named plains under uniform geological and geomorphologic conditions consists of rather uniform and flat background sprinkled with subsidence plots representing soil subsidence. They are round holes without prevailing orientation and aggregating into line systems. Nature conditions depend on climatic zone, material constitution of deposits, and other factors. The landscapes within the depressions are also determined by the same factors. They can be motley grass - grasses meadows with black soil, desert vegetation on gray-brown desert soil, bogged depressions, and others. In spite of different soils and vegetation covers geometric features of the landscape pattern remains similar (fig. 1).

Thus, the mathematical model of the landscape pattern under investigation treats it as a background with random circles corresponding to soil subsidence depressions. The model is based on the following assumptions:

• The occurrence of a new hole is a probabilistic process and within non-overlapping areas is an event independent of other depression occurrences

- The probability of depression occurrence is directly proportional to the time period (Δt) and size of the area (Δs) under study. The probability that two or more holes occur is negligibly small compared to the occurrence of only one depression:

$$p_1 = \lambda \Delta s \Delta t + o(\Delta s \Delta t)$$

$$p_k = o(\Delta s \Delta t) \quad k = 2,3... , \tag{1}$$

where λ is a parameter.

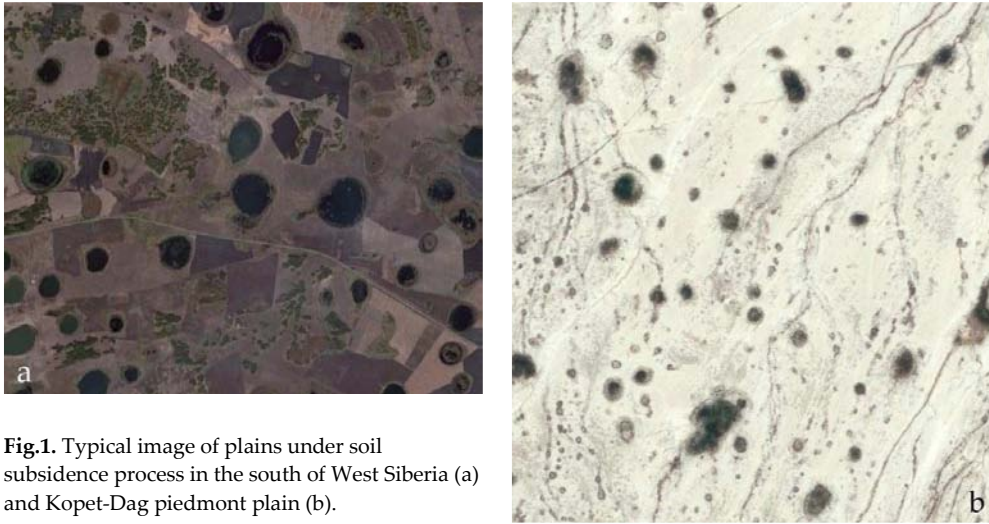- The growth rate of a depression is directly proportional to it actual size.

Fig.1. Typical image of plains under soil subsidence process in the south of West Siberia (a) and Kopet-Dag piedmont plain (b).

These assumptions are natural. Depression occurrences resulting after substratum changes are simultaneous and independent from each other in case of uniform nature conditions. Occurrence more then one depression within certain area is substantially rarer compared with appearance of a single site caused by soil subsidence. Growth rate of any depression is directly dependent on runoff concentration, which is determined by the actual size of the hole. This characteristic relation between runoff concentration and size of the hole is substantially independent on climatic, geological, and geomorphologic conditions of any area though the actual runoff and weathering parameters deepened on them.

The mathematical analysis of the model assumptions enable us to draw uniquely (Victorov, 1998, 2006) the general equations of the landscape pattern model for plains

under soil subsidence process. Let us demonstrate the radii distribution density of subsidence holes as an example.

Let us examine area change for a hole of soil subsidence genesis. The second assumption of the model causes proportional dependence of a generally random change of the hole radii during i section of time from the actual radii

$$\Delta u_i = \xi_i u_i \qquad (2)$$

The proportional indices ξ_i are random parameters resulting from atmospheric characteristics of every year such as precipitation, storm run-off, melting water volume and others. These indices are independent from year to year but they have got the same distribution, so we can get their mathematical expectation and variance as follows:

$$M\xi_i = a_0, \quad D\xi_i = \sigma_0^2. \qquad (3)$$

By transposing the terms dealing with the hole size to the left, then summing and replacing of the left part with the integral we get

$$\int_{x_0}^{x} \frac{du}{u} = \sum_{i=1}^{t} \xi_i \qquad (4)$$

After integration we get

$$\ln x - \ln x_0 = \sum_{i=1}^{t} \xi_i \qquad (5)$$

where x is a hole size at time t, x_0 is an initial hole size, which can be taken as 1 for simplification. In this case the expression becomes simpler

$$\ln x = \sum_{i=1}^{t} \xi_i \qquad (6)$$

Taking into account the sum of a great number of independent summands on the left we get according to the central limit theorem the size logarithm as a random variable with normal distribution. The mathematical expectation of this variable and its variance grow lineally by time due to independence of the summands

$$a(t) = a_0 t, \quad \sigma^2(t) = \sigma_0^2 t \qquad (7)$$

Thus the radii distribution density of subsidence holes (and their area correspondingly) after time t since the depression appearance is

$$f_r(x,t) = \frac{1}{\sqrt{2\pi}\sigma_0 x\sqrt{t}} \exp[-\frac{(\ln x - a_0 t)^2}{2\sigma_0^2 t}],$$ (8)

where a_0, σ_0 are parameters.

Using the mathematical analysis we can easily get that the probability of k initial sites occurring within area s during time t is described by (Victorov, 1998):

$$P(k) = \frac{(\lambda vt)^k}{k!} e^{-\lambda vt}$$ (9)

where v is the trial plot area, t is time, λ is the average number of sites appearing within a unit area per unit time.

The research shows that the mathematical models of landscape patterns are characterized with an amazing feature. It is invariant (Victorov, 1998, 2006). This feature means that the model equations remain true for different areas of the same genetic type, i.e. areas developing under the same process. The form of the equations is the same, while specific conditions of every area, such as, deposits, rainfall, and so on, determine only the values of certain parameters of the equations.

For example, one of the equations of the mathematical model for plains with soil subsidence (8 and 9) has the same form for different nature conditions. The chi-square test for this distribution is given below (tab. 1, 2)

Tab. 1. Comparison between actual and theoretical distribution of lognormal distributions for area of soil subsidence sites.

Location of a testing region	a, average logarithm of depression	σ, standard deviation for logarithm of depressions	N	χ^2	$\chi^2_{0.99}$
The south of West Siberia	12.685	0.734	78	1.329	9.210
Russian plain (Byelorussia)	3,980	0,409	61	3.889	9.210
	1,294	0,522	59	2.469	9.210
	0,153	0,635	52	4.163	9.210
Kopet-Dag piedmont plain	0.668	0.290	84	1.731	11.341
The south of West Siberia	12.077	1.325	44	4.739	6.635
Caspian lowland	4.537	0.955	34	0.763	9,210

Tab. 2. Comparison between actual and theoretical spatial distribution (Poisson Law) for soil subsidence sites.

Location of a testing region	Average spatial density of depressions	N	χ^2	$\chi^2_{0.99}$
Caspian lowland	0.556	63	0.086	6.635
Caspian lowland	0.548	62	0.056	6.635
Caspian lowland	0.912	80	0.262	6.635
Kopet-Dag piedmont plain	1,654	130	0.494	13,277
	3.308	65	2.736	15,086
	1.720	100	1.858	11.341
North Africa (Algeria)	0.622	98	2.924	6.635
Barabinskaya steppe	1.174	69	1.056	9,210

The following general directions within the frames of the mathematical morphology of landscape have given the most interesting results up to now:

- natural risks assessment,
- landscape metrics selection
- analysis of natural dynamics,
- new landscape laws, invariants
- laws for age differentiation of natural units,
- new software for RSD processing.

The mathematical morphology of landscape has got important results solving the **problem of natural risk assessment**. The goal is to design the theory and predicting evaluation methods for probability and damage risk to an engineering construction by nature hazards. Though many works (Natural risk assessment and management, 2003; Kolluru, Bartell, Pitblado, a. o., 1996 and etc.) are devoted to damage risk evaluation, the problem is still among topical ones. The difficulty is in the little knowledge about those peculiarities of the process, which allow us to estimate probability and damage values. The statistical approach needs too many observation data, and in this case the estimation loses it significance due to limited lifetime of engineering constructions.

This problem can be solved for a wide spectrum of hazardous exogenous processes (karst, soil subsidence, thermokarst, etc) within the frames of the mathematical morphology of landscape. The decision is based on the mathematical models of landscape patterns, the core of the mathematical morphology of landscape.

Consider natural risk assessment for a thermokarst area as an example. Thermokarst plains with fluvial erosion are slight wavy subhorizontal areas covered by tundra

vegetation, interspersed with lakes and khasyreis (a khasyrei is a drained lake) and rather rare fluvial erosion network. The main parts of the region include (fig. 2):
- flat tundra plain,
- thermokarst lakes,
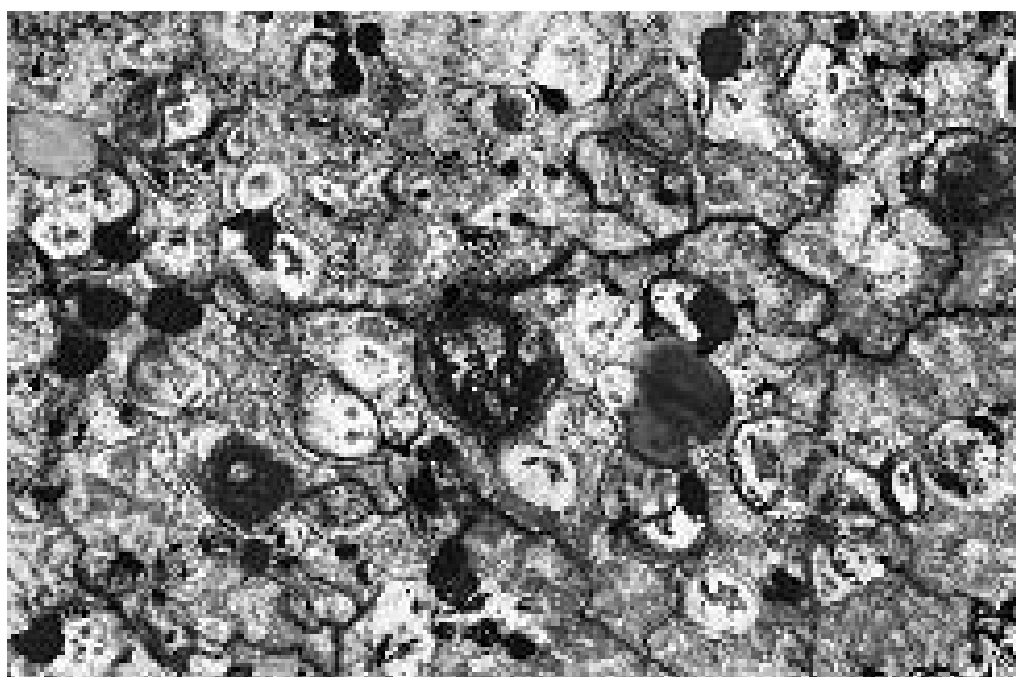- khasyreis,
- fluvial erosion network.

Fig. 2. Typical image of thermokarst plains with fluvial erosion in the north of West Siberia.

Thermokarst, thermo-abrasion, and thermo-erosion processes perform in complex interrelations and determine the type of the area.

The assumptions of the model include (Victorov, 2001, 2005):
1. Appearance of a thermokarst depression within every taken area is an occasional event, which probability is directly proportional to the size of the area.
2. Radius of an initial thermokarst depression is a random variable with distribution density $\varphi(x)$.
3. Growth of an appeared thermokarst depression is a random variable; it is independent of other lakes and the growth rate is directly proportional to

heat storage in the lake water; it is inversely proportional to the total surface area of the water body.

4. Depth of a lake is in direct proportion to its diameter.
5. In the course of its growth, a lake can turn into a khasyrei after draining by the fluvial erosion network; probability of this does not depend on other lakes. If it happens the depression stops to grow.
6. Appearance of new fluvial erosion sources within an occasional area is a random quantity, which probability is directly proportional to the size of the area.

Hence, using the described above model we can get impact probability for different types of constructions (Victorov, 2003, 2006):

circular constructions (a cycle with radius *l*):

$$P_{ds}(l) = 1 - \exp[-\pi\gamma(t)[(\bar{r}(t)+l)^2 + \sigma^2(t)] - \pi\gamma_h(t)[(\bar{r_h}(t)+l)^2 + \sigma_h^2(t)]]$$

(10)

linear constructions (with length *L*)

$$P_{dl}(L) = 1 - \exp[-2[\gamma(t)\bar{r}(t) + \gamma_h(t)\bar{r}_h(t)]L]$$

(11)

small area (point) constructions

$$P_d(t) = 1 - e^{-\gamma(t)s(t) - \gamma_h(t)s_h(t)}$$

(12)

where $\gamma(t)$, $\bar{r}(t)$, $\sigma(t)$, $\gamma_h(t), \bar{r_h}(t), \sigma_h(t)$ are average area density, average radius, and radius standard deviation for active thermokarst sites (lakes) and khasyreis respectively.

The result of the analysis was tested empirically. A region of ancient thermokarst in Central Siberia is one of the testing areas. It is a flat tilled surface with slightly represented round depressions, which are distinctly revealed at air-borne photos. They are spread all over the area in disorder. Their average area density, average radius and standard deviation are detected from the air-borne photos or maps and used in the equation (10) for round engineering constructions to estimate impact probability. Empirical values result from placing circles simulating engineering constructions of different size within the area according to a uniform grid and counting a number of impacts.

The tab. 3. gives the comparison of the obtained theoretical and empirical values for the Central Siberia area with permissible distinctions 0,100 at significance level 0, 95. All empirical values agree with the calculated.

Tab. 3. An Impact Probability for Round Constructions of Different Size (ancient thermokarst, Central Siberia).

Dimension of construction	Theoretical probability	Empirical frequency	Difference	Permissible distinction at significance level 0, 95.
10	0,367	0,393	0,026	0,100
20	0,477	0,505	0,028	0,100
30	0,582	0,639	0,057	0,100
40	0,677	0,75	0,073	0,100
50	0,76	0,835	0,075	0,100
60	0,826	0,882	0,056	0,100
70	0,878	0,916	0,038	0,100

Landscape metrics selection is among the most interesting problems in landscape science.

Many researchers suggest for analysis quantitative parameters of landscape mosaics formed by natural units on the Earth surface. They are called landscape metrics (Ivashutina, Nikolaev, 1971; Victorov, 1986; Riitters, O'Neill, Hunsaker a. o., 1995; Jaeger, 2000). Landscape metrics can be used for different tasks. At the same time any case of landscape metrics application causes some general fundamental issues:

- How could we optimize landscape metrics selection?
- Does optimizing selection of landscape metrics depend on physiographical conditions?
- Does a limited set of metrics exist, which contains large majority of data concerning other landscape metrics behaving ("complete set of landscape pattern metrics")?

The last issue a priori seems to be unsolvable.

The mathematical models of landscape patterns is a base for optimizing selection of landscape metrics for a great variety of landscapes. Plains affected by soil subsidence are taken as an example. General assumptions of the model were given above.

Optimizing selection of metrics. Let the following set of rather common metrics is taken for quantitative analysis of the landscape pattern for the plain under soil subsidence:

Average spatial density of depressions (Patch Density, class level calculation)

$$m_1 = \frac{n}{S}, \tag{13}$$

Variation coefficient for spatial density of depressions

$$m_2 = \frac{\sigma_n}{n_0} \tag{14}$$

Average area of a depression[1] (mean patch size, class level calculation)

$$m_3 = \frac{\sum_{i=1}^{n} q_i}{n} \tag{15}$$

Area share taken by the depressions

$$m_4 = \frac{S_1}{S} \tag{16}$$

Average distance to the nearest center of a depression

$$m_5 = \frac{\sum_{i=1}^{n} r_i}{n} \tag{17}$$

where n is a number of holes within a plot of area S, n_0 is average number of holes at the trial plot, σ_n is a standard deviation of hole number within the trial plot, S_1 is total area taken by the holes within the plot, q_i is the area of the depression (i), r_i is a distance between the center of depression (i) to the closest one.

The question is, is this set of metrics suitable for joint use?

Let us consider the general expressions of the mathematical model of the landscape pattern for plains under soil subsidence process (8) and (9). We should express the given metrics by model parameters in order to find interrelations among them.

The first metric is a number of holes within the unit area, 1 square km for instance. According to the model a number of depressions within the plot is stochastic and at any time satisfies Poisson distribution:

$$P(k) = \frac{(\gamma S)^k}{k!} e^{-\gamma S} \tag{18}$$

where γ is average number of hole centers within the unit area at the given moment, S is the plot area.

It is natural that in our analysis we are interested not in statistic fluctuations but in real difference, which are reflected with average value (mathematical expectation). Taking into account the expression for the mathematical expectation of the para-

---

[1] Without taking into account hole fusions

meter distributed according the Poisson law one can get an average number for depressions within the plot:

$$n_S = \gamma S. \tag{19}$$

One can get the first metric by dividing the average number of the depressions within the plot on its area (S)

$$m_1 = \gamma. \tag{20}$$

The second metric is a ratio between the standard deviation of a number of depressions within the trial plot to their mathematical expectation. The trial plot is a circle of a unit area (1 square km for instance). As far as a a number of depressions within the trial plot also satisfies the Poisson law (18), their standard deviation and expectation are given with the following expressions according the Poisson distribution:

$$\sigma_n = \sqrt{\gamma}, \ n_0 = \gamma \tag{21}$$

After dividing and simplifying we get the following relation for the second metric:

$$m_2 = \frac{1}{\sqrt{\gamma}}. \tag{22}$$

The third metric is an average area for a depression

$$m_3 = \mu \tag{23}$$

where μ is the mathematical expectation of depression area. The holes are round, so the area and radius logarithms are linear relate

$$\ln q_i = \ln \pi + 2 \ln r_i. \tag{24}$$

It means that at any time the area of the depression has lognormal distribution after the radius with a little bit different from radius distribution parameters,

$$f_q(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\ln x - a)^2}{2\sigma^2}} \tag{25}$$

where a, σ average logarithm and standard deviation of the logarithm of depression area at present.

According to the well-known expression for mathematical expectation of lognormal distribution (Koroljuk, Portenko, Skorokhod et al., 1985) it is related with the model parameters a, σ by the following expression:

$$\mu = \exp\left(0.5\sigma^2 + a\right) \qquad (26)$$

The share area of the depressions within the plot in question (the forth metric) is also connected with the model parameters. This relationship can be find taking into account the fact that the share of area covered by the holes is a probability that a random point of the plot is affected by the soil subsidence process and hence appeared within any hole. According the results of the risk assessment (Victorov, 2003), this probability and also the forth metric is connected with the parameters of the landscape pattern by the following relation:

$$m_4 = 1 - \exp(-\gamma\,\mu) \qquad (27)$$

The fifth metric is an average distance from the center of the hole to the center of the nearest hole. The distance distribution between centers of nearest depressions can be get using the following consideration (fig. 3).
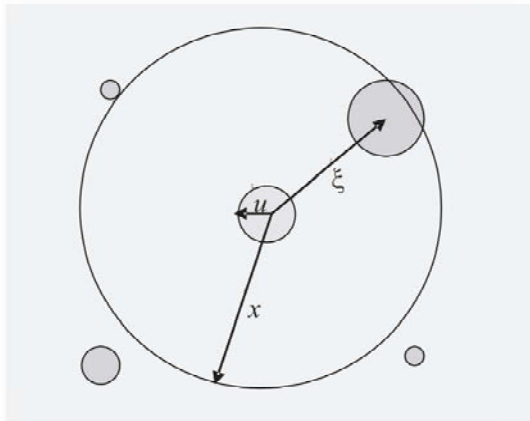

Fig.3. The illustration for derivation of the formula of average minimal distance between soil subsidence holes (explanation in the text).

The probability that the mi-nimal distance between depressions ξ would not exceed the given value x is equal the probability that there is at least a single depression within the ring area between the boundary of the given hole of ra-dius u and a circle of radius x. Thus we take into account that no depression can appear within the existing one. Basing on the Poisson distribution for centers of depressions this probability and hence the conditional distribution of the minimal distance are

$$F_\xi(x|u) = 1 - \exp[-\pi\gamma(x^2 - u^2)] \qquad (28)$$

Hence the conditional expectation for the minimal distance is

$$M(\xi|u) = \int_u^{+\infty} \exp[-\pi\gamma(x^2 - u^2)]dx \qquad (29)$$

Using the Mills equation (Koroljuk, Portenko, Skorokhod et al., 1985) we obtain

$$M(\xi|u) = u + \frac{1}{\sqrt{\gamma}} \exp(\pi\gamma u^2)[1 - \Phi(\sqrt{2\pi\gamma}u)], \qquad (30)$$

where $\Phi(t)$ is the Laplas function.

First terms expansion in series by u gives us approximate expression for the conditional mathematical expectation of the minimal distance

$$M(\xi|u) \approx \frac{1}{2\sqrt{\gamma}} + \frac{\pi\sqrt{\gamma}}{2}u^2 . \qquad (31)$$

At least, after examining all possible radius values of the depression under consideration u, taking into account the whole set of possibilities (25) and integrating the resulted expression one can finally get the complete mathematical expectation corresponding the fifth metric of the landscape pattern

$$m_5 = \frac{1}{2\sqrt{\gamma}} + \frac{\sqrt{\gamma}}{2}\mu . \qquad (32)$$

Thus, the mathematical analysis of the expressions corresponding to the metrics shows hidden independencies within the set of the metrics in question.

$$m_2 = \frac{1}{\sqrt{m_1}} , \ m_4 = 1 - \exp(-m_1 m_3) \ m_5 = \frac{1 + m_1 m_3}{2\sqrt{m_1}} \qquad \qquad (33)$$

The obtained theoretical conclusion is illustrated at fig.4 and tables 4 and 5. They present theoretical and empirical values of the metrics for areas under different nature conditions. The theoretical values result from metrics m_1 and m_2 according to the above equations. The depression characteristics including area, center coordinates, and dimensions were obtain by measuring at air-borne and space images. The empirical values of the forth metric were obtained using the analytical software package for the quantitative parameters of the landscape patterns designed in the Institute of Environmental Geoscience RAS by A.S. Victorov and A.A. Victorov. In majority of cases the data from the graph (fig.4) and tables prove the theoretically predicted interrelations among the metrics.

It means that the set of metrics in question is not optimal because the second, forth and fifth metrics are non-informative because they related in unobvious way with the first and third metrics and are under their control.

116

Fig. 4. Comparison of theoretical relations between metrics m_1 and m_2 with empirical data.
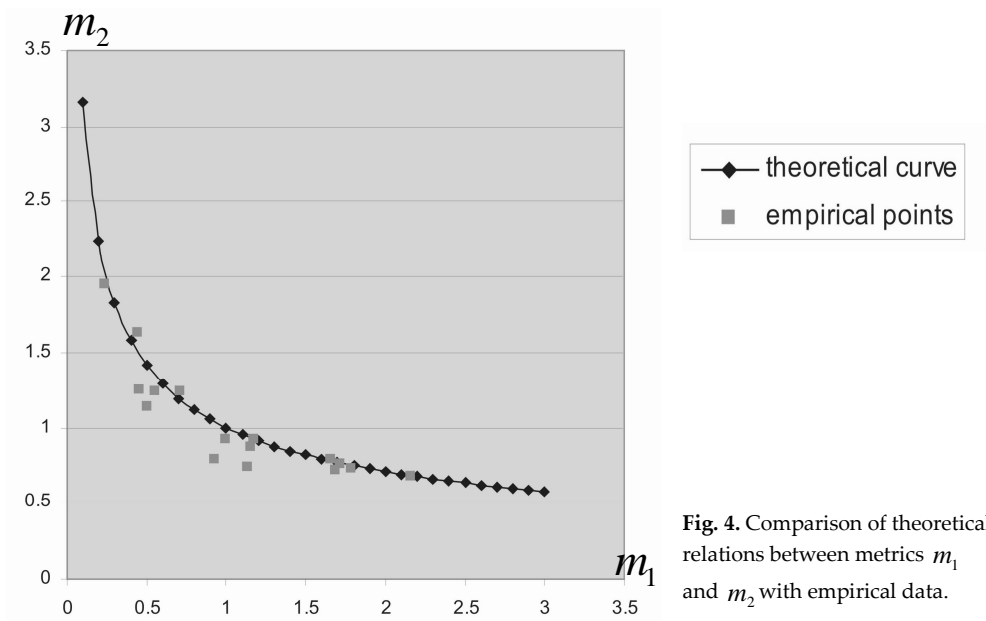
Tab. 4. Comparison between theoretical dependence of m_1, m_3 and m_4 metrics and empirical data.

Location of a testing region	Metric m_1, km⁻²	Metric m_3, km²	Metric m_4	
			Empirical value	Theoretical (calculated) value
Turgai table land	0,111	0,820	0,106	0,087
Caspian lowland	1388,889	0,000	0,209	0,188
Barabinskaya steppe	0,899	0,307	0,198	0,241
Caspian lowland	11,364	0,008	0,070	0,090
Kopet-Dag piedmont plain	81,439	0,001	0,053	0,073
Russian plain (Byelorussia)	148,448	0,002	0,250	0,224
The south of West Siberia	0,272	0,434	0,093	0,111
The south of West Siberia	0.364	0.354	0.053	0.129

Tab. 5. Comparison between theoretical dependence of m_1, m_3 and m_5 metrics and empirical data

Location of a testing region	Metric m_1, km^{-2}	Metric m_3, km^2	Metric m_5, m	
			Empirical value	Theoretical (calculated) value
Russian plain (Byelorussia)	148.448	0.002	60	51
Kopet-Dag piedmont plain	81.439	0.001	68	60
The south of West Siberia	0.364	0.355	1048	936
The south of West Siberia	0.272	0.434	1152	1072

Existence of the complete set of metrics. There is an infinite number of the metrics, which can be used for landscape pattern analysis of the plains under soil subsidence process. Even if one takes the most common metrics their number is very great. At the same time as it was shown above some metrics can be non-informative in case of mutual application and their mutual use is needless. The question is: does a limited set of metrics exist, which makes needless *all other* metrics? It is possible if the given landscape pattern has a limited set of the metrics, which controls all other possible metrics of the pattern. In this case using the other metrics does not give new infor-mation because their values are determined by the metrics of the set.

Hence this hypothetic set of metrics (let us call it "the complete set of metrics") contains the total information about the landscape pattern, for sure without taking into account statistical fluctuation.

The analysis of the model of plains under soil subsidence predicts behavior of any quantitative characteristic of the landscape pattern. Actually any landscape metric of the pattern under consideration (landscape pattern of plains under soil subsidence) use in the metric formulas only the following independent variables[2]:

Numerical characteristics concerning relative position of the sites (holes) and (or)

Numerical characteristics concerning dimensions of the sites.

It is not necessary to use the metrics, which describes shape of the sites because all the holes are cycles.

But the first assumption of the model and the law of spatial distribution of soil subsidence holes give all the pattern features behavior describing relative position of the sites and allow us to find probability for any spatial arrangement of the sites. The second assumption and the law of radius distribution give all the pattern features

---

[2] Here we do not consider the metrics using additional data such as classification distance (Николаев; Викторов).

118

behavior describing the site dimensions, such as area, diameter, perimeter of the holes and others. It means that assuming the model about independence of growth and location of the sites it is possible to predict metric values using the mathematical model of landscape pattern.

For example, let us examine the widely used metric, the average area of sites (mean patch size, landscape level calculation)

$$m_6 = \frac{S}{m},$$ (34)

where m is total number of sites.

Let us find this metric as the mathematical expectation of the value. Assuming the Poisson law for a number of holes distribution and that the total number of sites within the area in question is greater by one at the expense of background area as additional site one can get the equation

$$m_6 = \sum_{k=0}^{+\infty} \frac{S}{k+1} \frac{(\gamma S)^k}{k!} e^{-\gamma S}.$$ (35)

Factoring out from the summation sign and assuming that the obtained series differ from exponent expansion by one after simplifying we get

$$m_6 = \frac{1}{\gamma}(1 - e^{-\gamma S}).$$ (36)

The number of examples can be increased but it is clear from the demonstrated statements that the assumptions of the model are sufficient for behavior prediction of any characteristic of the pattern. At the same time some of the obtained expressions are simple, the other ones cannot be simplified but the relation between the model and the metric remains in any case.

But the assumptions and main equations of the model use only three parameters. It means that any metric of the landscape pattern of the plains under soil subsidence process is determined only by these three parameters of the model a, σ, γ. Hence we demonstrated existence of the finite set of quantitative parameters, which can express any metric of the landscape pattern under consideration. It means the above parameters of the model form the hypothetical complete set of metrics, which is dealt in the issue. They are

γ is average spatial density of soil subsidence depressions (holes),

a is average logarithm of depression area,

σ - is standard deviation for logarithm of depressions.

The fulfilled study allows us to make the following conclusions:

1. The optimizing selection of landscape metrics for different purposes can be done theoretically using the mathematical models of landscape patterns.

2. The optimizing decisions are common for areas of the same genetic type under wide spectrum of nature conditions; it results from generality of the mathematical model of the landscape patterns.

3. There are the finite sets of metrics, i.e. a low number of metrics determining all other metrics accurate to statistic fluctuations.

The approach evolved by the mathematical morphology of landscape patterns gives **techniques for predicting processes development** in very important practical tasks. At the same time one of the most interesting results concerns environment dynamics during long period. It is especially important because every researcher usually deals with old nature processes. In this case three variants of process development are of special interest:

1. The process unlimitedly develops (either increases or decreases),
2. The process once reaches a certain stationary state,
3. The process turns into continuous waving one.

Determining process development as the second type is very efficient because in this case description of the process behavior can be essentially simplified and in turn the predicting procedure becomes simpler.

Consider analytical capabilities of the model for process dynamics with long time of development taking as an example thermokarst plains with fluvial erosion. The main assumptions of the model are described above.

Thus, the territory can be regarded as a certain flow of developing sites of the thermokarst process. Every site appears at a random moment independently from each other, enlarges under influence of different factors including random ones and finally reaches a critical value at a random moment and comes to the stage of degenerate sites.

Every thermokarst depression passes through the following stages:

1. Origin of depressions
2. These depressions are filled up with water as lakes and increase in size independently of each other due to thermo-abrasion process.
3. Occasionally any lake can be drained due to erosion process and turn to a khasyrei. Size increasing stops.

The mathematical analysis of the model gives the main equations describing thermokarst developing (Victorov, 2001, 2006). They are:

The Distribution of a number of thermokarst depressions which originated for time t at a site s

120

$$P(k,t) = \frac{[\eta(t)s]^k}{k!} e^{-\eta(t)s}$$

(37)

The growth of lake radii can be considered as a Markov random process with transition probabilities

$$f(x,t,u) = \frac{1}{\sqrt{2\pi}\sigma x \sqrt{t}} e^{-\frac{(\ln x - \ln u - at)^2}{2\sigma^2 t}}$$

(38)

The radius distribution density for a thermokarst lake since a period t after its appearance

$$f(x,t,u) = \int_0^x \frac{1}{\sqrt{2\pi}\sigma x \sqrt{t}} e^{-\frac{(\ln x - \ln u - at)^2}{2\sigma^2 t}} \varphi(u)du$$

(39)

The radius distribution density for a thermokarst lake since a period t after its appearance (for simplifying the model let us take a unit radius for originated initial thermokarst depressions).

$$f_0(x,t) = \frac{1}{\sqrt{2\pi}\sigma x \sqrt{t}} e^{-\frac{(\ln x - at)^2}{2\sigma^2 t}}$$

(40)

where $\eta(t)$ is a distribution density for active thermokarst sites at moment t, a, σ - parameters.

The analysis of the model assumptions gives us essential conclusions concerning the developing process, particularly, after a long period of development. In this very case it possible to show that under rather general conditions ((Victorov, 2003, 2006) the territory in question reaches a certain state close to stationary one with character-ristics of dynamic balance. Thus, after a long period of development a number of quantitative characteristics become stable:

- process impact (P_d^*) , share of area affected by the dangerous process

$$\ln[1 - P_d^*] = -\lambda\pi[1 - P_d^*]I ,$$

(41)

- radii distribution for the active sites

$$f(x,\infty) = -\frac{2}{xEi(-\pi\gamma)} e^{-\pi\gamma x^2},$$ (42)

- average area density of the active sites

$$\eta(\infty) = -\frac{\lambda}{2a} Ei(-\pi\gamma),$$ (43)

- radii distribution for the for khasyreis

$$F(x) = 1 - e^{-\pi\varepsilon x^2}.$$ (44)

In other words, new active sites of thermokarst processes continuously appear and disappear within the observed area, their dimensions change, new degenerate sites appear but the main quantitative statistic characteristics, such as area density of the active sites, the dimensions of degenerate and active sites remain stable.

Therefore, the mathematical morphology of landscape gives us analytical prediction of the area dynamic state after a long time of development.

Age differentiation laws for natural units are one more interesting result of the mathematical morphology of landscape. The landscape age differentiation analysis shows obligatory determination of two types of differentiation:

- Macro age differentiation
- Micro age differentiation.

The macro age differentiation has started since the beginning of the landscape differentiation of the whole territory. It is characterized with rather simultaneous beginning of development of natural units within large territories. Their age can be defined as macro-age. Age differentiation caused by successive glacial or marine recession can be taken as an example.

The micro age differentiation is caused by asynchronous start of different natural units of the same type within the even macro-age area. One can take as an example age differentiation of soil subsidence depressions, appeared at different time within a uniform plain. This type of age differentiation is more usual for small natural units, such as facie, stow, and complex stow.

The micro age differentiation reflects the fact that natural units of the same type, which origin resulted from developing nature processes forming corresponding landscape pattern of the area, exist different time and, hence, their soil and vegetation covers, local ground waters, and permafrost conditions developed during different time in every case. So, in spite of their common genesis they cannot be regarded as totally similar.

For example, separate khasyreis (drained thermokarst lakes) are characterized with different micro-age if they appeared in different time and, thus, their soil and vegetation covers as well as new permafrost developed here for different time. Another example of micro age differentiation deals with micro depressions of Caspian Lowland. Time of vegetation and soil development varies from one depression to another one, processes resulting in under depression water lenses and solonetz edges have also different duration.

The mathematical morphology of landscapes has got results about quantitative laws of micro age differentiation for a number of genetic types of territory. It was found out that large areas appeared to be a sort of "patchwork" formed by adjacent nature units differed by their age.

The micro age differentiation of alluvial plains was examined, for instance. Landscape pattern of an alluvial plain is formed by alternation of ridges and interridge depressions with corresponding soil and vegetation covers. The ridges and depressions appear as a result of meander development and river-channel straightening (fig. 5). Let time period between 2 successive straightenings be a cycle of meander development (**meandering cycle**), while a set of nature units based on ridges and depressions appeared during a cycle is a **package**. It is obvious that ridges' age within a package uniformly changes according to their contiguous alternation from the starting to the enclosing arcs of a single package. As straightenings repeat from time to time every younger package razes either a part or the whole previous one. Several contiguous packages can be razed, so we call the corresponding cycles **razed cycles**.

Total or partial razing depends on ratio between meandering cycle durations. If younger cycle was longer than the older one, only part of the package is razed, and a fragment of the package remains[3]. In other case the whole package is razed. Those cycles, which are represented with unerased fragments of packages forming the landscape pattern of the plain, are called **represented cycles**.

Let us call the number of a represented cycle from younger to older "the order of a cycle".

Hence, in general **two contiguous fragments of packages do not directly follow for each other in time but are separated by a certain time interval**. One can regard an alluvial plain as a set of multiple-aged fragments of packages, which origins are separated by a certain time interval from each other. The detail analysis shows that within a plot corresponding to a single meander the fragments are separated in time equal to a sum of erased package durations with that of erased part of the later

---

[3] It is supposed that alluvial processes remain stable during the investigated period, including stable rate of package growth.

package, represented by the unerased fragment. This very compound process results in micro age differentiation of alluvial plains.

Fig. 5. Typical image of an alluvial plain landscape pattern at RSD.

The mathematical model of the alluvial plain morphological pattern (Victorov, 1998, 2006) can be a base for the revealing the micro age differentiation. The following assumptions of the model are sufficient for the analysis:

- river-channel straightenings are independent from each other and
- straightening probability for a small time interval Δt is straightly proportional to the interval value and much greater than a probability of one more straightening.

The mathematical analysis gives the following main laws of micro age differentiation for alluvial plains (Victorov, 2007):

- duration distribution for represented meandering cycles obeys gamma distribution

$$f_{k+1}(x) = \frac{\lambda^{k+1}}{k!} x^k e^{-\lambda x} ,$$ (45)

- distribution of a number of erased cycles between contiguous represented meandering cycles corresponds to geometrical distribution and depends on a cycle order

$$P(v_k = m) = \left(1 - \frac{1}{2^k}\right)^m \frac{1}{2^k},$$ (46)

- an average number of erased cycles between two represented cycles make a sequence common to different alluvial plains

$$Mv_k = 2^k - 1,$$ (47)

- duration distribution for forming unerased package fragment corresponds to exponential distribution

$$F_{\varepsilon_k}(x) = 1 - e^{-\lambda x},$$ (48)

- average length of time intervals dividing packages of contiguous orders obeys the equation

$$M\mu_k = \frac{k-1}{\lambda} + \frac{(2^k - 1)}{\lambda}\left[1 - k\sum_{n=2}^{+\infty} \frac{1}{n^{k+1}}\right].$$ (49)

where k is a cycle order, λ - is a parameter inverse to average duration of cycles.

The obtained results allow us to determine a concept of a landscape age spectrum for alluvial plains.

When the channel is stable and development results only from meander growth and channel straightening, the channel is a generator of ridges and depressions packages. As a result a series of the represented package fragments of a certain age sequence (generation) appears. According to the given above laws these fragments are divided with rather definite time intervals, depending on groups of erased cycles first of all. These series of time intervals dividing represented fragments package of contiguous orders, which change according the package order, can be called **age spectrum** of a landscape for alluvial plains.

Thermokarst plains with fluvial erosion are characterized with their own micro age differentiation resulting from draining thermokarst lakes and hence their turn into khasyreis due to fluvial erosion ((Victorov, 2006). It is possible to show that after a long period of development under diffuse processes a landscape appear to be a "patchwork" of multiage segments (micro age differentiation). Age distribution of segments corresponds to exponential distribution with a shift

$$F_\tau(y) = 1 - e^{-ay-b},$$ (50)

where a and b are parameters.

Therefore, the mathematical morphology of landscape gives new approaches in researching age differentiation of landscapes.

In general the up-to-date state of the mathematical morphology of landscape allows us to state that the obtained results can be regarded as new steps in decision different tasks of landscape science. The mathematical models of landscape patterns are a common base for the tasks.

Hence, the mathematical morphology of landscapes is a perspective way for solving various tasks basing on the mathematical analysis. The mathematical models of landscape patterns is the theoretical basis for these decisions.

REFERENCES

Fridland V.M., 1972: Soil cover pattern. Moscow, Publ. House "Mysl", p.423 (in Ru-ssian).

Ivashutina L.I., Nikolaev V.A. 1971: Landscape pattern contrast and some aspects of its study [in:] Vestnik M.S.U, geogr. # 5, pp 77–81 (in Russian).

Jaeger JAG, 2000: Landscape division, splitting index, and effective mesh size: new measures of landscape fragmentation. Landscape Ecol 15(2): 115-130.

Kolluru R.V., Bartell S.M., Pitblado R.M., Stricoff R.S., 1996: Risk assessment and management handbook for environmental, health, and safety professionals. McGraw-Hill, New York, p 550.

Koroljuk, Portenko, Skorokhod et al.,1985: Handbook for probability theory and mathematical statistics. Moscow, Publ. House "Nauka"', p. 640 (in Russian).

Leitao A.B. et al. , 2006: Measuring landscapes: a planner's handbook. Island press, p.245.

McGarigal K, Cushman S.A., Neel M.C., Ene E., 2002: FRAGSTATS: Spatial Pattern Analysis Program for Categorical Maps. Computer software program produced by the authors at the University of Massachusetts, Amherst, MA: University of Massachusetts. www.umass.edu/landeco/research/fragstats/fragstats.htm.

Moser B., Jaeger J. A.G., Tappeiner U., Tasser E., Eiselt B., 2006: Modification of the effective mesh size for measuring landscape fragmentation to solve the boundary problem. [in:] Landscape Ecology 3/22.

Pshenitchnikov A.E., 2003: Thematic morphometry – general directions and u used parameters [in:] Vest-nik MSU, geogr. # 5 pp 42–46 (in Russian).

Ragozin A.L. (ed.), 2003: Natural risk assessment and management. Proceedings of the all-Russia confe-rence RISK-2003, vol 1. PFUR Publishing, Moscow, p 413 (in Russian).

Riitters K.H., O'Neill R.V., Hunsaker C.T., Wickham J.D., Yankee D.H., Timmins S.P., Jones K.B., Jackson B.L., 1995: A vector analysis of landscape pattern and structure metrics. Landscape Ecol 10:23 – 39.

Simonov YU.G., 1970: Geographical neighborhood and methods of its measuring [in:]

Vestnik MSU, geogr. # 4, pp 13–19 (in Russian).

Victorov A.S., 1986: Landscape pattern. Moscow, Publ. House "Mysl", p. 179 (in Russian).

Victorov A.S., 1998: Matematicheskaya morfologiya landshafta (Mathematical morphology of landscape). Tratek, Moscow, p 180 (in Russian).

Victorov A.S. Zaitzev M.L., 2000: Mathematical model of ridge-cellular and cellular sand plains as a base for remote sensing data interpretation [in:] Issledovanie Zemli iz Kosmosa, # 4 (in Russian).

Victorov A.S. 2001: A model of dynamics for thermokarst plains with fluvial erosion [in:] Sergee-vskie chteniya, Issue 3, Moscow, Publ. House GEOS, pp 351–356 (in Russian).

Victorov A.S., 2003: An integrated mathematical model for diffuse exogenous geolo-gical processes. Proceeding of the 9th Annual Conference of the IAMG, Port-smouth, GB.

Victorov A.S. 2005: Mathematical models of thermokarst and fluvial erosion plains [in:] Proceedings of the IAMG' 05, vol 1. York University, Toronto, pp 62–67.

Victorov A.S., 2006: General problems of the mathematical morphology of landscape. Moscow, Publ. House "Nauka'", p. 180 (in Russian).

Victorov A.S. 2007: Age differentiation model for alluvial plains. In Geoekologiya, #4, pp 302–309 (in Russian).

SUMMARY

The paper discusses the mathematical morphology of landscape, which is a branch of the landscape science dealing with quantitative laws for the Earth surface mosaics, formed by natural units (bogs, aeolian ridges, karst, etc.), and mathematical methods of their analysis. The mathematical models are a core of the mathematical morphology of landscape. A mathematical model of a landscape pattern is a set of mathematical dependences reflecting most essential geometric characteristics of a landscape. The theory of random processes is a base for the mathematical models of morphological patterns. They appear to be an indivisible integral base for solving various tasks. The landscape researches, which deal with landscape metrics, new landscape laws, risk assessment, and development dynamics, are in need of the quantitative analysis of spatial landscape patterns.