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NONLINEAR CHOQUARD EQUATIONS ON HYPERBOLIC SPACE

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Abstract. In this paper, our purpose is to prove the existence results for the following nonlinear Choquard equation

$$-\Delta_{\mathbb{B}^N} u = \int\limits_{\mathbb{B}^N} \frac{|u(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\, dV_y \cdot |u|^{p-2}u + \lambda u$$

on the hyperbolic space $\mathbb{B}^N$, where $\Delta_{\mathbb{B}^N}$ denotes the Laplace–Beltrami operator on $\mathbb{B}^N$,

$$\sinh\frac{\rho(T_y(x))}{2} = \frac{|T_y(x)|}{\sqrt{1-|T_y(x)|^2}} = \frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}},$$

λ is a real parameter, $0 < \mu < N$, $1 < p \le 2^*_\mu, N \ge 3$ and $2^*_\mu := \frac{2N-\mu}{N-2}$ is the critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.

Keywords: nonlinear Choquard equation, hyperbolic space, existence solutions, Hardy–Littlewood–Sobolev inequality.

Mathematics Subject Classification: 35A01, 35J60.

1. INTRODUCTION

In this article, we investigate the nonlinear Choquard equation

$$-\Delta_{\mathbb{B}^N} u = \int\limits_{\mathbb{B}^N} \frac{|u(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\, dV_y \cdot |u|^{p-2}u + \lambda u \tag{1.1}$$

on the hyperbolic space $\mathbb{B}^N$, where $\Delta_{\mathbb{B}^N}$ denotes the Laplace–Beltrami operator on $\mathbb{B}^N$,

$$\sinh\frac{\rho(T_y(x))}{2} = \frac{|T_y(x)|}{\sqrt{1-|T_y(x)|^2}} = \frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}},$$

λ is a real parameter, $1 < p \leq 2_\mu^*, 0 < \mu < N, N \geq 3$ and $2_\mu^* := \frac{2N-\mu}{N-2}$ is the critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.

When posed in the Euclidean space $\mathbb{R}^N$, problem (1.1) is closely related to the nonlinear Choquard or the Choquard–Pekar equation

$$-\Delta u + V(x)u = \left(\frac{1}{|x|^\mu} * |u|^p\right)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

In the physical case $N = 3$, $p = 2$, $\mu = 1$, the problem

$$-\Delta u + V(x)u = \left(\frac{1}{|x|} * |u|^2\right)u \quad \text{in } \mathbb{R}^3 \tag{1.3}$$

appeared in the work [18] by S.I. Pekar describing the quantum mechanics of a polaron. P. Choquard [7] used it to describe an electron trapped in its own hole, in a certain approximation to the Hartree–Fock theory of one component plasma in 1976. In some particular cases, equation (1.3) is also called the nonlinear Schrödinger–Newton equation [19]. For more related results, we refer to [5,8,12,13,20,21,26] and reference therein. Mathematically, the existence and qualitative properties of solutions of Choquard equation (1.2) have been widely studied, see [6,11,14–17,23–25]

It is also interesting to study nonlocal problem (1.2) with respect to different ambient geometries, in particular to see how curvature properties affect the existence and nature of solutions. In the last decades, Mancini and Sandeep investigated in [10] the existence/nonexistence and uniqueness of a positive solution of the following local elliptic equation

$$-\Delta_{\mathbb{B}^N} u = |u|^{p-1}u + \lambda u \tag{1.4}$$

in the subcritical case for every $\lambda < (\frac{N-1}{2})^2$ and in critical exponent case for $\frac{N(N-1)}{4} < \lambda \leq (\frac{N-1}{2})^2$ with $N \geq 4$ on the hyperbolic space $\mathbb{B}^N$. Moreover, they proved that if $\lambda = 0$ and $1 < p < \frac{N+2}{N-2}$, then problem (1.4) has a positive solution. Afterward, Bhakta and Sandeep investigated in [1] the priori estimates, existence/nonexistence of radial sign changing solutions of problem (1.4). In [2], the classification of radial solutions of problem (1.4) was done by Bonforte *et al.* Such a problem has been extensively studied in recent years, see for instance [4,22] and references therein.

Motivated by the above papers, we will study the existence results for problem (1.1). The difficulties in treating nonlinear Choquard problem (1.1) originate in at least two facts. Firstly, the nonlocal term appears in the nonlinearity. Secondly, there is a lack of compactness due to the fact that we are working in $\mathbb{B}^N$ which is a noncompact manifold. Thus the starting point of variational approach to problem (1.1) is the following Hardy–Littlewood–Sobolev inequality on the hyperbolic space $\mathbb{B}^N$ (see [9]).

Proposition 1.1. *Let $0 < \mu < N$ and $q = \frac{2N}{2N-\mu}$. Then $f, g \in L^q(\mathbb{B}^N)$,*

$$\left| \int_{\mathbb{B}^N} \int_{\mathbb{B}^N} \frac{f(x)g(y)}{|2\sinh \frac{\rho(T_y(x))}{2}|^\mu} \, dV_y \, dV_x \right| \leq C(N,\mu)\|f\|_q \|g\|_q, \tag{1.5}$$

where

$$C(N,\mu) = \pi^{\mu/2} \frac{\Gamma(N/2 - \mu/2)}{\Gamma(N-\mu/2)} \left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{-1+\mu/N}$$

is the best constant for the classical Hardy–Littlewood–Sobolev constant on $\mathbb{R}^N$. Furthermore, the constant $C(N,\mu)$ is the sharp for the inequality (1.5) *and there is nonzero extremal function for the inequality* (1.5).

The integral

$$\int_{\mathbb{B}^N} \int_{\mathbb{B}^N} \frac{|u(x)|^p |u(y)|^p}{|2\sinh \frac{\rho(T_y(x))}{2}|^\mu} \, dV_y \, dV_x$$

from the inequality (1.5) is well defined if $|u|^p \in L^{\frac{2N}{2N-\mu}}(\mathbb{B}^N)$. Thus, for $u \in H^1(\mathbb{B}^N)$, by the Sobolev embedding theorem, we know that $2 \leq p \cdot \frac{2N}{2N-\mu} \leq 2^* = \frac{N}{N-2}$. It implies that

$$\frac{2N-\mu}{N} \leq p \leq 2^*_\mu = \frac{2N-\mu}{N-2}.$$

The main results of this paper are the following:

Theorem 1.2. *Let $0 < \mu < N$ and $\frac{2N-\mu}{N} < p < 2^*_\mu = \frac{2N-\mu}{N-2}$ if $N \geq 3$. Then there exists a positive solution of* (1.1) *for any $\lambda \leq \frac{(N-1)^2}{4}$.*

In the critical case the situation is more complicated. Namely, we have:

Theorem 1.3. *Let $0 < \mu < N$, $N \geq 4$, $p = 2^*_\mu = \frac{2N-\mu}{N-2}$ and $\frac{N(N-2)}{4} < \lambda \leq \frac{(N-1)^2}{4}$. Then* (1.1) *has a positive solution.*

This paper is organized as follows. In Section 2, we will introduce some notations and preliminary results. In Section 3, we will present the proof of Theorem 1.2. The proof of Theorem 1.3 will be discussed in Section 4.

2. NOTATIONS AND PRELIMINARY RESULTS

The hyperbolic space $\mathbb{H}^N$ is a complete simply-connected Riemannian manifold which has constant sectional curvature equal to -1. There are several models for $\mathbb{H}^N$ and we will use the Poincaré ball model $\mathbb{B}^N$ in this paper.

The Poincaré ball model for the hyperbolic space is

$$\mathbb{B}^N = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^N : |x| < 1\}$$

endowed with a Riemannian metric g given by $g_{ij} = (p(x))^2 \delta_{ij}$, where $p(x) = \frac{2}{1-|x|^2}$.

We denote the hyperbolic volume by $dV_{\mathbb{B}^N}$ and it is given by $dV_{\mathbb{B}^N} = (p(x))^N\ dx$. The hyperbolic gradient and the Laplace–Beltrami operator are

$$\Delta_{\mathbb{B}^N} = (p(x))^{-N}\mathrm{div}((p(x))^{N-2}\nabla u), \quad \nabla_{\mathbb{B}^N} u = \frac{\nabla u}{p^2(x)}$$

where ∇ and div denote the Euclidean gradient and divergence in $\mathbb{R}^N$, respectively.

For each $a \in \mathbb{B}^N$, we define the Möbius transformations T_a by

$$T_a(x) = \frac{(1-|a|^2)(x-a) - |x-a|^2 a}{1 - 2x\cdot a + |x|^2|a|^2},$$

where $x \cdot a = x_1 a_1 + x_2 a_2 + \cdots + x_N a_N$ denotes the scalar product in $\mathbb{R}^N$. It is known that the measure on $\mathbb{B}^N$ is invariant with respect to the Möbius transformations. Using the Möbius transformations, we can define the distance from x to y in $\mathbb{B}^N$ as follows:

$$\rho(x,y) = \rho(O, T_x(y)) := \rho(T_x(y)) = \rho(T_y(x)) = \log\frac{1+|T_y(x)|}{1-|T_y(x)|}.$$

A simple calculation shows that

$$\begin{aligned} T_a(T_a(x)) = x, \quad 1 - |T_a(x)|^2 &= \frac{(1-|a|^2)(1-|x|^2)}{1-2x\cdot a + |x|^2|a|^2}, \\ |T_a(x)| &= \frac{|x-a|}{\sqrt{1-2x\cdot a+|x|^2|a|^2}}, \\ \sinh\frac{\rho(T_a(x))}{2} &= \frac{|T_a(x)|}{\sqrt{1-|T_a(x)|^2}} = \frac{|x-a|}{\sqrt{(1-|x|^2)(1-|a|^2)}}. \end{aligned}$$

Thanks to the Poincaré inequality

$$\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2\ dV_x \geq \frac{(N-1)^2}{4}\int_{\mathbb{B}^N} |u|^2\ dV_x, \quad u \in H^1(\mathbb{B}^N),$$

then if $\lambda < \frac{(N-1)^2}{4}$, it follows that

$$\|u\|_\lambda = \left(\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u(x)|^2 - \lambda u^2(x)\ dV_x\right)^{1/2}, \quad u \in C_0^\infty(\mathbb{B}^N),$$

is a norm equivalent to the $H^1(\mathbb{B}^N)$ norm. If $\lambda = \frac{(N-1)^2}{4}$, by the sharp Poincaré inequality

$$S_{\lambda,q}\left(\int_{\mathbb{B}^N} |u|^q\ dV_x\right)^{\frac{2}{q}} \leq \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u(x)|^2\ - \frac{(N-1)^2}{4}u^2(x)\ dV_x, \quad u \in C_0^\infty(\mathbb{B}^N).$$

it follows that $\|u\|_{\frac{(N-1)^2}{4}}$ is a norm as well on $C_0^\infty(\mathbb{B}^N)$, where $N \geq 3, q \in (2, \frac{2N}{N-2}]$ and $S_{\lambda,q} > 0$.

When $\lambda \leq \frac{(N-1)^2}{4}$, let $\mathcal{H}_\lambda(\mathbb{B}^N)$ denote the completion of $C_0^\infty(\mathbb{B}^N)$ with respect to the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u(x)|^2 - \lambda u^2(x)\ dV_x.$$

Observe that

$$S_{\lambda,q} \left(\int_{\mathbb{B}^N} |u|^q\ dV_x \right)^{\frac{2}{q}} \leq \|u\|_\lambda, \quad q \in \left(2, \frac{2N}{N-2}\right] \quad \text{for } u \in \mathcal{H}_\lambda(\mathbb{B}^N).$$

Denote

$$I(u) = \frac{\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u(x)|^2 - \lambda u^2(x)\ dV_x}{\left(\int_{\mathbb{B}^N} \int_{\mathbb{B}^N} \frac{|u(x)|^p |u(y)|^p}{|2\sinh \frac{\rho(T_y(x))}{2}|^\mu}\ dV_x\ dV_y \right)^{\frac{1}{p}}}, \quad u \in \mathcal{H}_\lambda(\mathbb{B}^N) \setminus \{0\},$$

and

$$\xi_{\mu,p}(\mathbb{B}^N) = \inf_{u \in \mathcal{H}_\lambda(\mathbb{B}^N)} I(u).$$

By the Hardy–Littlewood–Sobolev inequality (1.5) and the Poincaré–Sobolev inequality, it holds

$$\xi_{\mu,p}(\mathbb{B}^N) \geq C(N,\mu) S_{\lambda, p\cdot\frac{2N}{2N-\mu}} > 0$$

for $\frac{2N-\mu}{N} < p \leq \frac{2N-\mu}{N-2}$, $N \geq 3$.

Because problem (1.1) is invariant under isometry groups of $\mathbb{B}^N$ and the conformal group of $\mathbb{B}^N$ is the same as the isometry group, firstly for $r > 0$, define

$$S_r = \{x \in \mathbb{R}^N : |x|^2 = 1 + r^2\}$$

and for $a \in S_r$ define

$$A(a,r) = B(a,r) \cap \mathbb{B}^N$$

where $B(a,r)$ is the open ball in the Euclidean space with center a and radius $r > 0$. Moreover, for the choice of a and r, $\partial B(a,r)$ is orthogonal to S^{N-1}.

Similarly to [1], we have the following result:

Lemma 2.1. *Let $r_1 > 0$, $r_2 > 0$ and $A(a_i, r_i)$, $i = 1, 2$, be as in the above definition, then there exists $\tau \in \mathcal{I}(\mathbb{B}^N)$ such that $\tau(A(a_1, r_1)) = A(a_2, r_2)$, where $\mathcal{I}(\mathbb{B}^N)$ is the isometry group of $\mathbb{B}^N$.*

Secondly, we need the following Brezis–Lieb lemma:

Lemma 2.2. *Let* $0 < \mu < N, p \in \left(\frac{2N-\mu}{N}, 2_\mu^*\right]$ *and* $\{u_n\}$ *be a bounded sequence in* $L^{p\cdot\frac{2N}{2N-\mu}}(\mathbb{B}^N)$ *such that* $u_n \to u$ *almost in* $\mathbb{B}^N$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty}\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N}\frac{|u_n(y)|^p|u_n(x)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x - \int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N}\frac{|u_n(y)-u(y)|^p|u_n(x)-u(x)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x = \int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N}\frac{|u(y)|^p|u(x)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x.$$

Proof. Firstly, we notice that

$$\begin{aligned}
&\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N}\frac{|u_n(y)|^p|u_n(x)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x\\
&-\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N}\frac{|u_n(y)-u(y)|^p|u_n(x)-u(x)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x\\
&=\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N}\frac{(|u_n(y)|^p-|u_n(y)-u(y)|^p)(|u_n(x)|^p-|u_n(x)-u(x)|^p)}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x\\
&\quad+2\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N}\frac{(|u_n(y)|^p-|u_n(y)-u(y)|^p)(|u_n(x)-u(x)|^p)}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x.
\end{aligned}\tag{2.1}$$

Secondly, from $u_n \in L^{p\cdot\frac{2N}{2N-\mu}}(\mathbb{B}^N)$, similarly as in the Brézis–Lieb lemma, one has

$$|u_n-u|^p-|u_n|^p \to |u|^p \quad \text{in } L^{\frac{2N}{2N-\mu}}(\mathbb{B}^N). \tag{2.2}$$

By the Hardy–Littlewood–Sobolev inequality (1.5), this implies that

$$\int\limits_{\mathbb{B}^N}\frac{|u_n(y)-u(y)|^p-|u_n(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y \to \int\limits_{\mathbb{B}^N}\frac{|u|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y \quad \text{in } L^{\frac{2N}{\mu}}(\mathbb{B}^N). \tag{2.3}$$

Moreover, for $u_n \in L^{p\cdot\frac{2N}{2N-\mu}}(\mathbb{B}^N)$ and $u_n(x) \to u(x)$ a.e., we have

$$|u_n-u|^p \rightharpoonup 0 \quad \text{weakly in } L^{\frac{2N}{2N-\mu}}(\mathbb{B}^N). \tag{2.4}$$

From (2.1)–(2.4) we obtain the conclusion. □

3. EXISTENCE RESULT FOR $\frac{2N-\mu}{N} < p < \frac{2N-\mu}{N-2}$

In this section, we will be concerned with the proof of Theorem 1.2.

Lemma 3.1. *Let* $0 < \mu < N$ *and* $\frac{2N-\mu}{N} < p < 2^*_\mu = \frac{2N-\mu}{N-2}$ *if* $N \geq 3$. *Then for* $\lambda \leq \frac{(N-1)^2}{4}$, $\xi_{\mu,p}(\mathbb{B}^N)$ *is attained by some nonnegative function in* $\mathcal{H}_\lambda(\mathbb{B}^N)$.

Proof. Let

$$\mathrm{K} := \left\{ u \in \mathcal{H}_\lambda(\mathbb{B}^N) \,:\, \|u\|_\lambda^2 = \int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|u(x)|^p|u(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\, dV_x\, dV_y,\ u \neq 0 \right\}$$

be the Nehari manifold. We have that

$$\xi_{\mu,p}(\mathbb{B}^N) = \inf_{u\in\mathcal{H}_\lambda(\mathbb{B}^N)} I(u) = \inf_{u\in \mathrm{K}} I(u),$$

$$I(u) = \|u\|_\lambda^{\frac{2(p-1)}{p}} = \left(\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|u(x)|^p|u(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\, dV_x\, dV_y \right)^{\frac{p-1}{p}}.$$

Now we claim that $\xi_{\mu,p}(\mathbb{B}^N)$ is achieved. Choosing a minimizing sequence $\{u_n\}$ of $\xi_{\mu,p}(\mathbb{B}^N)$ in K. Clearly, $\{u_n\}$ is bounded in $\mathcal{H}_\lambda(\mathbb{B}^N)$ and

$$\|u_n\|_\lambda^{\frac{2(p-1)}{p}} = \left(\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\, dV_x\, dV_y \right)^{\frac{p-1}{p}} \to \xi_{\mu,p}(\mathbb{B}^N) \quad \text{as } n\to\infty.$$

In order to show that $\xi_{\mu,p}(\mathbb{B}^N)$ is achieved, it is enough to exhibit a minimizing sequence $\{u_n\} \subset \mathrm{K}$ such that $u_n(x) \to u(x)$ for a.e. x for some $u \in \mathrm{K}$.

Then, we will show that, up to isometry group of $\mathbb{B}^N$, $\{u_n\}$ converges weakly, and pointwise, to some $u \in \mathrm{K}$.

(i) In this step we will prove the theorem when $u = 0$.

Since $\{u_n\}$ is the minimizing sequence of $\xi_{\mu,p}(\mathbb{B}^N) > 0$ and u_n does not converge strongly to zero we get

$$\liminf_{n\to\infty} \int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|u_n(x)|^p|u_n(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\, dV_x\, dV_y > \delta_1 > 0.$$

Then by the Hardy–Littlewood–Sobolev inequality (1.5), it implies that

$$\liminf_{n\to\infty} \int\limits_{\mathbb{B}^N} |u_n(x)|^{p\cdot\frac{2N}{2N-\mu}}\, dV_x > \delta_2 > 0.$$

Let us fix $\delta > 0$ such that

$$0 < 2\delta < \delta_2 < \left((C(N,\mu))^{-1} S^{1+p}_{\lambda,p\cdot\frac{2N}{2N-\mu}}\, \xi_{\mu,p}^{-\frac{p^2}{4(p-1)}} \right)^{\frac{p}{p-2}\frac{2N}{2N-\mu}}.$$

Now, we define the concentration function

$$Q_n : (0,+\infty) \to \mathbb{R},$$
$$Q_n(r) = \sup_{x\in S_r} \int\limits_{A(x,r)} |u_n|^{p\cdot\frac{2N}{2N-\mu}}\ dV_x.$$

Then, $\lim\limits_{r\to 0} Q_n(r) = 0$, and $\lim\limits_{r\to\infty} Q_n(r) > \delta$ for large r. $A(x,r)$ approximates the intersection of $\mathbb{H}^N$ with a half space $\{y \in \mathbb{R}^N : \langle y, x\rangle > 0\}$. Therefore, we can choose $R_n > 0$ and $x_n \in S_{R_n}$ such that

$$\sup_{x\in S_{R_n}} \int\limits_{A(x,R_n)} |u_n|^{p\cdot\frac{2N}{2N-\mu}}\ dV_x = \delta.$$

For $x_0 \in S_{\sqrt{3}}$, using Lemma 2.1 let us choose $T_n \in I(\mathbb{B}^N)$ such that

$$A(x_n, R_n) = T_n(A(x_0, \sqrt{3})).$$

Now define $v_n(x) = u_n \circ T_n(x)$. Since T_n is an isometry one can easily see that $\{v_n(x)\} \subset$ K and is again minimizing

$$\|v_n\|_\lambda^{\frac{2(p-1)}{p}} = \left(\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|v_n(x)|^p|v_n(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\ dV_x\ dV_y\right)^{\frac{p-1}{p}} \to \xi_{\mu,p}(\mathbb{B}^N) \quad \text{as } n\to\infty.$$

Moreover,

$$\begin{aligned}\int\limits_{A(x_0,\sqrt{3})} |v_n|^{p\cdot\frac{2N}{2N-\mu}}\ dV_x &= \int\limits_{A(x_n,R_n)} |u_n|^{p\cdot\frac{2N}{2N-\mu}}\ dV_x \\ &= \sup_{x\in S_{\sqrt{3}}} \int\limits_{A(x,\sqrt{3})} |v_n|^{p\cdot\frac{2N}{2N-\mu}}\ dV_x = \delta.\end{aligned} \tag{3.1}$$

By the Ekeland principle, we may assume $\{v_n\}$ is a Palais–Smale sequence, i.e.

$$\langle v_n, u\rangle_\lambda = \int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|v_n(y)|^p|v_n(x)|^{p-2}v_n(x)u}{|2\sinh\frac{p(T_y(x))}{2}|^\mu}\ dV_y\ dV_x + o(1)$$

uniformly for u in bounded sets of $\mathcal{H}_\lambda$. Thus, up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $\mathcal{H}_\lambda(\mathbb{B}^N)$. Moreover, we get

$$\|v\|_\lambda^2 = \int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|v(x)|^p|v(y)|^p}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\ dV_x\ dV_y.$$

Therefore, it remains to show that $v \neq 0$.

(ii) Assume, by contradiction, that $v = 0$. We claim that for any $1 > r > 2 - \sqrt{3}$,

$$\int\limits_{\mathbb{B}^N \cap \{|x| \geq r\}} |v_n|^{p \cdot \frac{2N}{2N-\mu}} \, dV_x = o(1).$$

To show this, let us fix a point $a \in S_{\sqrt{3}}$. Let $\Phi \in C_0^\infty(A(a, \sqrt{3}))$ be such that $0 \leq \Phi \leq 1$, where $A(a, \sqrt{3}) = B(a, \sqrt{3}) \cap \mathbb{B}^N$ and $B(a, \sqrt{3})$ is the Euclidean ball with the center a and the radius $\sqrt{3}$, and

$$\begin{aligned} &\int\limits_{\mathbb{B}^N} \nabla_{\mathbb{B}^N} v_n \nabla_{\mathbb{B}^N} \Psi - \lambda v_n \Psi \, dV_x \\ &= \int\limits_{\mathbb{B}^N} \int\limits_{\mathbb{B}^N} \frac{|v_n(y)|^p |v_n(x)|^{p-2} v_n(x) \Psi}{|2 \sinh \frac{p(T_y(x))}{2}|^\mu} \, dV_y \, dV_x + o(1) \|\Psi\| \end{aligned}$$

for $\Psi \in \mathcal{H}_\lambda(\mathbb{B}^N)$.

Now putting $\Psi = \Phi^2 v_n$, in the above identity, we get

$$\begin{aligned} &\int\limits_{\mathbb{B}^N} \nabla_{\mathbb{B}^N} v_n \nabla_{\mathbb{B}^N} (\Phi^2 v_n) - \lambda v_n \Phi^2 v_n \, dV_x \\ &= \int\limits_{\mathbb{B}^N} \int\limits_{\mathbb{B}^N} \frac{|v_n(y)|^p |v_n(x)|^{p-2} v_n(x) \Phi^2 v_n}{|2 \sinh \frac{p(T_y(x))}{2}|^\mu} \, dV_y \, dV_x + o(1). \end{aligned}$$

From the fact $v_n \rightharpoonup v = 0$, a simple computation gives

$$\begin{aligned} &\int\limits_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} (\Phi v_n)|^2 - \lambda (\Phi v_n)^2 \, dV_x \\ &= \int\limits_{\mathbb{B}^N} \int\limits_{\mathbb{B}^N} \frac{|v_n(y)|^p |v_n(x)|^{p-2} (v_n(x) \Phi)^2}{|2 \sinh \frac{p(T_y(x))}{2}|^\mu} \, dV_y \, dV_x + o(1). \end{aligned} \tag{3.2}$$

Now, using (3.2), the Hölder inequality, the Hardy–Littlewood–Sobolev inequality (1.5) and the Poincaré-Sobolev inequality, we get

$$\begin{aligned}
& S_{\lambda,p\cdot\frac{2N}{2N-\mu}}\left(\int\limits_{\mathbb{B}^N}|\Phi v_n(x)|^{p\cdot\frac{2N}{2N-\mu}}\,dV_x\right)^{\frac{2}{p\cdot\frac{2N}{2N-\mu}}}\\
&\leq C(N,\mu)\left(\int\limits_{\mathbb{B}^N}|v_n(y)|^{p\cdot\frac{2N}{2N-\mu}}\,dV_y\right)^{\frac{2N-\mu}{2N}}\\
&\quad\cdot\left(\int\limits_{\mathbb{B}^N}[|v_n(x)|^{p-2}(v_n(x)\Phi)^2]^{\frac{2N}{2N-\mu}}\,dV_x\right)^{\frac{2N-\mu}{2N}}\\
&\leq C(N,\mu)S^{-p}_{\lambda,p\cdot\frac{2N}{2N-\mu}}\|u_n\|_\lambda^{\frac{p}{2}}\cdot\left(\int\limits_{\mathbb{B}^N}[|v_n(x)|^{p-2}(v_n(x)\Phi)^2]^{\frac{2N}{2N-\mu}}\,dV_x\right)^{\frac{2N-\mu}{2N}}\\
&\leq C(N,\mu)S^{-p}_{\lambda,p\cdot\frac{2N}{2N-\mu}}\xi_{\mu,p}^{\frac{p^2}{4(p-1)}}\cdot\left(\int\limits_{\mathbb{B}^N}|\Phi v_n(x)|^{p\cdot\frac{2N}{2N-\mu}}\,dV_x\right)^{\frac{2}{p}\frac{2N-\mu}{2N}}\\
&\quad\cdot\left(\int\limits_{A(a,\sqrt{3})}|v_n(x)|^{p\cdot\frac{2N}{2N-\mu}}\,dV_x\right)^{\frac{p-2}{p}\frac{2N-\mu}{2N}}.
\end{aligned}$$

Now if

$$\int\limits_{\mathbb{B}^N}|\Phi v_n(y)|^{p\cdot\frac{2N}{2N-\mu}}\,dV_y\not\to 0,$$

we get

$$\begin{aligned}
\delta^{\frac{p-2}{p}\frac{2N-\mu}{2N}} &> \left(\int\limits_{A(a,\sqrt{3})}|v_n(x)|^{p\cdot\frac{2N}{2N-\mu}}\,dV_x\right)^{\frac{p-2}{p}\frac{2N-\mu}{2N}}\\
&\geq (C(N,\mu))^{-1}S^{1+p}_{\lambda,p\cdot\frac{2N}{2N-\mu}}\xi_{\mu,p}^{-\frac{p^2}{4(p-1)}} > \delta^{\frac{p-2}{p}\frac{2N-\mu}{2N}},
\end{aligned}$$

which is a contradiction. This implies that

$$\int\limits_{\mathbb{B}^N}|\Phi v_n(y)|^{p\cdot\frac{2N}{2N-\mu}}\,dV_y\to 0.$$

Since $a\in S_{\sqrt{3}}$ is arbitrary, the claim follows.

If $\frac{2N-\mu}{N} < p < \frac{2N-\mu}{N-2}$, this together with the fact that $v_n \to 0$ in $L_{loc}^{p\cdot\frac{2N}{2N-\mu}}(\mathbb{B}^N)$ immediately gives a contradiction to (3.1). Hence $v \neq 0$ and $v \in \mathrm{K}$. □

Proof of Theorem 1.2. It is easy to see that the minimizer u (also $|u|$) for $\xi_{\mu,p}(\mathbb{B}^N)$, up to a constant multiplier, satisfies equation (1.1). By the strong maximum principle, either $|u| > 0$ or $|u| = 0$. Since $u \neq 0$, we conclude that there exists a positive solution for equation (1.1). □

4. EXISTENCE RESULT FOR $p = \frac{2N-\mu}{N-2}$

Notice that, by the Hardy-Littlewood–Sobolev inequality (1.5), there holds

$$\left(\int_{\mathbb{B}^N}\int_{\mathbb{B}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x\right)^{\frac{N-2}{2N-\mu}} \leq (C(N,\mu))^{\frac{N-2}{2N-\mu}}\,\|u\|^2_{L^{\frac{2N}{N-2}}(\mathbb{B}^N)}. \tag{4.1}$$

From the Poincaré inequality, $(\int_{\mathbb{B}^N}|\nabla_{\mathbb{B}^N}u|^2\,dV_x)^{\frac{1}{2}}$ is a norm equivalent to the $H^1(\mathbb{B}^N)$ norm. Now, we denote

$$S_{H,\mathbb{B}^N} = \inf_{u\in H^1(\mathbb{B}^N)\setminus\{0\}} \frac{\int_{\mathbb{B}^N}|\nabla_{\mathbb{B}^N}u|^2\,dV_x}{\left(\int_{\mathbb{B}^N}\int_{\mathbb{B}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu}\,dV_y\,dV_x\right)^{\frac{N-2}{2N-\mu}}}.$$

Then we have the following lemma:

Lemma 4.1. *Let $N \geq 3$, we have*

$$\begin{aligned} S_{H,\mathbb{B}^N} &= \frac{S}{(C(N,\mu))^{\frac{N-2}{2N-\mu}}} \\ &= S_{H,\mathbb{R}^N} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}|\nabla u|^2\,dx}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\,dx\right)^{\frac{N-2}{2N-\mu}}} \end{aligned}$$

and $S_{H,\mathbb{B}^N}$ is never achieved on the hyperbolic space $\mathbb{B}^N$, where S is the best Sobolev constant on $\mathbb{R}^N$.

Proof. By the Hardy–Littlewood–Sobolev inequality (1.5) and Lemma 1.2 of [6], we obtain

$$S_{H,\mathbb{B}^N} \geq \frac{1}{(C(N,\mu))^{\frac{N-2}{2N-\mu}}} \inf_{u \in H^1(\mathbb{B}^N)\setminus\{0\}} \frac{\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u|^2 \, dV_x}{\left(\int_{\mathbb{B}^N} |u(x)|^{\frac{2N}{N-2}} \, dV_x \right)^{\frac{N-2}{N}}}$$

$$= \frac{1}{(C(N,\mu))^{\frac{N-2}{2N-\mu}}} S = S_{H,\mathbb{R}^N},$$

where S is the best Sobolev constant in $\mathbb{R}^N$.

Now we will prove that

$$S_{H,\mathbb{B}^N} \leq \frac{1}{(C(N,\mu))^{\frac{N-2}{2N-\mu}}} S.$$

Let $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ be a minimizing sequence for $S_{H,\mathbb{R}^N}$, we make translations and dilations for $\{u_n\}$ by choosing $y_n \in B_1(0)$ and $\tau_n > 0$ such that

$$u_n^{y_n,\tau_n}(x) = \tau_n^{\frac{N-2}{2}} u_n(\tau_n x + y_n) \in C_0^1(B_1(0))$$

which satisfies

$$\int_{B_1(0)} \int_{B_1(0)} \frac{|u_n^{y_n,\tau_n}(x)|^{2_\mu^*} |u_n^{y_n,\tau_n}(y)|^{2_\mu^*}}{|x-y|^\mu} \, dy \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^*}}{|x-y|^\mu} \, dy \, dx, \tag{4.2}$$

$$\int_{B_1(0)} |\nabla u_n^{y_n,\tau_n}(x)|^2 \, dx = \int_{\mathbb{R}^N} |\nabla u_n(x)|^2 \, dx, \quad \int_{B_1(0)} \left(\frac{2}{1-|x|^2} \right)^2 |u_n^{y_n,\tau_n}(x)|^2 \, dx \to 0. \tag{4.3}$$

We denote

$$v_n^{y_n,\tau_n}(x) = \left(\frac{1-|x|^2}{2} \right)^{\frac{N-2}{2}} u_n^{y_n,\tau_n}(x).$$

Then

$$\int_{\mathbb{B}^N} \int_{\mathbb{B}^N} \frac{|v_n^{y_n,\tau_n}(x)|^{2_\mu^*} |v_n^{y_n,\tau_n}(y)|^{2_\mu^*}}{|2\sinh \frac{\rho(T_y(x))}{2}|^\mu} \, dV_y \, dV_x = \int_{B_1(0)} \int_{B_1(0)} \frac{|u_n^{y_n,\tau_n}(x)|^{2_\mu^*} |u_n^{y_n,\tau_n}(y)|^{2_\mu^*}}{|x-y|^\mu} \, dy \, dx \tag{4.4}$$

and

$$\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} v_n^{y_n,\tau_n}(x)|^2 \, dV_x = \int_{B_1(0)} |\nabla u_n^{y_n,\tau_n}(x)|^2 \, dx + \frac{N(N-2)}{4} \int_{B_1(0)} \left(\frac{2}{1-|x|^2} \right)^2 |u_n^{y_n,\tau_n}(x)|^2 \, dx. \tag{4.5}$$

From (4.4) and (4.5), we have

$$S_{H,\mathbb{B}^N} \le \frac{\int\limits_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} v_n^{y_n,\tau_n}(x)|^2 \, dV_x}{\left(\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|v_n^{y_n,\tau_n}(x)|^{2^*_\mu}|v_n^{y_n,\tau_n}(y)|^{2^*_\mu}}{|2\sinh\frac{\rho(T_y(x))}{2}|^\mu} \, dV_y \, dV_x\right)^{\frac{N-2}{2N-\mu}}}$$
$$\to S_{H,\mathbb{R}^N} = \frac{1}{(C(N,\mu))^{\frac{N-2}{2N-\mu}}} S.$$

Due to the fact $S_{H,\mathbb{R}^N}$ is never achieved except when $\Omega = \mathbb{R}^N$, then $S_{H,\mathbb{B}^N}$ is never achieved on the hyperbolic space $\mathbb{B}^N$. □

Now, we shall establish the existence results for weak solutions to (1.1) for $p = \frac{2N-\mu}{N-2}$. To this end, we consider

$$\xi_\mu(\mathbb{B}^N) := \xi_{\mu,\frac{2N-\mu}{N-2}}(\mathbb{B}^N) = \inf_{u\in\mathcal{H}_\lambda(\mathbb{B}^N)} I(u).$$

First, we show the following lemma.

Lemma 4.2. *Let $N \ge 4$, and $0 < \mu < N$. Then*

$$\xi_\mu(\mathbb{B}^N) < S_{H,\mathbb{B}^N}$$

for all $\frac{N(N-2)}{4} < \lambda \le \frac{(N-1)^2}{4}$.

Proof. Let $\phi \in C_0^\infty(\mathbb{B}^N)$ such that $0 \le \phi \le 1$ and $\phi = 1$ on $|x| < r$, where $0 < r < 1$. Define v_ε as

$$v_\varepsilon(x) = \phi(x)(N(N-2))^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-2}{2}}.$$

Define

$$u_\varepsilon(x) = \left(\frac{1-|x|^2}{2}\right)^{\frac{N-2}{2}} v_\varepsilon(x).$$

Then we recall some results from [1,3,22]:

(1) $\int\limits_{\mathbb{B}^N} |\nabla v_\varepsilon|^2 \, dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2}) = C(N,\mu)^{\frac{(N-2)N}{(2N-\mu)2}} S_{H,\mathbb{B}^N}^{\frac{N}{2}} + O(\varepsilon^{N-2}),$

(2) $\int\limits_{\mathbb{B}^N} v_\varepsilon^2 \Big(\frac{2}{1-|x|^2}\Big)^2 \, dx = \begin{cases} \varepsilon^2 + O(\varepsilon^{N-2}), & N \ge 5, \\ d\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2), & N = 4, \end{cases}$ where d is a positive constant,

(3) $\left(\int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|v_\varepsilon(x)|^{2^*_\mu}|v_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu} \, dx \, dy\right)^{\frac{N-2}{2N-\mu}} = C(N,\mu)^{\frac{(N-2)N}{(2N-\mu)2}} S_{H,\mathbb{B}^N}^{\frac{N}{2}} + O(\varepsilon^{N-2}).$

Then, it holds

$$
\begin{aligned}
&\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u_\varepsilon(x)|^2 \, dV_x - \lambda \int_{\mathbb{B}^N} u_\varepsilon^2(x) \, dV_x \\
&= \int_{\mathbb{B}^N} |\nabla v_\varepsilon(x)|^2 \, dx - \Big(\lambda - \frac{N(N-2)}{4}\Big) \int_{\mathbb{B}^N} \Big(\frac{2}{1-|x|^2}\Big)^2 v_\varepsilon^2(x) \, dx \\
&= C(N,\mu)^{\frac{(N-2)N}{(2N-\mu)2}} S_{H,\mathbb{B}^N}^{\frac{N}{2}} - \Big(\lambda - \frac{N(N-2)}{4}\Big) d\varepsilon^2 + O(\varepsilon^{N-2}), \text{ if } N \geq 5,
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
&\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u_\varepsilon(x)|^2 \, dV_x - \lambda \int_{\mathbb{B}^N} u_\varepsilon^2(x) \, dV_x \\
&= \int_{\mathbb{B}^N} |\nabla v_\varepsilon(x)|^2 \, dx - \Big(\lambda - \frac{N(N-2)}{4}\Big) \int_{\mathbb{B}^N} \Big(\frac{2}{1-|x|^2}\Big)^2 v_\varepsilon^2(x) \, dx \\
&= C(N,\mu)^{\frac{(N-2)N}{(2N-\mu)2}} S_{H,\mathbb{B}^N}^{2} - \Big(\lambda - \frac{N(N-2)}{4}\Big) d\varepsilon^2 |ln\varepsilon| + O(\varepsilon^2), \text{ if } N = 4,
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
&\left(\int_{\mathbb{B}^N} \int_{\mathbb{B}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*} |u_\varepsilon(y)|^{2_\mu^*}}{|2\sinh \frac{p(T_y(x))}{2}|^\mu} \, dV_x \, dV_y \right)^{\frac{N-2}{2N-\mu}} \\
&= \left(\int_{\mathbb{B}^N} \int_{\mathbb{B}^N} |u_\varepsilon(x)|^{2_\mu^*} \Big(\frac{2}{1-|x|^2}\Big)^{N-\mu/2} |x-y|^{-\mu} \Big(\frac{2}{1-|y|^2}\Big)^{N-\mu/2} |u_\varepsilon(y)|^{2_\mu^*} \, dy \, dx \right)^{\frac{N-2}{2N-\mu}} \\
&= \left(\int_{\mathbb{B}^N} \int_{\mathbb{B}^N} |v_\varepsilon(x)|^{2_\mu^*} |x-y|^{-\mu} |v_\varepsilon(y)|^{2_\mu^*} \, dy \, dx \right)^{\frac{N-2}{2N-\mu}} \\
&= \left(C(N,\mu)^{\frac{N}{2}} S_{H,\mathbb{B}^N}^{\frac{2N-\mu}{2}} + O(\varepsilon^{\frac{N-2}{2N-\mu}}) \right)^{\frac{N-2}{2N-\mu}}.
\end{aligned} \tag{4.8}
$$

If $N \geq 5$, using (4.6) and (4.8), we have

$$
\begin{aligned}
&\frac{\displaystyle\int_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u_\varepsilon(x)|^2 \, dV_x - \lambda \int_{\mathbb{B}^N} u_\varepsilon^2(x) \, dV_x}{\left(\displaystyle\int_{\mathbb{B}^N} \int_{\mathbb{B}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*} |u_\varepsilon(y)|^{2_\mu^*}}{|2\sinh \frac{p(T_y(x))}{2}|^\mu} \, dV_x \, dV_y \right)^{\frac{2}{8-\mu}}} \\
&\leq S_{H,\mathbb{B}^N} - \Big(\lambda - \frac{N(N-2)}{4}\Big) d\varepsilon^2 |\ln\varepsilon| + O(\varepsilon^2) < S_{H,\mathbb{B}^N}.
\end{aligned} \tag{4.9}
$$

If $N = 4$, we can also get

$$\begin{aligned}
&\frac{\int\limits_{\mathbb{B}^N} |\nabla_{\mathbb{B}^N} u_\varepsilon(x)|^2 \, dV_x - \lambda \int\limits_{\mathbb{B}^N} u_\varepsilon^2(x) \, dV_x}{\left(\int\limits_{\mathbb{B}^N} \int\limits_{\mathbb{B}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*} |u_\varepsilon(y)|^{2_\mu^*}}{|2 \sinh \frac{p(T_y(x))}{2}|^\mu} \, dV_x \, dV_y \right)^{\frac{N-2}{2N-\mu}}} \\
&= \frac{C(N,\mu)^{\frac{(N-2)N}{(2N-\mu)2}} S_{H,\mathbb{B}^N}^{\frac{N}{2}} - (\lambda - \frac{N(N-2)}{4}) d\varepsilon^2 + O(\varepsilon^{N-2})}{\left(C(N,\mu)^{\frac{N}{2}} S_{H,\mathbb{B}^N}^{\frac{2N-\mu}{2}} + O(\varepsilon^{\frac{N-2}{2N-\mu}}) \right)^{\frac{N-2}{2N-\mu}}} \\
&\le S_{H,\mathbb{B}^N} - \Big(\lambda - \frac{N(N-2)}{4} \Big) d\varepsilon^2 + O(\varepsilon^{\frac{N}{2}}) < S_{H,\mathbb{B}^N}.
\end{aligned} \tag{4.10}$$

By (4.9) and (4.10), we finish the proof. □

To show the existence of weak solutions for (1.1), we will prove the following lemma for the existence of a minimizer for $\xi_\mu(\mathbb{B}^N)$.

Lemma 4.3. *Let $N \ge 4$, $0 < \mu < N$ and $\frac{N(N-2)}{4} < \lambda \le \lambda_1 := \frac{(N-1)^2}{4}$. If $\xi_\mu(\mathbb{B}^N) < S_{H,\mathbb{B}^N}$, then $\xi_\mu(\mathbb{B}^N)$ is achieved by a positive function $u \in \mathcal{H}_\lambda(\mathbb{B}^N)$.*

Proof. Similar as Lemma 3.1, we also have

$$\int\limits_{\mathbb{B}^N} |\Phi v_n(y)|^{\frac{2N}{N-2}} \, dV_y = \int\limits_{\mathbb{B}^N} |\Phi v_n(y)|^{2_\mu^* \cdot \frac{2N}{2N-\mu}} \, dV_y \to 0. \tag{4.11}$$

However, we do not have $v_n \to v = 0$ in $L_{loc}^{p \cdot \frac{2N}{2N-\mu}}(\mathbb{B}^N)$ if $p = \frac{2N-\mu}{N-2}$.

If $v = 0$, fix $2 - \sqrt{3} < r < R < 1$ and $\varphi \in C_0^\infty(\mathbb{B}^N)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| < r$ and $\varphi(x) = 0$ for $|x| > R$. Define

$$w_n = \varphi v_n,$$

then by (4.11), we have

$$I(w_n) \to \xi_\mu(\mathbb{B}^N)$$

and w_n has compact support in $\mathbb{B}^N$, hence $\omega_n \in H^1(\mathbb{B}^N)$ (see Lemma 2.3 of [10]) and

$$\lim_{n \to \infty} \int\limits_{\mathbb{B}^N} w_n^2 \, dV_x = 0.$$

Now, we denote

$$\tilde{w}_n(x) = \left(\frac{2}{1 - |x|^2} \right)^{\frac{N-2}{2}} w_n(x).$$

Then $\tilde{w}_n(x) \in H_0^1(B_R(0))$ and

$$\xi_\mu(\mathbb{B}^N) = \lim_{n\to\infty} I(w_n) = \frac{\int\limits_{B_R(0)} |\nabla \tilde{w}_n(x)|^2 - \Big(\lambda - \frac{N(N-2)}{4}\Big)\Big(\frac{2}{1-|x|^2}\Big)^2 \tilde{w}_n^2(x)\, dx}{\left(\int\limits_{B_R(0)}\int\limits_{B_R(0)} \frac{|\tilde{w}_n(x)|^{2^*_\mu}|\tilde{w}_n(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy\, dx\right)^{\frac{N-2}{2N-\mu}}}.$$

Since

$$\lim_{n\to\infty}\int\limits_{B_R(0)} \Big(\frac{2}{1-|x|^2}\Big)^2 \tilde{w}_n^2(x)\, dx = \lim_{n\to\infty}\int\limits_{\mathbb{B}^N} w_n^2\, dV_x = 0,$$

and

$$\int\limits_{B_R(0)}\int\limits_{B_R(0)} \frac{|\tilde{w}_n(x)|^{2^*_\mu}|\tilde{w}_n(y)|^{2^*_\mu}}{|2\sinh\frac{p(T_y(x))}{2}|^\mu}\, dV_y\, dV_x$$
$$= \int\limits_{\mathbb{B}^N}\int\limits_{\mathbb{B}^N} \frac{|w_n(x)|^{2^*_\mu}|w_n(y)|^{2^*_\mu}}{|x-y|^\mu}\, dy\, dx \to \big(\xi_\mu(\mathbb{B}^N)\big)^{\frac{p}{p-1}},$$

then we obtain

$$\xi_\mu(\mathbb{B}^N) \geq S_{H,\mathbb{R}^N} = S_{H,\mathbb{B}^N}.$$

It is a contradiction. Thus $v \neq 0$, similarly as in [1] and [10], this implies that $\xi_\mu(\mathbb{B}^N)$ is achieved.

□

Proof of Theorem 1.3. Similarly as in the proof of Theorem 1.2, there exists a positive solution for equation (1.1) if $p = \frac{2N-\mu}{N-2}$. □

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