ISSN: 1231-4005
e-ISSN: 2354-0133
DOI: 10.5604/12314005.1161825

IMPACT OSCILLATORS: FUNDAMENTALS AND APPLICATIONS

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Abstract

The paper presents results of the analysis of nonlinear oscillators with nonsmooth elements and nonlinear systems with nonsmooth forcing components. Main attention has been focused on highlighting specific properties of nonsmooth systems compared to their smooth counterparts. Nonsmooth transformation of the time variable and the replacement of initial issues by boundary problems have been taken as the base for the analytical method. Results of numerical simulations and computing in the form of graphs of displacements and velocity waveforms and attractors are presented. To fully identify the system's behaviour and meet high performance specifications recourse to model all dynamics together with their interactions has been taken into account. Strong interactions among the parts of the system are considered and the phenomenon of the impact is exhibited. It has been found that non-smooth dynamical systems reveal significant wealth of nonlinear phenomena, including a chaotic, that are unique to this potentially important class of nonlinear systems. In non-smooth systems at small change of parameters, a sudden transition from a stable periodic oscillation to the full range of chaotic oscillations may often occur. The dynamics of nonsmooth oscillations with shock external forcing is analysed by using a relatively new mathematical tool, which appears to be hyperbolic algebra. The key idea of this tool is steeped in of non-smooth time transformations (NSTT) for strongly nonlinear, but still smooth models.

Keywords: *nonlinear systems, impact oscillators, nonsmooth excitations, nonsmooth time transformation, hyperbolic numbers*

1. Introduction

Nonlinear dynamical systems are common models for many problems in physics, engineering, chemistry, biology, medicine and social sciences [1–3]. The rapid development of the present technology and the ever-increasing requirements for installed devices imply stimulation of research centres not only to design new practical systems, but also to search for new components with improved operational characteristics compared to the previously used. A special attention of the search is focused on an analysis of the nonlinear low dimensional systems, which is motivated by the following attempts:

- ❖ the low dimensional systems may exhibit very complex dynamics,
- ❖ the fundamental behaviour of nonlinear high dimensional systems can be successfully modelled by the systems of low dimensions,
- ❖ a concept of the nonlinear normal modes very often allows to reduce a high dimensional system to that with a few degree of freedom only,
- ❖ recent results show that a nonsmooth transformation of the time variable can be effective in an exact analysis of nonsmooth nonlinear systems. In addition, the nonsmooth solutions with a constant period of oscillations can be established.

It is well known that possible transitions to nonsmooth limits can make investigations especially difficult. This is because the dynamic methods were originally developed within the paradigm of smooth motions based on the classical theory of differential equations. The corresponding solutions often include quasi-harmonic expansions as a generic feature that explicitly points to the physical basis of these methods namely - the harmonic oscillator. Some of the techniques are also applicable to dynamical systems close to integrable but not necessarily linear.

The main objective of this paper is to introduce a unified physical basis for analyses of vibrations with essentially unharmonic, non-smooth or may be discontinuous time shapes.

2. Basic properties of nonsmooth dynamical systems

Over the past decade, dynamical systems, in which nonsmooth signals are generated, have gained increasing recognition in electromechanical engineering and other applied sciences, as the nonsmooth effects can already be really taken into account and there is no need of smoothing. This is due to the fact that have been developed new analytical and numerical tools suited for testing of nonsmooth systems. This problem has steadily gaining. In importance as the applications for the construction smart devices, electrical and electronic components with increasingly merge, as well as less and less resistance to current and voltage surge pulses are growing constantly [11, 13]. A key feature of the shock dynamics of nonlinear systems is the phenomenon of sudden change, when the periodic orbit reaches the barrier at zero speed, but nonzero acceleration. Then a small nonzero perturbation of such orbits may disclose slight irregular oscillations that bring different forms compared to the smooth nonlinear dynamics.

Particularly vulnerable are the controllers (PLC) acquiring signals (data) from sensors spread over large areas and long lines combined with other controllers, control apparatus in the control room, etc. In the group of dynamic interactions, we introduce the basic division:

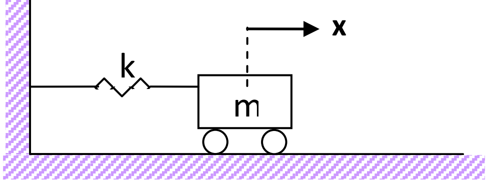
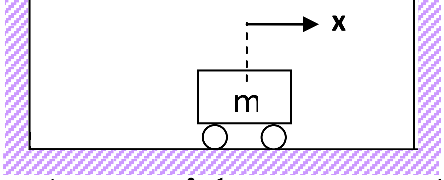
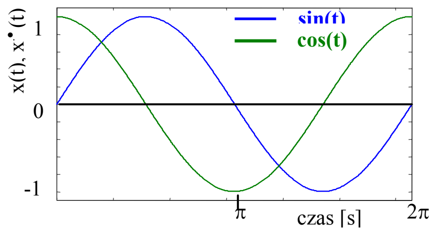
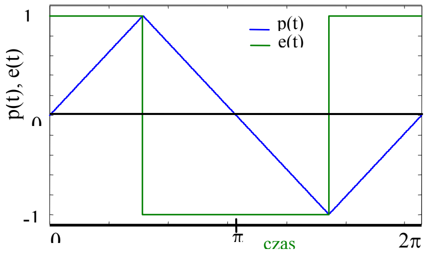
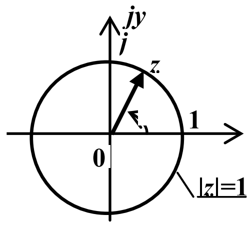
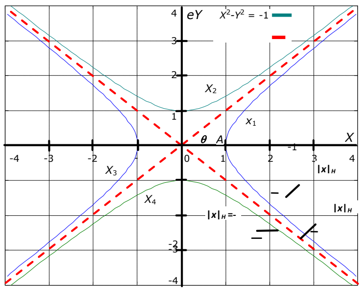
- oscillations, characterized by a long duration of action, involving many cycles and the limited amplitude of characteristic signals (charge, flux, current, voltage),
- single shock excitations of operations with short duration pulses and high amplitudes (e.g. the impact of an aircraft at the power line, the impact of electromagnetic waves during *E*-bomb explosions),
- shock incident signals of repeated action, in which the strong pulse interactions occur periodically (e.g. ground fault arising during persistent short-circuit system with the ground, resonance resulting at favourable conditions for the formation of resonance and ferroresonance, defibrillation, or stop of harmful ventricular flickers by using current pulses delivered by the defibrillator electrodes) [2, 4, 8, 9, 12, 15].

Nonsmooth dynamical systems exhibit more complex and enriched dynamics, when compared with their smooth counterparts. However, the qualitative analysis and design is still the subject of intensive research. Recently, it has been found that nonsmooth dynamical systems reveal significant wealth of nonlinear phenomena, including a chaotic, that are unique to this potentially important class of nonlinear systems. Furthermore, in the case of highly nonlinear systems a significant unpredictability appears in the course of their dynamics. For instance, a sudden transition from a stable periodic oscillation to the full range of chaotic oscillations may often occur in nonsmooth system at small change of parameters, while such phenomenon is not observed in smooth configurations, if they are not in series with period doubling bifurcation [1, 3, 4, 6, 9]. The dynamics of nonsmooth oscillations with shock forcing is analysed in the sequel by using a relatively new mathematical tool, which appears to be hyperbolic algebra [7, 10, 13]. The key idea of this tool is steeped in of nonsmooth time transformations (NSTT) proposed originally in [11] for strongly nonlinear, but still smooth models. The NSTT is based on the algebra of hyperbolic numbers, an approach corresponding to the algebra of complex numbers and functions in the case of smooth excitations. The solution efficiency of NSTT results from explicit links between impact dynamics and hyperbolic algebras analogously to the link between harmonic oscillations and conventional complex analyses. Presently, this is one of the principal challenges at the crossroad between non-smooth dynamical systems, mathematics and computer science [5, 8, 16]. Basic details in this direction are presented in the next Section.

3. Hyperbolic numbers and nonlinear phenomena

The classical theory of differential equations usually avoids non-differentiable and discontinuous functions. In many such cases, it is still possible to adapt different smooth methods of the dynamic analyses through strongly non-linear algebraic manipulations with state vectors

Tab. 1. Representations of smooth linear and nonsmooth nonlinear dynamical systems

	Harmonic state	Impact state
1	***Hamonic oscillator***  $\ddot{x} + x = 0$	***Impact oscillator***  x=-1 $\ddot{x} + x^{2n-1} = 0,\ n \to \infty$ x=1
2	***Signals: sin(t) and cos(t)*** 	***Signals: p = sin$_T$(t) and e = cos$_p$ (t)*** 
3	***Classic complex numbers*** $z = x + jy$ $j^2 = -1$ $v(t) = \Re e\{z\exp(j\omega z\} + \Re e\{\mid z \mid \exp(j\omega x + \varphi)\}$ $= \mid z \mid \sin(\omega t + \varphi);\ \varphi = \arctan(y/x)$	***Hyperbolic numbers*** $h(t) = U(p(t)) + W(p(t))e(t)$ $e(t) = \dot{p}(t),\ e^2(t) = 1$
4	***Clasic complex plane***  $\mid z \mid^2 = x^2 + y^2$ $\varphi = \arctan(\frac{y}{x})$ $z = \exp(j\varphi)$	***Hyperbolic plane***  $-\infty < \theta < \infty$ $x_{1,3} = \pm\exp(e\theta),\ \theta = \mathrm{arctanh}\ (Y/X);$ $x_{2,4} \pm e \cdot \exp(e\theta),\ \theta = \mathrm{arctanh}\ (X/Y)$
5	***Fourier series*** $\sum_{k=0}^{\infty}[A_k\cos(kt) + B_k\sin(kt)]$	***Saw tooths power series*** $\sum_{k=0}^{\infty}[\frac{1}{k!}X^{(k)}(0)p^k + \frac{1}{k!}Y^{(k)}(0)p^k\dot{e}]$

or by splitting the phase space into multiple domains based on the system details. Possible alternatives to such approaches can be built on generating models developing essentially nonlinear/unharmonic behaviours as their inherent properties. Such models must be general and simple enough in order to play the role of physical basis. As shown in the sequel, new generating systems can be found by intentionally imposing the 'worst case scenario' on conventional methods in anticipation that failure of one asymptotic may point to its complementary counterpart. The tool presented here employs nonsmooth (impact) systems as a basis to describing not only impact but also smooth or even linear dynamics. This is built on the idea of nonsmooth time substitutions/transformations (NSTT) proposed originally for strongly nonlinear but still smooth models. The methodological role of NSTT is to reveal explicit links between impact dynamics and hyperbolic algebras analogously to the link between harmonic vibrations and conventional complex analyses.

The hyperbolic numbers also called the "perplex numbers," serve as coordinates in the Lorentzian plane in much the same way that the classic complex numbers serve as coordinates in the Euclidean plane. Such models appear to be general and simple enough in order to exhibit the role of physical basis of the studied problem. In the sequel, we are focused on the analysis of a class of nonconstant solutions of state variable equation, which is next related to fixed points in the scale of complexity, namely periodic orbits. Further, the studied system with discontinuities can be simplified by means of appropriate nonsmooth transformations of variables. The idea is that simplicity of a mathematical formalism is caused by hidden links between the corresponding generating models and subgroups of rigidbody motions [4, 12]. The present approach employs time histories of impact systems as new time arguments. The occurrence of such algebraic structures seems to be essential feature of the approach since it justifies and simplifies analytical manipulations with noninvertible temporal substitutions such as NSTT.

The hyperbolic numbers called also perplex numbers, or split-complex numbers, are a two-dimensional commutative algebra over the real numbers different from the complex numbers [5]. Every hyperbolic number has the form

$$w = x + uy, \tag{1}$$

where x and y are real numbers. The number u is similar to the imaginary unit $j = \sqrt{-1}$, except that

$$u^2 = +1. \tag{2}$$

Just as for complex numbers, one can define the notion of a hyperbolic conjugate number as

$$w^* = x - uy. \tag{3}$$

The modulus of a hyperbolic number $w = x + uy$ is given by the isotropic quadratic form

$$|w| = \sqrt{w \cdot w^*} = \sqrt{x^2 - y^2}\ . \tag{4}$$

There are two nontrivial idempotents given by $q = (1 - u)/2$ and $q^* = (1 + u)/2$. Recall that idempotent means that $qq = q$ and $q^* \cdot q^* = q^*$. Modules of both these elements are null

$$|q| = |q^*| = |q \cdot q^*| = 0. \tag{5}$$

Very often it is convenient to use q and q^* as an alternate basis for the hyperbolic plane. This basis is called the diagonal basis or null basis. The hyperbolic complex number w can be written in the null basis as

$$w = x + uy = (x - y)q + (x + y)q^*. \tag{6}$$

Figures given in table 1 illustrate the hyperbolic plane. Note, that in contrast to the circle, each of the hyperbola branches is covered exactly once as the hyperbolic angle θ is varying in the infinite interval. Respective portions of the hyperbolic plane show subsets with modulus zero (red), one (blue), and minus one (green). The analog of Euler's formula for the hyperbolic numbers is

$$exp(u\theta) = cosh(\theta) + u\ sinh(\theta), \tag{7}$$

where θ is standing for the hyperbolic angle.

The above equality can be derived from a power series expansion using the fact that *cosh* has only even powers, while *sinh* has odd powers only. The hyperbolic angle θ is twice the area of the sector A0x in figure presented in 2nd column and 5th position of the table 1. In addition, to hyperbolic angles θ, we can give the geometrical meaning of an area $\theta = 2\cdot area(\text{A0x})$ and this area has the same value measured in both "hyperbolic" and "Euclidean" way. By analogy with the circular angles φ defined on the unitary circle $|z| = 1$, we can define $cosh(\theta)$ and $sinh(\theta)$ as the abscissa and the ordinate of the hyperbola point defined by θ, respectively. Then, such an approach still works for general cases by generating specific algebraic structures in terms of the coordinates.

In our case, the unipotent u is not a number but the discontinuous function of certain physical nature i.e., the rectangular cosine wave $e(t)$. Indeed, since t is running then there is no unique choice for the magnitude e, whereas always $e^2 = 1$. Therefore, identity (1) generates the hyperbolic structure from the very general properties of periodic processes.

Despite the strong nonlinearity caused by impacts, the generic oscillator is also described by quite simple elementary functions such as triangular sine and rectangular cosine, say $p(t)$ and $\dot{p}(t) = e(t)$, respectively, which are presented in figure given in 2nd column and 2nd position in table 1.

Finally, let us mention that the hyperbolic plane has another natural basis associated with the two isotropic lines separating the hyperbolic quadrants as shown in 2nd column and 5th position of the table 1. The transition from one basis to another is given by $e_{\pm} = (1 \pm e)/2$ or, inversely, $1 = e_+ + e_-$ and $e = e_+ - e_-$. The elements e_+ and e_- are mutually annihilating (idempotents) so that $e_+e_- = 0$, $e_-^2 = e_-$ and $e_+^2 = e_+$. It is clear also that $ee_+ = e_+$, and $ee_- = -e_-$. Note that this basis usually couples the corresponding smoothness (boundary) conditions.

Therefore, for any periodic function $x = x(t)$ whose period is T, we can write

$$x = X + Ye = X(e_+ + e_-) + Y(e_+ - e_-) = (X + Y)e_+ + (X - Y)e_- = X_+(p)e_+ + X_-(p)e_-, \tag{8}$$

where

$$X_+(p) = X(p) + Y(p) \qquad \text{and} \qquad X_-(p) = X(p) - Y(p). \tag{9}$$

This suggests possible recipes for effective dealing with the differential equations of oscillation on entire time intervals, despite discontinuity and/or non-smoothness points.

4. Fundamental properties of impact oscillators

Impact oscillator is a term used herein to represent a system, which is periodically driven in a specific way, which also is an intermittent or continuous time-varying sequence of switchings with the limit restrictions. This important structure of non-smooth dynamical systems shows not only the typical features of smooth nonlinear systems, such as generic bifurcations, multiple solutions and chaos. Moreover, it also displays new phenomena appearing as, for instance, the sudden change of the system state, where a periodic orbit reaches the barrier at zero speed, but nonzero acceleration. Small nonzero perturbations of such orbits may or may not have to disclose the impact.

An impact oscillator is represented by

$$\ddot{x}(t) + x^{2n-1}(t) = 0, \quad n \to \infty, \tag{10}$$

where the upper dot stands for the time derivative. This mathematical model describes nonlinear oscillator shown in Table 1, right panel, first position. At the limit $n = \infty$, when the restoring force vanishes inside the interval $-1 < x < 1$ but becomes infinitely growing as the system reaches the potential barriers at $x = \pm 1$, the body displacement and its velocity are described by signals shown in table 1, right panel, second position. Alongside the above mathematical challenges, this case admits interpretation by means of the total energy

$$\frac{\dot{x}^2}{2} + \frac{x^{2n}}{2n} = \frac{1}{2}, \tag{11}$$

where the number 1/2 on the right-hand side corresponds to the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. Taking into account that the state variable of the oscillator reaches its amplitude value at zero kinetic energy, gives the estimate $-n^{-1/(2n)} \leq x(t) \leq n^{1/(2n)}$ for any time t. Since $n^{1/(2n)} \to 1$ as $n \to \infty$ then the limit oscillation is restricted by the interval $-1 \leq x(t) \leq 1$. Inside of this interval, the second term on the left-hand side of expression (39) vanishes and hence, x & $= \pm 1$or x $= \pm t + \alpha_{\pm}$, where $\alpha_{\pm}$ are constants. By manipulating with the signs and constants, one can construct the sawtooth sine $p(t)$ – triangular wave – since there is no other way to providing the periodicity condition. Therefore, one has the couple of periodic functions

$$\{x(t), \dot{x}(t)\} = \{p(t), \dot{p}(t) = e(t)\}, \tag{12}$$

where $\dot{p}\,(t) = e(t)$ is a generalized derivative of the sawtooth sine and will be named as a rectangular cosine. The analogy of respective relations in the hyperbolic and the conventional complex planes is shown in table 1.

The presence of functions $p(t)$ and $e(t)$ in further developed analytical algorithm is not a simple match of different pieces of solutions but it has its real physical basis and invokes specific mathematical tools. Their effectiveness is determined by the following statement:

Any periodic process x(t) of the period T can be expressed through the dynamic state of the impact oscillator, *{p(t), e(t)}*, in the form of ‘ hyperbolic complex number’

$$x(t) = X(p) + Y(p)e(t), \tag{13}$$

where the functions *X(p)* and *Y(p)* on the right-hand side are easily expressed through the original function *x(t)*, if this function is known.

The expression (13) can be qualified as non-smooth time transformation (NSTT), $t \to p$, on the manifold of periodic oscillations. In a case when $x(t)$ is an unknown periodic oscillation of some dynamical system, equations for $X(p)$ and $Y(p)$ components are obtained by substituting (13) into the corresponding differential equation of oscillation. Then either analytical or numerical procedures can be applied. To illustrate the above statement let us consider a first order nonlinear system representing a simple model of gas pipe flow driven by a regularly repeating rectangular pressure wave (Fig. 1).

a) b)

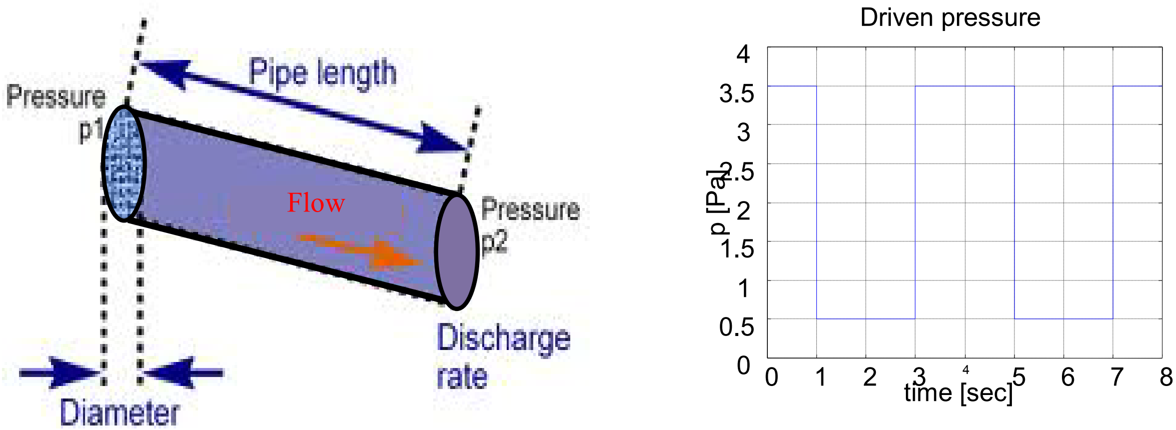


Fig .1. Gas pipe: a) scheme, b) repeating rectangular wave driving pressure

The flow $q(t)$ is described by first-order non-linear differential equation

$$\dot{q}(t) + kq^2(t) = p_0 + p_1 e(t/a), \tag{14}$$

where constants $k = K/M$, $p_0 = P_0/M$, $p_1 = P_1/M$ denote the pipe parameters related to the pipe flow inertance and $a = T/4$ with T as a period of driving pressure.

The corresponding periodic solution can be represented in the form

$$q(t) = X(p) + Y(p)e(t/a), \tag{15}$$

where p(t/a) and e(t/a) are triangular and rectangular waves with the period $T = 4a$.

Substituting (15) in (14), gives

$$Y' + ak(X^2 + Y^2) = ap_0, \quad X' + 2akXY = ap_1, \quad Y(\pm 1) = 0, \tag{16}$$

where upper sign ‘ denotes derivative with respect to $p(t)$.

Introducing the new unknowns $U = X + Y$ and $V = X - Y$, brings the boundary value problem (16) to the form

$$U' + akU^2 = aG, \quad V' - akV^2 = -aH, \quad U(\pm 1) = V(\pm 1), \tag{17}$$

where $G = p_0 + p_1$ and $H = p_0 - p_1$ are constant.

Both equations in (17) are separable and thus admit general solutions of the form

$$U(p,c_1)=\sqrt{\frac{G}{k}}\left[1-\frac{2}{1+c_1e^{2a\sqrt{kG}p}}\right],\quad V(p,c_2)=-\sqrt{\frac{H}{k}}\left[1-\frac{2}{1+c_2e^{2a\sqrt{kH}p}}\right], \tag{18}$$

where c_1 and c_2 are arbitrary constants of integration to be determined from the boundary conditions

$$U(1,c_1)=V(1,c_2),\quad U(-1,c_1)=V(-1,c_2). \tag{19}$$

Each real solution for the constants c_1 and c_2 gives a periodic solution of differential equation (14) and it is determined as follows

$$q(t)=\frac{1}{2}[U+V]+\frac{1}{2}[U-V]e(t). \tag{20}$$

Fig. 2 shows what happens to the steady state flow profile as the period of pressure wave becomes twice longer. The model parameters are k = 1.5, p_0 = 2.0, and p_1=1.5. In cases T = 4 s and T = 8 s, the arbitrary constants are c_1 = 222.15, c_2= 0.08 and c_1 = 17594.31, c_2 = 65.77. To illustrate the above statement let us consider an oscillator with constants mass M = 1 kg connected with a spring exhibiting a hardening characteristic given by

$$f=\tan(x)\cos^{-2}(x), \tag{21}$$

where x denotes the non-dimensional state variable.

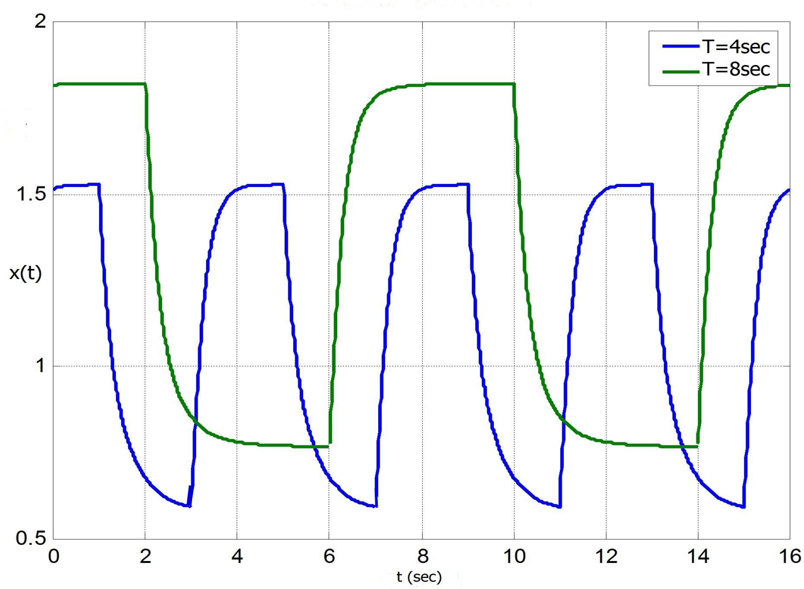


Fig. 2. Solutions of (14) for two periods of driven pressure

Such an oscillator is described by the following equation

$$\ddot{x}+\frac{\tan(x)}{\cos^2(x)}=0. \tag{22}$$

Following the introduced rules and making the substitutions

$$t\to p(t)\quad \text{and}\quad x(t)=X(p)+Y(p)e(t), \tag{23}$$

yields a solution to the resulting boundary value problem and then substituting the result into (22) gives

$$x(t)=\arcsin(\alpha\sin(p(t)/\sqrt{1-\alpha^2})), \tag{24}$$

where $\alpha=\sin(A)$ with A as amplitude of oscillations. Then, the corresponding temporal mode of oscillation changes its shape from smooth quasi harmonic to nonsmooth triangular sine.

6. Conclusions

In this paper, a version of nonsmooth substitution, specifically the nonsmooth time variable transformation is applied to analyse waveforms of currents and voltages in nonlinear nonsmooth dynamical systems. The basic rules for algebraic and differential manipulations are presented to apply the nonsmooth argument substitutions in differential equation on the entire time interval.

The two main features of the presented approach are to generate a particular algebraic structure and switching the initial value formulation to a boundary value problem. The nonsmooth time transformation shows an explicit link between the underlying dynamics and hyperbolic algebra, analogously to the link between the harmonic approach and complex analysis. These impose two principal features on the dynamical systems by generating specific algebraic structures and switching formulations to boundary-value problems. The effectiveness of the presented method has been illustrated by several examples of analysis of basic nonlinear nonsmooth circuits constituting starting points for more complex systems. The case of rectangular cosine waveform of incident excitation has been also considered and illustrated.

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