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ON THE SOLVABILITY OF SOME PARABOLIC EQUATIONS INVOLVING NONLINEAR BOUNDARY CONDITIONS WITH L^1 DATA

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Abstract. We analyze the existence of solutions for a class of quasilinear parabolic equations with critical growth nonlinearities, nonlinear boundary conditions, and L^1 data. We formulate our problems in an abstract form, then using some techniques of functional analysis, such as Leray–Schauder's topological degree associated with the truncation method and very interesting compactness results, we establish the existence of weak solutions to the proposed models.

Keywords: quasilinear parabolic equation, nonlinear boundary conditions, weak solutions, Leray–Schauder topological degree, L^1-data.

Mathematics Subject Classification: 35K59, 35K55, 35A01, 35B09, 35D30.

1. STATEMENT OF THE PROBLEM

In this work, we are concerned with a class of quasi-linear parabolic problems with nonlinear boundary conditions, whose model reads as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + G(t,x,u,\nabla u) = f(t,x) & \text{in } Q_T, \\ u(0,x) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma(t,\sigma,u) = g(t,\sigma) & \text{on } \Sigma_T, \end{cases} \tag{1.1}$$

where Ω is an open regular bounded subset of $\mathbb{R}^N$ for $N \geq 1$, with smooth boundary $\partial\Omega$, $T > 0$, $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$. ν denotes the unit normal vector to the boundary $\partial\Omega$ and Δ is the Laplacian operator. The nonlinearity G is a Carathéodory function and γ represents the nonlinearity defined at the boundary Σ_T. The given data f, g are two measurable functions. This kind of problems arises from the modeling of the standard nonlinear heat-transfer equation following the Kirchhoff, and enthalpy transformations. For further elaboration, we refer the readers to see [4, 24, 30–32].

The studies of nonlinear partial differential equations have attracted much attention over the last twenty years. Several works have been interested in the existence, uniqueness and asymptotic behaviors of solutions to such problems under different conditions and using various techniques [1, 3, 19, 20, 23, 29]. In particular, PDEs involving nonlinear boundary conditions are of special interest to various areas of applied mathematics, physics and control theory. The book by Rădulescu et al [25] is one of the most suitable references to understanding theoretical analysis of nonlinear PDEs. This book can be considered as the starting to study the complex behaviors of nonlinear PDEs via different approaches such as fixed point methods, topological degree, penality method, etc. Systems of the form (1.1) are used to model various problems issued from different fields, such as biology, chemistry, and physics. Therefore, several researchers have treated this type of problem under either Dirichlet or Neumann boundary conditions, and different analytical and numerical techniques and methods for the existence problem have been used [2, 4, 7, 12–16, 28].

In order to trace the objectives of our work, we propose to start by recalling some previous studies which strongly rely upon our problem. One can mention the following: The work by Amann [5] focuses on the existence problem of nonlinear elliptic equations with nonlinear boundary conditions. Completely continuous maps in ordered Banach spaces are used to extend the applicability of the general existence and uniqueness theorem, by obtaining sufficient criteria for the existence of sub and super-solutions. The result developed in this work was based on the transformation of the proposed nonlinear elliptic boundary value problem into an equivalent fixed point equation in $\mathcal{C}(\overline{\Omega})$. In [11], the authors deal with the existence of a solution of a ϕ-Laplacian problem with nonlinear boundary conditions. Using the sub- and super-solution method combined with the Nagumo condition, the authors established the existence of solutions to the studied problem and derived a-priori bounds for the derivatives of the solutions. Using topological methods, Amster [6] proved the existence of solutions that belong to the space $\mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ of a nonlinear elliptic second order problem, with nonlinear boundary conditions under a variant of the so-called Hartman condition. Their proof relies on the maximum principle and the unique solvability of the associated linear Robin problem. The work [8] concerned by the existence and uniqueness of entropy solutions for a quasilinear parabolic equation under a nonlinear boundary condition. Their proof was based on the nonlinear semi-group theory. Last but not least, it is worth mentioning that the authors in [18] used a combination of the truncation method, and Schauder's fixed point theorem to obtain an existence result for a class of the periodic version of quasilinear parabolic problems under nonlinear boundary conditions.

In this paper, we aim to establish the existence of solutions to (1.1) with weak regularities on data f and g using general assumptions on the nonlinearities. We shall assume that (f, g) belongs only to $L^1(Q_T) \times L^1(\Sigma_T)$, and we will suppose that $G(t, x, u, \nabla u)$ enjoy some growth structure and involves a sign conditions. Further, the nonlinear boundary term $\gamma(t, \sigma, u)$ will be assumed to meet some specific growth conditions. The last one will be used to derive some coercivity property of the considered operator. We would like to mention that the major difficulties in our work lie not only in consideration of irregular data but also in the absence of the initial condition ($u_0 = 0$). In general, the standard theory developed for initial boundary value problems does not be applied. By taking into account the nature of our assumptions, we can see that the present work contributes to enriching the existing literature not only on nonlinear PDEs with L^1 data but also on those having Neumann and Robin-type boundary conditions.

In the following points, we will highlight our main objectives:

- establish the existence and uniqueness of weak solution to the heat equation with nonlinear boundary conditions and L^2 data,
- develop Leray–Schauder's fixed point method to investigate the existence of a weak solution to (1.1) with bounded nonlinearities,
- show the existence of a weak solution to (1.1) with L^1 data,
- prove compactness results which will be applicable to parabolic equations with nonlinear boundary conditions.

We have arranged the rest of our paper as follows: In Section 2, we put forward mathematical preliminaries and assumptions used to solve problem (1.1). Section 3 tackles interest in a modified version of problem (1.1) by taking the case of $G \equiv 0$, f and g are regular enough. We will use the monotone operator theory to investigate the existence and uniqueness of a weak solution when the data belongs to L^2. In Section 4, we employ Leray–Schauder's topological degree to obtain the existence of a weak solution to problem (1.1) in the particular case, where the nonlinearity G is bounded. Last but not least, in the fifth section, we prove the existence of a weak solution to problem (1.1) when the data belonging only to L^1 and the nonlinearity has quadratic growth with respect to the gradient. We close our paper with Section 6 which takes an interesting compactness result of the heat operator with nonhomogeneous Neumann boundary conditions and L^1 data.

2. MATHEMATICAL PRELIMINARIES AND ASSUMPTIONS

For the reader's convenience, we propose to exhibit the functional framework which involves the L^2 setting. Let $0 < T < +\infty$, we define the functional space

$$\mathcal{V} := L^2(0, T; H^1(\Omega)).$$

It is well known that

$$||| \, u \, |||_{\mathcal{V}} := \left(\int\limits_{Q_T} |\nabla u(t,x)|^2 dx\, dt + \int\limits_{Q_T} |u(t,x)|^2 dx\, dt \right)^{\frac{1}{2}},$$

stands the standard norm of the space $\mathcal{V}$. In addition, we can notice that the following norm

$$\|u\|_{\mathcal{V}} := \left(\int\limits_{Q_T} |\nabla u(t,x)|^2 dx\, dt + \int\limits_{\Sigma_T} |u(t,\sigma)|^2 d\sigma\, dt \right)^{\frac{1}{2}}$$

is an equivalent norm to the standard norm $||| \cdot |||_{\mathcal{V}}$. As we can see, $\mathcal{V}$ is a Banach, reflexive and separable space. We will denote by $\mathcal{V}^*$ the topological dual space of $\mathcal{V}$

which reads as $\mathcal{V}^* := L^2(0,T;(H^1(\Omega))^*)$, and we designate by $\langle \cdot, \cdot \rangle$ the duality pairing between $\left(H^1(\Omega)\right)^*$ and $H^1(\Omega)$. Now, we are ready to define the space

$$\mathcal{W}(0,T) = \left\{ u \in \mathcal{V} \, : \, \frac{\partial u}{\partial t} \in \mathcal{V}^* \right\}.$$

We equip it with the norm

$$\|u\|_{\mathcal{W}(0,T)} = \|u\|_{\mathcal{V}} + \left\| \frac{\partial u}{\partial t} \right\|_{\mathcal{V}^*}.$$

From the classical result [21], we have the following embeddings:

$$\mathcal{W}(0,T) \hookrightarrow \mathcal{C}\left([0,T]; L^2(\Omega)\right), \tag{2.1}$$

$$\mathcal{W}(0,T) \overset{\text{compact}}{\hookrightarrow} L^2(Q_T). \tag{2.2}$$

Now we will recall a surjectivity result for monotone operators. This result will be used in the following.

Proposition 2.1 ([21])**.** *If $\mathcal{X}$ is a reflexive Banach space, $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subseteq \mathcal{X} \to \mathcal{X}^*$ is a linear maximal monotone operator and $\mathcal{A} : \mathcal{X} \to \mathcal{X}^*$ is a hemicontinuous, monotone and coercive operator (i.e. $\frac{\langle \mathcal{A}(u), u \rangle_{\mathcal{X}^*, \mathcal{X}}}{\|u\|_{\mathcal{X}}} \to +\infty$ as $\|u\|_{\mathcal{X}} \to \infty$), then $\mathcal{L}+\mathcal{A}$ is surjective.*

2.1. ASSUMPTIONS

In this paper, we will examine the existence of solutions to problem (1.1) under the following assumptions:

(A_1) The functions f and g are measurable functions belonging to certain Lebesgue spaces.

(A_2) $\gamma : \Sigma_T \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function which satisfies for almost (t, σ) in Σ_T and for all s in $\mathbb{R}$

$$s \longmapsto \gamma(t,\sigma,s) \text{ is a nondecreasing function,} \tag{2.3}$$

$$s\gamma(t,\sigma,s) \geq \gamma_0 s^2, \tag{2.4}$$

$$|\gamma(t,\sigma,s)| \leq \gamma_1(1+|s|^\theta), \tag{2.5}$$

where γ_0 and γ_1 are nonnegative constants and $1 \leq \theta < \frac{N+2}{N}$.

(A_3) $G : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies some growth conditions to be specified later.

2.2. SOME TRUNCATIONS FUNCTIONS

For any $k > 0$, we define the standard truncation function

$$T_k(s) = \min\{k, \max\{s, -k\}\}$$

and we note $S_k(v) = \int_0^v T_k(s)ds$. Furthermore, we can construct $\tau_k \in \mathcal{C}^2$ a more regular truncation function such that

$$\begin{cases} \tau_k(r) = r & \text{if } 0 \le r \le k, \\ \tau_k(r) \le k+1 & \text{if } r \ge k, \\ 0 \le \tau_k'(r) \le 1 & \text{if } r \ge 0, \\ \tau_k'(r) = 0 & \text{if } r \ge k+1, \\ 0 \le -\tau_k''(r) \le C(k). \end{cases}$$

Typically, the construction of such truncation τ_k can be given as follows:

$$\tau_k(s) = \begin{cases} s & \text{in } [0,k], \\ \frac{1}{2}(s-k)^4 - (s-k)^3 + s & \text{in } [k,k+1], \\ \frac{1}{2}(k+1) & \text{for } s > k+1. \end{cases}$$

For $\epsilon > 0$, we define the following convex function:

$$j_\epsilon(s) = \begin{cases} -\frac{1}{\epsilon} + \frac{1}{\epsilon}\exp\left(-\epsilon s - \epsilon^2 \ln\left(\left|\frac{s-\epsilon}{\epsilon}\right|\right)\right) & \text{if } s < 0, \\ 0 & \text{if } s \ge 0. \end{cases}$$

By a simple computation, we can easily show that j_ϵ satisfies the following properties:

(a) $j_\epsilon'(s)$ is bounded for all $s \in \mathbb{R}$,
(b) $j_\epsilon'(s) \to \mathrm{sign}^-(s)$ when $\epsilon \to 0$,
(c) $j_\epsilon(s) \to (s)^-$ when $\epsilon \to 0$,

where sign^- is the following "sign" function:

$$\mathrm{sign}^-(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r \ge 0. \end{cases}$$

In the remainder of this paper, we denote by C every generic and nonnegative constant. The value of this constant can change in different situations. It may depend on the given data but always remains independent of the estimated sequence index.

3. EXISTENCE RESULTS IN L^2 FRAMEWORK

The aim of this section is to provide the existence and uniqueness result of the weak solution to (1.1) without nonlinearity ($G(t,x,u,\nabla u) = 0$). Then the studied problem reads as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(t,x) & \text{in } Q_T, \\ u(0,x) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma(t,\sigma,u) = g(t,\sigma) & \text{on } \Sigma_T. \end{cases} \tag{3.1}$$

Here, we will assume that the term source f belongs to $L^2(Q_T)$, and the data g stands an element of $L^2(\Sigma_T)$. For the nonlinearity $\gamma(\cdot)$, we shall assume that hypotesis (A_2) holds. Then, our purpose is to investigate the existence and uniquness of weak solution to problem (1.1) in the following sense.

Definition 3.1. A measurable function $u : Q_T \to \mathbb{R}$ is said to be a weak solution to problem (3.1) if it satisfies the following conditions:

$$u \in \mathcal{W}(0,T), \quad u(0,x) = 0 \text{ in } L^2(\Omega),$$

$$\int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} \nabla u \nabla \varphi \, dxdt + \int_{\Sigma_T} \gamma(t,\sigma,u)\varphi \, d\sigma dt = \int_{Q_T} f\varphi \, dxdt + \int_{\Sigma_T} g\varphi \, d\sigma dt, \tag{3.2}$$

for all test functions $\varphi \in \mathcal{V}$.

We exhibit the main result of this section in the following theorem.

Theorem 3.2. *Let $f \in L^2(Q_T)$, $g \in L^2(\Sigma_T)$ and assume that assumption (A_2) holds, then problem* (3.1) *admits a unique weak solution u in the sense of Definition 3.1. Furthermore, the weak solution u satisfies the following energy estimate:*

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{\mathcal{V}} \leq C \left(\|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)} \right), \tag{3.3}$$

where C is a constant depending only on T, Ω, N and γ_0.

3.1. PROOF OF THEOREM 3.2

We will detail the proof of Theroem 3.2 in the following subsections.

3.1.1. Existence result

In this step, we are aiming to establish the existence of weak solution to (3.1). To do so, we will formulate the parabolic partial differential equation (3.1) into an equivalent abstract equation posed in the Banach space $\mathcal{V}$. Thereafter, we shall use the result of Proposition 2.1 to study the existence of solution to the last one. Let us start by introducing

$$\mathcal{D}(\mathcal{L}) \coloneqq \{u \in \mathcal{W}(0,T) \, : \, u(0) = 0\}.$$

By virtue of the density property of $\mathcal{C}_c^\infty(Q_T)$ in $\mathcal{V}$, and using the fact that $\mathcal{C}_c^\infty(Q_T) \subset \mathcal{D}(\mathcal{L})$, one can conclude that $\mathcal{D}(\mathcal{L})$ is dense in $\mathcal{V}$.

Here, we introduce the operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \longrightarrow \mathcal{V}^*$ such that

$$\langle \mathcal{L}u, \varphi \rangle \coloneqq \int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle dt, \quad \text{for all } \varphi \in \mathcal{V}.$$

Then, the result given in [21, Lemma 1.1, p. 313] leads to conclude that $\mathcal{L}$ is a closed, skew-adjoint and maximal monotone operator. Now, we consider on $\mathcal{V}$ the following operator

$$\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}^*,$$

satisfying

$$\langle \mathcal{A}u, \varphi\rangle := \int\limits_{Q_T} \nabla u \nabla \varphi \, dxdt + \int\limits_{\Sigma_T} \gamma(t,\sigma,u)\varphi \, d\sigma dt, \qquad \text{for all } \varphi \in \mathcal{V}. \tag{3.4}$$

One can therefore verify that the existence of a weak solution of (3.1) is equivalent to finding a solution to the abstract equation written as follows:

$$\mathcal{L}u + \mathcal{A}u = \mathcal{F}, \tag{3.5}$$

where $\mathcal{F}$ stands for an element of $\mathcal{V}^*$ defined as

$$\langle \mathcal{F}, \varphi\rangle := \int\limits_{Q_T} f\varphi \, dxdt + \int\limits_{\Sigma_T} g\varphi \, d\sigma dt, \quad \text{for all } \varphi \in \mathcal{V}.$$

Now, we are in the setting to verify the conditions of Proposition 2.1. We have the following results:

(a) The operator $\mathcal{A}$ is hemicontinuous. Let $u, v \in \mathcal{V}$, we have

$$\langle \mathcal{A}u, v\rangle \leq \int\limits_{Q_T} |\nabla u||\nabla v| \, dxdt + \int\limits_{\Sigma_T} |\gamma(t,\sigma,u)||v| \, d\sigma dt. \tag{3.6}$$

Using Hölder's inequality and assumption (A_2), it is straightforward to get

$$\langle \mathcal{A}u, v\rangle \leq \|\nabla u\|_{L^2(Q_T)} \|\nabla v\|_{L^2(Q_T)} + \gamma_1 \int\limits_{\Sigma_T} |v| \, d\sigma dt + \gamma_1 \int\limits_{\Sigma_T} |u|^\theta |v| \, d\sigma dt.$$

Applying Hölder's inequality again, one obtains the following

$$\langle \mathcal{A}u, v\rangle \leq \|\nabla u\|_{L^2(Q_T)} \|\nabla v\|_{L^2(Q_T)} + C\|v\|_{L^2(\Sigma_T)} + \gamma_1 \|v\|_{L^2(\Sigma_T)} \||u|^\theta\|_{L^2(\Sigma_T)}.$$

According to the trace theorem (see for example [17, Theorem 4.1.3]), and since $1 \leq \theta < \frac{N+2}{N} < \frac{N}{N-2}$, we obtain

$$\langle \mathcal{A}u, v\rangle \leq \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} + C\|v\|_{\mathcal{V}} + C\|v\|_{\mathcal{V}} \|u\|_{\mathcal{V}}^\theta,$$

hence

$$\|\mathcal{A}u\|_{\mathcal{V}^*} \leq C \max\left\{\|u\|_{\mathcal{V}}, \|u\|_{\mathcal{V}}^\theta\right\}.$$

By applying the results of Theorems 2.1 and 2.3 from [21], we derive that the operator $\mathcal{A}$ is hemicontinuous.

(b) The operator $\mathcal{A}$ is monotone. For any u, $\hat{u} \in \mathcal{V}$, we have

$$\langle \mathcal{A}(u) - \mathcal{A}(\hat{u}), u - \hat{u}\rangle = \int\limits_{Q_T} |\nabla(u-\hat{u})|^2 \, dx \, dt + \int\limits_{\Sigma_T} (\gamma(t,\sigma,u) - \gamma(t,\sigma,\hat{u}))(u-\hat{u}) \, d\sigma dt.$$

By using (2.3), we conclude that

$$\langle \mathcal{A}(u) - \mathcal{A}(\hat{u}), u - \hat{u} \rangle \geq 0,$$

which proves that $\mathcal{A}$ is a monotone operator.

(c) The operator $\mathcal{A}$ is coercive. Thanks to the hypothesis (A_2), we have

$$\langle \mathcal{A}u, u \rangle \coloneqq \int_{Q_T} |\nabla u|^2 \, dx \, dt + \int_{\Sigma_T} \gamma(t, \sigma, u) u \, d\sigma dt \geq \min(1, \gamma_0) \|u\|_{\mathcal{V}}^2.$$

Therefore,

$$\lim_{\|u\|_{\mathcal{V}} \to \infty} \frac{\langle \mathcal{A}u, u \rangle}{\|u\|_{\mathcal{V}}} \geq \lim_{\|u\|_{\mathcal{V}} \to \infty} \min(1, \gamma_0) \|u\|_{\mathcal{V}} = \infty,$$

which is equivalent to saying that $\mathcal{A}$ is a coercive operator.

With the help of Proposition 2.1, we conclude the existence of $u \in \mathcal{D}(\mathcal{L})$ solution to the abstract equation (3.5). Therefore, we obtain the existence of a weak solution to the parabolic equation (3.1).

3.1.2. Uniqueness result

Let u_1 and u_2 be two weak solutions of (3.1). By taking the difference between the weak formulations (3.4) of u_1 and u_2, respectively, we obtain that, for all $\varphi \in \mathcal{V}$,

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial(u_1 - u_2)}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} (\nabla u_1 - \nabla u_2) \nabla \varphi \, dx \, dt \\ &+ \int_{\Sigma_T} (\gamma(t, \sigma, u_1) - \gamma(t, \sigma, u_2)) \varphi \, d\sigma dt = 0. \end{aligned} \tag{3.7}$$

Let us take $\varphi = (u_1 - u_2)\chi_{(0,t)}$ as a test function in the weak formulation (3.7). By using the monotony assumption (2.3), one gets

$$\int_0^t \left\langle \frac{\partial(u_1 - u_2)}{\partial t}, u_1 - u_2 \right\rangle dt + \int_{Q_t} |\nabla(u_1 - u_2)|^2 dx dt \leq 0, \tag{3.8}$$

where $Q_t \coloneqq (0, t) \times \Omega$. We deal with left-hand side of (3.8) via the integration by parts formula, one gets for almost every $t \in (0, T)$

$$\int_{\Omega} (u_1 - u_2)^2(t) \, dx \leq 0.$$

Therefore, we get $u_1 = u_2$ a.e in Q_T, which shows that the obtained weak solution u of (3.1) is unique.

3.1.3. Energy estimate

In this step, we shall show that every weak solution u of (3.1) satisfies the energy estimate (3.3). To do this, we start by taking $\varphi = u\chi_{(0,t)}$ in the weak formulation associated to (3.1) with $0 < t < T$. We have

$$\begin{aligned} &\int_0^t \left\langle \frac{\partial u}{\partial t}, u \right\rangle dt + \int_{Q_t} |\nabla u|^2 \, dxdt + \int_{\Sigma_t} \gamma(t,\sigma,u)u \, d\sigma dt \\ &= \int_{Q_t} fu \, dxdt + \int_{\Sigma_t} gu \, d\sigma dt, \end{aligned} \tag{3.9}$$

where $\Sigma_t := (0,t) \times \partial\Omega$. Now we employ assumption (A_2) in (3.9) to get

$$\begin{aligned} &\frac{1}{2}\int_{\Omega} u^2(t) \, dt + \int_{Q_t} |\nabla u|^2 \, dxdt + \gamma_0 \int_{\Sigma_t} |u|^2 \, d\sigma dt \\ &\leq \int_{Q_t} |fu| \; dxdt + \int_{\Sigma_t} |gu| \; d\sigma dt. \end{aligned} \tag{3.10}$$

From Young's inequality and by applying trace theorem, it follows that

$$\int_{\Omega} u^2(t) \, dt \leq \int_{Q_t} |f|^2 \, dxdt + \int_{\Sigma_t} |g|^2 \, d\sigma dt + C \int_{Q_t} |u|^2 \, dxdt. \tag{3.11}$$

By applying Gronwall's lemma, we obtain

$$\int_{Q_T} u^2 \, dxdt \leq (\exp(T) - 1)\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\Sigma_T)} \right). \tag{3.12}$$

We substitute the result of (3.12) in (3.11) to get

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_{\Omega} u^2(t) &\leq \|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\Sigma_T)} \\ &\quad + C\exp(T)\left(\|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)} \right). \end{aligned} \tag{3.13}$$

Therefore,

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C(T,\Omega)\left(\|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)} \right). \tag{3.14}$$

On the other hand, combining the inequalities (3.10), (3.11) and (3.12) we obtain

$$\min\{1,\gamma_0\}\|u\|_{\mathcal{V}} \leq C(T,\Omega)\left(\|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)} \right). \tag{3.15}$$

In view to the estimates (3.14) and (3.15), we conclude that (3.3) holds, which completes the proof.

4. EXISTENCE RESULTS FOR BOUNDED NONLINEARITY

In this section, we study the existence of solutions to problem (1.1) when the non-linearity $G(t,x,u,\nabla u)$ is bounded. We will state the main result of this section in the following theorem.

Theorem 4.1. *Along hypothesis* (A_2)–(A_3)*, we assume the existence of a nonnegative function* $\mathfrak{M} \in L^2(Q_T)$ *such that for a.e.* (t,x) *in* Q_T*,*

$$|G(t,x,s,r)| \leq \mathfrak{M}(t,x), \quad \textit{for all } (s,r) \in \mathbb{R} \times \mathbb{R}^N. \tag{4.1}$$

Then for any $(f,g) \in L^2(Q_T) \times L^2(\Sigma_T)$*, problem* (1.1) *has a weak solution* u *satisfying*

$$\begin{gathered} u \in \mathcal{W}(0,T), \quad u(0,x) = 0 \textit{ in } L^2(\Omega), \\ \int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} \nabla u \nabla \varphi \, dxdt + \int_{\Sigma_T} \gamma(t,\sigma,u)\varphi \, d\sigma dt + \int_{Q_T} G(t,x,u,\nabla u)\varphi \, dxdt \\ = \int_{Q_T} f\varphi \, dxdt + \int_{\Sigma_T} g\varphi \, d\sigma dt, \end{gathered} \tag{4.2}$$

for all test function $\varphi \in \mathcal{V}$*.*

4.1. PROOF OF THEOREM 4.1

To establish the result of Theorem 4.1, we propose to use Leray–Schauder topological degree. As a first step, let us formulate the existence question of weak solutions to (1.1) into the seeking of a fixed point for a well-posed mapping. Hence, let us consider the following mapping

$$\begin{aligned} \mathcal{H} : [0,1] \times \mathcal{V} &\longrightarrow \mathcal{V} \\ (\lambda, v) &\longmapsto u, \end{aligned}$$

where u is a weak solution to the following nonlinear parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \lambda G(t,x,v,\nabla v) = \lambda f(t,x) & \text{in } Q_T, \\ u(0,x) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma(t,\sigma,u) = \lambda g(t,\sigma) & \text{on } \Sigma_T. \end{cases} \tag{4.3}$$

According to hypothesis (4.1), we can notice that the function $\lambda G(t,x,v,\nabla v)$ belongs to $L^2(Q_T)$. Therefore, the result of Theorem 3.2 assures that for any fixed (λ, v) in $[0,1]\times\mathcal{V}$, problem (4.3) has a unique weak solution u in the sense that

$$\begin{aligned} &u \in \mathcal{W}(0,T), \quad u(0,x)=0 \text{ in } L^2(\Omega),\\ &\int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} \nabla u \nabla\varphi \, dxdt + \int_{\Sigma_T} \gamma(t,\sigma,u)\varphi \, d\sigma dt \\ &+ \int_{Q_T} \lambda G(t,x,v,\nabla v)\varphi \, dxdt = \lambda \int_{Q_T} f\varphi \, dxdt + \lambda \int_{\Sigma_T} g\varphi \, d\sigma dt, \end{aligned} \tag{4.4}$$

for all test function $\varphi \in \mathcal{V}$. Consequently, we derive that the mapping $\mathcal{H}$ is well-defined. In addition, using again the result of Theorem 3.2, we conclude from (3.3) that u the weak solution of (4.3) satisfies the following energy estimate

$$\begin{aligned} \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{\mathcal{V}} \leq C \big[&\lambda \|G(t,x,v,\nabla v)\|_{L^2(Q_T)} \\ &+ \lambda \left(\|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)} \right) \big] . \end{aligned} \tag{4.5}$$

Using assumption (4.1) with the fact that λ belongs to $[0,1]$, the inequality (4.5) becomes

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{\mathcal{V}} \leq C \left(\|\mathfrak{M}\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)} \right). \tag{4.6}$$

Let us mention that inequality (4.6) will play a crucial in the sequel. It will be very helpful in the proclamation of some interesting properties of the mapping $\mathcal{H}$. As well known, to apply Leray–Schauder topological degree, we must proceed by following three steps:

Step 1. The mapping $\mathcal{H}$ is continuous. To study the continuity of the mapping $\mathcal{H}$, we take (λ_n, v_n) a sequence in $[0,1]\times\mathcal{V}$ such that (λ_n, v_n) converges strongly to some (λ, v) in $[0,1]\times\mathcal{V}$. And, we shall prove that $\mathcal{H}(\lambda_n, v_n)$ converges strongly to $\mathcal{H}(\lambda, v)$ in $\mathcal{V}$. Let us start by setting

$$u_n = \mathcal{H}(\lambda_n, v_n), \quad u = \mathcal{H}(\lambda, v). \tag{4.7}$$

From (4.7), we know that for a fixed n the function u_n stands to satisfy the following weak formulation

$$\begin{aligned} &u_n \in \mathcal{W}(0,T), \quad u_n(0,x)=0 \text{ in } L^2(\Omega),\\ &\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} \nabla u_n \nabla\varphi \, dxdt + \int_{\Sigma_T} \gamma(t,\sigma,u_n)\varphi \, d\sigma dt \\ &+ \int_{Q_T} \lambda_n G(t,x,v_n,\nabla v_n)\varphi \, dxdt = \lambda_n \int_{Q_T} f\varphi \, dxdt + \lambda_n \int_{\Sigma_T} g\varphi \, d\sigma dt. \end{aligned} \tag{4.8}$$

Furthermore, by estimate (4.7), it follows that

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} + \|u_n\|_{\mathcal{V}} \leq C\left(\|\mathfrak{M}\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)}\right) \tag{4.9}$$

where C is a constant depending only on T, Ω, N and γ_0. In view of (4.9), we deduce that the sequence (u_n) is bounded in $\mathcal{V}$. Let us return back to estimate the time derivative $\frac{\partial u_n}{\partial t}$ in the space $\mathcal{V}^*$. By taking into account the weak formulation (4.8), we get

$$\begin{aligned}\left|\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi \right\rangle dt\right| \leq & \int_{Q_T} |\nabla u_n \nabla \varphi| \, dxdt + \int_{\Sigma_T} |\gamma(t,\sigma,u_n)\varphi| \, d\sigma dt \\ & + \int_{Q_T} |G(t,x,v_n,\nabla v_n)\varphi| \, dxdt + \int_{Q_T} |f\varphi| \, dxdt \\ & + \int_{\Sigma_T} |g\varphi| \, d\sigma dt.\end{aligned} \tag{4.10}$$

With the help of (4.1) and assumption (A_3), we may use Hölder inequality on the right-hand side of (4.10). Then

$$\begin{aligned}\left|\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi \right\rangle dt\right| \leq & \|\nabla u_n\|_{L^2(Q_T)} \|\nabla \varphi\|_{L^2(Q_T)} \\ & + \gamma_1 \left(1 + \||u_n|^\theta\|_{L^2(\Sigma_T)}\right) \|\varphi\|_{L^2(\Sigma_T)} \\ & + \left(\|\mathfrak{M}\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)}\right) \|\varphi\|_{L^2(Q_T)} \\ & + \|g\|_{L^2(\Sigma_T)} \|\varphi\|_{L^2(\Sigma_T)}.\end{aligned} \tag{4.11}$$

Thanks to the trace theorem (see for example [17, Theorem 4.1.3]), and since $1 \leq s < \frac{N+2}{N} < \frac{N}{N-2}$, we derive from (4.11) the following inequality

$$\begin{aligned}\left|\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi \right\rangle dt\right| \leq & \|u_n\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}} + \gamma_1 \left(1 + \|u_n\|_{\mathcal{V}}^\theta\right) \|\varphi\|_{\mathcal{V}} \\ & + C\left(\|\mathfrak{M}\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)}\right) \|\varphi\|_{\mathcal{V}} \\ & + C\|g\|_{L^2(\Sigma_T)} \|\varphi\|_{\mathcal{V}},\end{aligned} \tag{4.12}$$

where C is a constant independent of n. By using the estimates (4.9) and (4.12), we conclude that $\frac{\partial u_n}{\partial t}$ is uniformly bounded in $\mathcal{V}^*$. At this stage, we are able to use the compactness result of (2.2) and therefore there exists a subsequence of (u_n), for simplicity denoted again by (u_n), such that

$$u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e in } Q_T. \tag{4.13}$$

In addition, the boundness of (u_n) in $\mathcal{W}(0,T)$ permits us to conclude the following weak convergence:

$$u_n \rightharpoonup u \text{ weakly in } \mathcal{W}(0,T). \tag{4.14}$$

Furthermore, by the trace theorem (see [22, Theorem 3.4.1]), we get

$$u_n \to u \text{ strongly in } L^2(\Sigma_T) \text{ and a.e in } \Sigma_T. \tag{4.15}$$

By employing the growth assumption (2.5) and (4.15), it results that

$$\gamma(t,\sigma,u_n) \to \gamma(t,\sigma,u) \text{ strongly in } L^2(\Sigma_T) \text{ and a.e in } \Sigma_T. \tag{4.16}$$

Let us recall that (v_n) converges strongly to v in $\mathcal{V}$. This fact allows us to have the following almost everywhere convergence:

$$(v_n, \nabla v_n) \to (v, \nabla v) \text{ a.e in } Q_T. \tag{4.17}$$

In accordance with (4.1) and (4.17), one may apply the Lebesgue dominated convergence theorem to obtain

$$\lambda_n G(t,x,v_n,\nabla v_n) \to \lambda G(t,x,v,\nabla v) \text{ strongly in } L^2(Q_T) \text{ and a.e in } Q_T. \tag{4.18}$$

As we can see, the continuity of the mapping $\mathcal{H}$ requires the strong convergence of (∇u_n) in $L^2(Q_T)^N$. To do so, we take the difference between the two weak formations (4.4) and (4.8) with $\varphi = (u_n - u)$ as a test function. Then we have

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial(u_n-u)}{\partial t}, (u_n-u)\right\rangle dt + \int_{Q_T} |\nabla u_n - \nabla u|^2 \, dxdt \\
&+ \int_{\Sigma_T} (\gamma(t,\sigma,u_n) - \gamma(t,\sigma,u))\,(u_n-u)\, d\sigma dt \\
&+ \int_{Q_T} (\lambda_n G(t,x,v_n,\nabla v_n) - \lambda G(t,x,v,\nabla v))\,(u_n-u)\; dxdt, \\
&= (\lambda_n - \lambda)\int_{Q_T} f(u_n-u)\, dxdt + (\lambda_n-\lambda)\int_{\Sigma_T} g(u_n-u)\, d\sigma dt.
\end{aligned} \tag{4.19}$$

With the help of the Hölder inequality, we derive from (4.19) the following inequality

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u_n-u)^2(T)\, dx + \int_{Q_T} |\nabla u_n - \nabla u|^2 \, dxdt \\
&\le \|\gamma(t,\sigma,u_n) - \gamma(t,\sigma,u)\|_{L^2(\Sigma_T)} \|u_n - u\|_{L^2(\Sigma_T)} \\
&\quad + \|\lambda_n G(t,x,v_n,\nabla v_n) - \lambda G(t,x,v,\nabla v)\|_{L^2(Q_T)} \|u_n-u\|_{L^2(Q_T)} \\
&\quad + 2\|f\|_{L^2(Q_T)}\|u_n-u\|_{L^2(Q_T)} + 2\|g\|_{L^2(\Sigma_T)}\|u_n-u\|_{L^2(\Sigma_T)}.
\end{aligned} \tag{4.20}$$

Thanks to the strong convergences (4.13), (4.15), (4.16) and (4.18), we pass to the limit in (4.20) as $n \to \infty$, one obtains

$$\lim_{n\to\infty} \int_{Q_T} |\nabla u_n - \nabla u|^2 \, dxdt \le 0.$$

This proves that

$$\nabla u_n \to \nabla u \text{ strongly in } L^2(Q_T)^N \text{ and a.e in } Q_T. \tag{4.21}$$

Therefore, by (4.13) and (4.21), we derive that

$$u_n \to u \text{ strongly in } \mathcal{V}. \tag{4.22}$$

The obtained convergences (4.14), (4.16), (4.18) allow us to pass to the limit in all the terms of (4.8). By the fact that problem (4.3) has a unique solution, one may deduce that the mapping $\mathcal{H}$ is continuous.

Step 2. The mapping $\mathcal{H}$ is compact. Here, we aim to show that $\mathcal{H}$ is compact. To do so, let us take (λ_n, v_n) a bounded sequence in $[0,1]\times\mathcal{V}$, we shall prove that $(\mathcal{H}(\lambda_n, v_n))$ is relatively compact in $\mathcal{V}$. We start by setting $u_n = \mathcal{H}(\lambda_n, v_n))$. By following the same reasoning as in the continuity step, we can easy to establish that (u_n) is bounded in $\mathcal{W}(0,T)$. Therefore, we can deduce the existence of a measurable function $u : Q_T \to \mathbb{R}$ and a sub-sequence of (u_n) denoted again by (u_n) for simplicity such that

$$u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e in } Q_T. \tag{4.23}$$

$$u_n \to u \text{ strongly in } L^2(\Sigma_T) \text{ and a.e in } \Sigma_T. \tag{4.24}$$

$$\gamma(t,\sigma,u_n) \to \gamma(t,\sigma,u) \text{ strongly in } L^2(\Sigma_T) \text{ and a.e in } \Sigma_T. \tag{4.25}$$

As can be seen, the compactness of the mapping $\mathcal{H}$ requires establishing that (∇u_n) converges strongly to ∇u in $L^2(Q_T)^N$. This convergence makes the difference between the compactness proof of $\mathcal{H}$ and that of the continuity which we have already established in step 1. At this stage, we do not have any information about the almost everywhere convergence of $(v_n, \nabla v_n)$ in Q_T, this fact presents major difficulties. And so, we cannot deal with the terms $G(t,x,v_n,\nabla v_n)$ by following the same argument of obtaining (4.18). To overcome these difficulties, let us remark that from [18, 27] (see also [1] and [15] for Dirichlet boundary case), we can derive that

$$\nabla u_n \to \nabla u \text{ a.e in } Q_T. \tag{4.26}$$

Then, to prove that (∇u_n) converges to ∇u strongly in $L^2(Q_T)^N$, we subtract the weak formulation (4.8) for different sequence indexes n and m with $\varphi = (u_n - u_m)$ as a test function. One obtains

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial(u_n-u_m)}{\partial t}, (u_n-u_m)\right\rangle dt + \int_{Q_T} |\nabla u_n - \nabla u_m|^2 \, dxdt \\
&+ \int_{\Sigma_T} (\gamma(t,\sigma,u_n)-\gamma(t,\sigma,u_m))(u_n-u_m)\, d\sigma dt \\
&+ \int_{Q_T} (\lambda_n G(t,x,v_n,\nabla v_n) - \lambda_m G(t,x,v_m,\nabla v_m))(u_n-u_m)\, dxdt, \\
&= (\lambda_n-\lambda_m)\int_{Q_T} f(u_n-u_m)\, dxdt + (\lambda_n-\lambda_m)\int_{\Sigma_T} g(u_n-u_m)\, d\sigma dt.
\end{aligned} \tag{4.27}$$

To deal with (4.27), we use (4.1) with Hölder's inequality. It results that

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}(u_n-u_m)^2(T)\,dx+\int_{Q_T}\left|\nabla u_n-\nabla u_m\right|^2\,dxdt\\
&\leq \|\gamma(t,\sigma,u_n)-\gamma(t,\sigma,u_m)\|_{L^2(\Sigma_T)}\|u_n-u_m\|_{L^2(\Sigma_T)}\\
&\quad+2\|\mathfrak{M}\|_{L^2(Q_T)}\|u_n-u_m\|_{L^2(Q_T)}\\
&\quad+2\|f\|_{L^2(Q_T)}\|u_n-u_m\|_{L^2(Q_T)}\\
&\quad+2\|g\|_{L^2(\Sigma_T)}\|u_n-u_m\|_{L^2(\Sigma_T)}.
\end{aligned}\tag{4.28}$$

By employing the convergences (4.23), (4.24), (4.25) and (4.26), we may use Fatou's Lemma to pass to the limit in (4.28) as $m\to\infty$, one gets

$$\begin{aligned}
&\int_{Q_T}\left|\nabla u_n-\nabla u\right|^2\,dxdt\\
&\leq \|\gamma(t,\sigma,u_n)-\gamma(t,\sigma,u)\|_{L^2(\Sigma_T)}\|u_n-u\|_{L^2(\Sigma_T)}\\
&\quad+2\|\mathfrak{M}\|_{L^2(Q_T)}\|u_n-u\|_{L^2(Q_T)}\\
&\quad+2\|f\|_{L^2(Q_T)}\|u_n-u\|_{L^2(Q_T)}\\
&\quad+2\|g\|_{L^2(\Sigma_T)}\|u_n-u\|_{L^2(\Sigma_T)}.
\end{aligned}\tag{4.29}$$

Applying again the convergences(4.23), (4.24) and (4.25) to pass to the limit in (4.29) as $n\to\infty$, we arrive at

$$\lim_{n\to\infty}\int_{Q_T}\left|\nabla u_n-\nabla u\right|^2\,dxdt\leq 0,$$

which is equivalent to say that ∇u_n converges strongly to ∇u in $L^2(Q_T)^N$. Consequently, the mapping $\mathcal{H}$ is compact.

Step 3. Existence of radius $\mathcal{R}$ such that $\deg(u-\mathcal{H}(\lambda,u),\mathcal{B}_{\mathcal{R}},0)=1$. The purpose of this step is the construction of a nonnegative radius $\mathcal{R}$ independent of λ such that $u\neq\mathcal{H}(\lambda,u)$ for any $u\in\partial\mathcal{B}_{\mathcal{R}}$, $\lambda\in[0,1]$. To do this, we take $u\in\mathcal{V}$ such that $u=\mathcal{H}(\lambda,u)$ for some $\lambda\in[0,1]$. Hence, the measurable function u subjects to the following weak formulation

$$\begin{aligned}
&u\in\mathcal{W}(0,T),\quad u(0,x)=0\text{ in }L^2(\Omega),\\
&\int_0^T\left\langle\frac{\partial u}{\partial t},\varphi\right\rangle dt+\int_{Q_T}\nabla u\nabla\varphi\,dxdt+\int_{\Sigma_T}\gamma(t,\sigma,u)\varphi\,d\sigma dt\\
&+\int_{Q_T}\lambda G(t,x,u,\nabla u)\varphi\,dxdt=\lambda\int_{Q_T}f\varphi\,dxdt+\lambda\int_{\Sigma_T}g\varphi\,d\sigma dt,
\end{aligned}\tag{4.30}$$

for all test function $\varphi\in\mathcal{V}$. Furthermore, by taking into account assumption (4.1), we deduce from (4.6) that u satisfies the following estimate

$$\|u\|_{L^\infty(0,T;L^2(\Omega))}+\|u\|_{\mathcal{V}}\leq C\left(\|\mathfrak{M}\|_{L^2(Q_T)}+\|f\|_{L^2(Q_T)}+\|g\|_{L^2(\Sigma_T)}\right)$$

where C is a constant independent of λ. Then, by choosing

$$\mathcal{R} > C\left(\|\mathfrak{M}\|_{L^2(Q_T)} + \|f\|_{L^2(Q_T)} + \|g\|_{L^2(\Sigma_T)}\right),$$

we ensure that the Leray–Schauder topological degree $\deg(\mathcal{I}_d - \mathcal{H}(\lambda,\cdot), \mathcal{B}_{\mathcal{R}}, 0)$ is well defined in the ball $\mathcal{B}_{\mathcal{R}}$. In addition, we derive that the Leray–Schauder topological degree satisfies the following homotopy invariance property

$$\deg(\mathcal{I}_d - \mathcal{H}(1,\cdot), \mathcal{B}_{\mathcal{R}}, 0) = \deg(\mathcal{I}_d - \mathcal{H}(0,\cdot), \mathcal{B}_{\mathcal{R}}, 0). \tag{4.31}$$

To conclude, we need to check that $\deg(\mathcal{I}_d - \mathcal{H}(1,\cdot), \mathcal{B}_{\mathcal{R}}, 0) \neq 0$. To this done, we consider $\zeta = \mathcal{H}(0,\zeta)$, which means that ζ satisfies for all test function $\varphi \in \mathcal{V}$ the following weak formulation

$$\begin{gathered} \zeta \in \mathcal{W}(0,T), \quad \zeta(0,x) = 0 \text{ in } L^2(\Omega) \\ \int_0^T \left\langle \frac{\partial \zeta}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} \nabla\zeta\nabla\varphi \, dxdt + \int_{\Sigma_T} \gamma(t,\sigma,\zeta)\varphi \, d\sigma dt = 0. \end{gathered} \tag{4.32}$$

By choosing $\varphi = \zeta$ in (4.32), we obtain

$$\frac{1}{2}\int_\Omega \zeta^2(T)\, dx + \int_{Q_T} |\nabla\zeta|^2 \, dxdt + \int_{\Sigma_T} \gamma(t,\sigma,\zeta)\zeta \, d\sigma dt = 0. \tag{4.33}$$

By employing $(\mathcal{A}_2)$, we deduce from (4.33) the following inequality

$$\frac{1}{2}\int_\Omega \zeta^2(T)\, dx + \min\{1,\gamma_0\}\|\zeta\|_{\mathcal{V}}^2 \leq 0,$$

which implies that $\zeta = 0$ a.e in Q_T. Therefore, we conclude that

$$\deg(\mathcal{I}_d - \mathcal{H}(0,\cdot), \mathcal{B}_{\mathcal{R}}, 0) = 1.$$

By combining this result with that of (4.31), we arrive at

$$\deg(\mathcal{I}_d - \mathcal{H}(1,\cdot), \mathcal{B}_{\mathcal{R}}, 0) \neq 0.$$

Hence, a direct application of Leray–Schauder topological degree (see, e.g., [25]) permits us to derive the existence of u a weak solution to (1.1) that satisfies the weak formulation (4.2).

5. EXISTENCE RESULTS IN L^1 FRAMEWORK

In this section, we study the existence of weak solution to (1.1) when the given data (f,g) are nonnegative functions belonging to $L^1(Q_T)\times L^1(\Sigma_T)$. Furthermore, we assume that the nonlinearity $G(t,x,u,\nabla u)$ satisfies general growth conditions which are:

$$G(t,x,s,r)s \geq 0, \tag{5.1}$$

$$|G(t,x,s,r)| \leq \mu(|s|)\left(L(t,x) + \|r\|^2\right), \tag{5.2}$$

for all (s, r) in $\mathbb{R} \times \mathbb{R}^N$ and for a.e (t, x) in Q_T. Where L is a nonnegative function belonging to $L^1(Q_T)$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

As a fisrt step, we propose to define the notion of weak solution to (1.1) for the L^1 setting.

Definition 5.1. A measurable function $u : Q_T \to \mathbb{R}$ is said to be a weak solution to problem (1.1) if it satisfies the following conditions:

$$\begin{aligned} &u \in \mathcal{C}\left([0,T], L^1(\Omega)\right) \cap L^1(0,T;W^{1,1}(\Omega)), \quad G(t,x,u,\nabla u) \in L^1(Q_T), \\ &-\int_{Q_T} u\frac{\partial \varphi}{\partial t}\,dxdt + \int_{Q_T} \nabla u \nabla \varphi\,dxdt + \int_{\Sigma_T} \gamma(t,\sigma,u)\varphi\,d\sigma dt \\ &+\int_{Q_T} G(t,x,u,\nabla u)\varphi\,dxdt = \int_{Q_T} f\varphi\,dxdt + \int_{\Sigma_T} g\varphi\,d\sigma dt, \end{aligned} \tag{5.3}$$

for all test function $\varphi \in \mathcal{C}^1\left(\overline{Q_T}\right)$ such that $\varphi(T, \cdot) = 0$.

At this stage, we state the main result of this section.

Theorem 5.2. *We assume that* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$*,* (5.1) *and* (5.2) *hold and* $T > 0$ *fixed. Then the problem* (1.1) *has a weak solution* u *satisfying* $0 \le u \le w$ *a.e. in* Q_T *where* w *is the weak solution to* (6.2) *given in* (*Lemma 6.2, Appendix*)*.*

In what follows, two typical examples are presented in which the various assumptions on the nonlinearities G and γ of Theorem 5.2 are satisfied.

Example 5.3. As the first example, we take:

$$\begin{aligned} \gamma(t,\sigma,s) &= s|s|^{\theta-2}\mathfrak{a}(t,\sigma) \quad \text{for all } s \in \mathbb{R} \text{ and for a.e } (t,\sigma) \in \Sigma_T, \\ G(t,x,s,r) &= \arctan(s)\left(1 + \|r\|^2\right) \quad \text{for all } (s,r) \in \mathbb{R} \times \mathbb{R}^N \text{ and for a.e } (t,x) \in Q_T, \end{aligned}$$

where $1 \le \theta < \frac{N+2}{N}$ and $\mathfrak{a}$ is a measurable function belonging to $L^\infty(\Sigma_T)$ such that $0 < \underline{\mathfrak{a}} \le \mathfrak{a}(t,\sigma) \le \overline{\mathfrak{a}}$ for a.e $(t,\sigma) \in \Sigma_T$.

Example 5.4. For the second example, we consider

$$\begin{aligned} \gamma(t,\sigma,s) &= s\mathfrak{a}(t,\sigma) \quad \text{for all } s \in \mathbb{R} \text{ and for a.e } (t,\sigma) \in \Sigma_T; \\ G(t,x,s,r) &= \mathfrak{b}(t,x)s\left(1 + \|r\|^2\right) \quad \text{for all } (s,r) \in \mathbb{R} \times \mathbb{R}^N \text{ and for a.e } (t,x) \in Q_T, \end{aligned}$$

where $\mathfrak{a}$ is the function stated as above and $\mathfrak{b}$ is a measurable function belonging to $L^\infty(\Sigma_T)$ such that $0 < \underline{\mathfrak{b}} \le \mathfrak{b}(t,x) \le \overline{\mathfrak{b}}$ for a.e $(t,x) \in Q_T$.

Remark 5.5. Note that in the first example, we chose $\mathfrak{a}(t,\sigma) \equiv 0$, hence the boundary conditions associated with (1.1) are those of inhomogeneous Neumann. In the second example, choosing $\mathfrak{a}(t,\sigma) \neq 0$, the boundary conditions are Robin's inhomogeneous ones.

5.1. PROOF OF THEOREM 5.2

In order to prove Theorem 5.2, we start by introducing an approximate scheme with a more regular data, namely in L^2, for which we prove the existence of weak solutions, using the previous Theorem 4.1. Finally, through some a-priori estimates that we obtain by adapting some very interesting techniques, one proves that the solution of the approximated problem converges indeed to the solution of the proposed problem (1.1).

5.1.1. Approximating scheme

Let w be the nonnegative weak solution to (6.2). For all $n \in \mathbb{N}^*$, we introduce $w_n = \tau_n(w)$ and we define the Carathédory function G_n by

$$G_n(t,x,s,\xi) = \frac{G(t,x,s,\xi)}{1+\frac{1}{n}|G(t,x,s,\xi)|}.\mathbf{1}_{\{w\le n\}} \quad \text{a.e in } Q_T.$$

After that, we set

$$\begin{aligned} f_n &= \tau_n(f).\mathbf{1}_{\{w\le n\}}, \quad g_n = \tau_n(g).\mathbf{1}_{\{w\le n\}},\\ \widetilde{f}_n &= f.\mathbf{1}_{\{w\le n\}}, \quad \widetilde{g}_n = g_n.\mathbf{1}_{\{w\le n\}}. \end{aligned}$$

It is clear that the sequences (f_n), $g_n)$ are nonnegative and satisfying

$$0 \le f_n \le f, \quad (f_n) \to f \text{ in } L^1(Q_T) \quad \text{and} \quad \|f_n\|_{L^1(Q_T)} \le \|f\|_{L^1(Q_T)}, \tag{5.4}$$

$$0 \le g_n \le g, \quad (g_n) \to g \text{ in } L^1(\Sigma_T) \quad \text{and} \quad \|g_n\|_{L^1(\Sigma_T)} \le \|g\|_{L^1(\Sigma_T)}. \tag{5.5}$$

Now, we introduce the approximate problem of (1.1) as follows:

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta u_n + G_n(t,x,u_n,\nabla u_n) = f_n(t,x) & \text{in } Q_T,\\ u_n(0,x) = 0 & \text{in } \Omega,\\ \frac{\partial u_n}{\partial \nu} + \gamma(t,\sigma,u_n) = g_n(t,\sigma) & \text{on } \Sigma_T. \end{cases} \tag{5.6}$$

As a first step, we shall ensure the existence of a weak solution to the approached problem (5.6). This is the objective of the following lemma.

Lemma 5.6. *For any $n \in \mathbb{N}^*$, problem (5.6) has a weak solution u_n in the sense that*

$$u_n \in \mathcal{W}(0,T), \quad u_n(0,x) = 0 \text{ in } L^2(\Omega),$$

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} \nabla u_n \nabla \varphi \, dxdt + \int_{\Sigma_T} \gamma(t,\sigma,u_n)\varphi \, d\sigma dt \\ &+ \int_{Q_T} G_n(t,x,u_n,\nabla u_n)\varphi \, dxdt = \int_{Q_T} f_n \varphi \, dxdt + \int_{\Sigma_T} g_n \varphi \, d\sigma dt, \end{aligned} \tag{5.7}$$

for all test function $\varphi \in \mathcal{V}$. Moreover, we have

$$0 \le u_n \le w_n \le w. \tag{5.8}$$

Proof. To establish the result of Lemma 5.6, we remark that the nonlinearity is bounded by n. Then, by using the existence result of Theorem 3.2, we derive that for any $n \in \mathbb{N}^*$ problem (5.6) has a weak solution u_n which satisfies the weak formulation (5.7). It remains to show that u_n satisfies (5.8). To do so, we start by proving that u_n is nonnegative. Let us take $j'_\varepsilon(u_n)$ as a test function in (5.7). Then by integrating over Q_t, we obtain

$$\int_\Omega j_\varepsilon(u_n(t,x))\,dx + \int_{Q_t} |\nabla u_n|^2 j''_\varepsilon(u_n)\,dxdt + \int_{Q_t} G_n\left(t,x,u_n,\nabla u_n\right) j'_\varepsilon(u_n)\,dxdt$$
$$+ \int_{\Sigma_t} \gamma\left(t,\sigma,u_n\right) j'_\varepsilon(u_n)\,d\sigma dt = \int_{Q_t} f_n j'_\varepsilon(u_n)\,dxdt + \int_{\Sigma_t} g_n j'_\varepsilon(u_n)\,d\sigma dt.$$

The convexity of j_ε implies

$$\int_{Q_t} |\nabla u_n|^2\, j''_\varepsilon(u_n)\,dxdt \geq 0.$$

For the other terms, we use the fact that $j'_\epsilon(u_n) = 0$ on the set where $u_n \geq 0$. We therefore have

$$\begin{aligned} &\int_\Omega j_\varepsilon(u_n(t,x))\,dx + \int_{Q_t\cap[u_n<0]} G_n\left(t,x,u_n,\nabla u_n\right) j'_\varepsilon(u_n)\,dxdt \\ &+ \int_{\Sigma_t\cap[u_n<0]} \gamma\left(t,\sigma,u_n\right) j'_\varepsilon(u_n)\,d\sigma dt \leq \int_{Q_t} f_n j'_\varepsilon(u_n)\,dxdt + \int_{\Sigma_t} g_n j'_\varepsilon(u_n)\,d\sigma dt. \end{aligned} \tag{5.9}$$

By letting $\epsilon \to 0$ in (5.9), one obtains

$$\begin{aligned} &\int_\Omega u_n^-(t,x)\,dx - \int_{Q_t\cap[u_n<0]} G_n\left(t,x,u_n,\nabla u_n\right)\,dxdt \\ &- \int_{\Sigma_t\cap[u_n<0]} \gamma\left(t,\sigma,u_n\right)\,d\sigma dt \leq -\int_{Q_t} f_n\,dxdt - \int_{\Sigma_t} g_n\,d\sigma dt \leq 0. \end{aligned} \tag{5.10}$$

By using the sign conditions (2.4) and (5.1), the inequality (5.10) becomes

$$\int_\Omega u_n^-(t,x)\,dx \leq 0.$$

This proves that $u_n \geq 0$ almost everywhere in Q_T. As a result, we can derive from the sign conditions (2.4) and (5.1) that

$$\gamma(t,\sigma,u_n) \geq 0, \tag{5.11}$$
$$G_n(t,x,u_n,\nabla u_n) \geq 0. \tag{5.12}$$

Now, we are in the setting to show that $u_n \leq w_n$. To do this, we use a simple computation to seek the equation satisfied by w_n, we have

$$\begin{aligned}
\partial_t w_n &= \partial_t w.\tau_n'(w) = \partial_t w.\mathbf{1}_{\{w \leq n\}},\\
\nabla w_n &= \nabla w.\tau_n'(w) = \nabla w.\mathbf{1}_{\{w \leq n\}},\\
\Delta w_n &= \Delta w.\mathbf{1}_{\{w \leq n\}} + |\nabla w|^2.\tau_n''(w).
\end{aligned}$$

Since $0 \leq -\tau_n''(s) \leq C(n)$ and by using the fact that w is a weak solution to (6.2), it follows that w_n satisfies

$$\begin{cases}
\frac{\partial w_n}{\partial t} - \Delta w_n \geq \widetilde{f}_n(t,x) & \text{in } Q_T,\\
w_n(0,\cdot) = 0 & \text{in } \Omega,\\
\frac{\partial w_n}{\partial \nu} + \gamma(t,\sigma,w).\mathbf{1}_{\{w \leq n\}} = \widetilde{g}_n(t,\sigma) & \text{on } \Sigma_T.
\end{cases} \tag{5.13}$$

Thus, we can directly deduce that $(u_n - w_n)$ verifies the following weak formulation:

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial(u_n - w_n)}{\partial t}, \varphi \right\rangle dt + \int_{Q_T} \nabla(u_n - w_n)\nabla\varphi \, dxdt\\
&+ \int_{\Sigma_T} \left(\gamma(t,\sigma,u_n) - \gamma(t,\sigma,w).\mathbf{1}_{\{w \leq n\}}\right) \varphi \, d\sigma dt \\
&+ \int_{Q_T} G_n(t,x,u_n,\nabla u_n)\varphi \, dxdt \leq \int_{Q_T} (f_n - \widetilde{f}_n)\varphi \, dxdt + \int_{\Sigma_T} (g_n - \widetilde{g}_n)\varphi \, d\sigma dt,
\end{aligned} \tag{5.14}$$

for all nonnegative test function $\varphi \in \mathcal{V}$. Let us choose $(u_n - w_n)^+$ as a test function in (5.14). By employing (5.12), one gets

$$\begin{aligned}
&\int_{\Omega} \Pi(u_n - w_n) \, dx + \int_{Q_T} |\nabla(u_n - w_n)^+|^2 \, dxdt\\
&+ \int_{\Sigma_T} \left(\gamma(t,\sigma,u_n) - \gamma(t,\sigma,w).\mathbf{1}_{\{w \leq n\}}\right) (u_n - w_n)^+ \, d\sigma dt \\
&\leq \int_{Q_T} (f_n - \widetilde{f}_n)(u_n - w_n)^+ \, dxdt + \int_{\Sigma_T} (g_n - \widetilde{g}_n)(u_n - w_n)^+ \, d\sigma dt,
\end{aligned} \tag{5.15}$$

where $\Pi(y) = \int_0^y s^+ \, ds \geq 0$. Let us remark that $f_n - \widetilde{f}_n \leq 0$ and $g_n - \widetilde{g}_n \leq 0$. Then the inequality (5.15) is reduced to

$$\begin{aligned}
&\int_{Q_T} |\nabla(u_n - w_n)^+|^2 \, dxdt\\
&+ \int_{\Sigma_T} \left(\gamma(t,\sigma,u_n) - \gamma(t,\sigma,w).\mathbf{1}_{\{w \leq n\}}\right) (u_n - w_n)^+ \, d\sigma dt \leq 0.
\end{aligned} \tag{5.16}$$

Let us remark that we can split the second integral as follows:

$$\begin{aligned}
&\int\limits_{\Sigma_T} \left(\gamma(t,\sigma,u_n) - \gamma(t,\sigma,w).\mathbf{1}_{\{w\le n\}}\right)(u_n - w_n)^+ \, d\sigma dt \\
&= \int\limits_{\Sigma_T \cap [w\le n]} \left(\gamma(t,\sigma,u_n) - \gamma(t,\sigma,w_n)\right)(u_n - w_n)^+ \, d\sigma dt \\
&\quad + \int\limits_{\Sigma_T \cap [w> n]} \left(\gamma(t,\sigma,u_n)\right)(u_n - w_n)^+ \, d\sigma dt.
\end{aligned} \tag{5.17}$$

From (5.12), (5.16) and (5.17), we deduce that

$$\begin{aligned}
&\int\limits_{Q_T} |\nabla(u_n - w_n)^+|^2 \, dxdt \\
&+ \int\limits_{\Sigma_T \cap [w\le n]} \left(\gamma(t,\sigma,u_n) - \gamma(t,\sigma,w_n)\right)(u_n - w_n)^+ \, d\sigma dt \le 0.
\end{aligned} \tag{5.18}$$

We can deduce from inequality (5.18) the existence of a constant c such that $(u_n - w_n)^+ = c$ a.e. in Q_T and by using again (5.18) and assumption (2.3), we arrive at $(u_n - w_n)^+ = 0$ a.e. in Q_T. This is equivalent to say that $u_n \le w_n$ a.e in Q_T. $\square$

5.1.2. A priori estimates

In the sequel, we shall proceed to derive some adequate a-priori estimates on the solution u_n, and the nonlinearity $G_n(t,x,u_n,\nabla u_n)$, to eventually prove that, under appropriate additional assumptions, (u_n) converges to a solution of (1.1) as n tends to ∞.

Lemma 5.7. *Let u_n be the weak solution to the approximate problem* (5.6). *Then*

(i)
$$\sup_{0\le t\le T} \int\limits_{\Omega} |u_n(t,x)| \, dx \le C,$$

(ii)
$$\|G_n(t,x,u_n,\nabla u_n)\|_{L^1(Q_T)} + \|\gamma(t,\sigma,u_n)\|_{L^1(\Sigma_T)} \le C,$$

(iii)
$$\int\limits_{Q_T} |\nabla T_k(u_n)|^2 \, dx \le Ck.$$

Proof. (i) Let us consider the equation satisfied by u_n over Q_T. We have

$$\frac{\partial u_n}{\partial t} - \Delta u_n + G_n(t,x,u_n,\nabla u_n) = f_n \text{ in } Q_T. \tag{5.19}$$

Using the results of (5.4) and (5.12), we get

$$\frac{\partial u_n}{\partial t} - \Delta u_n \leq f \text{ in } Q_T.$$

Next, we integrate over Q_t for all $t \in (0,T)$ to obtain

$$\int\limits_{\Omega} u_n(t,x)\,dxdt + \int\limits_{\Sigma_t} \gamma(t,x,u_n)\,d\sigma dt \leq \int\limits_{Q_T} f\,dxdt + \int\limits_{\Sigma_T} g\,d\sigma dt.$$

Thanks to (5.11), we obtain

$$\sup_{0\leq t\leq T} \int\limits_{\Omega} |u_n(t,x)|\,dx \leq \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)}.$$

(ii) We consider again the equation satisfied by u_n, we have

$$\frac{\partial u_n}{\partial t} - \Delta u_n + G_n(t,x,u_n,\nabla u_n) = f_n \text{ in } Q_T. \tag{5.20}$$

By integrating over Q_T, we get the following equality

$$\begin{aligned}&\int\limits_{\Omega} u_n(T,x)\,dx + \int\limits_{Q_T} G_n(t,x,u_n,\nabla u_n)\,dxdt + \int\limits_{\Sigma_T} \gamma(t,\sigma,u_n)\,d\sigma dt\\ &= \int\limits_{Q_T} f_n\,dxdt + \int\limits_{\Sigma_T} g_n\,d\sigma dt,\end{aligned}$$

which yields

$$\|G_n(t,x,u_n,\nabla u_n)\|_{L^1(Q_T)} + \|\gamma(t,\sigma,u_n)\|_{L^1(\Sigma_T)} \leq \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)}.$$

(iii) Multiplying the equation (5.19) by the truncated function $T_k(u_n)$, and integrating on Q_T to obtain

$$\begin{aligned}&\int\limits_{Q_T} \frac{\partial S_k(u_n)}{\partial t}\,dxdt + \int\limits_{Q_T} |\nabla T_k(u_n)|^2\,dxdt\\ &+ \int\limits_{Q_T} G_n(t,x,u_n,\nabla u_n)\,T_k(u_n)\,dxdt + \int\limits_{\Sigma_T} \gamma(t,\sigma,u_n)\,T_k(u_n)\,d\sigma dt\\ &= \int\limits_{Q_T} T_k(u_n)\,f_n\,dxdt + \int\limits_{\Sigma_T} T_k(u_n) g_n\,d\sigma dt.\end{aligned} \tag{5.21}$$

Thanks to (5.8), (5.11), (5.12) and the fact that $u_n(0) = 0$, we get

$$\int\limits_{\Omega} S_k(u_n(T))\,dxdt + \int\limits_{Q_T} |\nabla T_k(u_n)|^2\,dxdt \leq k\left(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)}\right),$$

which completes the proof. □

Remark 5.8. With the help of (ii) from Lemma 5.7, one can use the compactness results of Lemma 6.1. Then, we derive the existence of a subsequence of (u_n), for simplicity again denoted by (u_n), such that

$$\begin{aligned} u_n &\longrightarrow u \text{ strongly in } L^1(0,T,W^{1,1}(\Omega)), \\ (u_n, \nabla u_n) &\longrightarrow (u, \nabla u) \text{ a.e. in } Q_T. \end{aligned}$$

As a result, one may deduce that

$$\begin{aligned} T_k(u_n) &\longrightarrow T_k(u) \text{ a.e in } Q_T, \\ \gamma(t,\sigma,u_n) &\longrightarrow \gamma(t,\sigma,u) \text{ a.e in } \Sigma_T. \end{aligned}$$

Furthermore, by employing (2.3), (2.5) and (5.8), one may apply the Lebesgue convergence theorem to get

$$T_k(u_n) \longrightarrow T_k(u) \text{ strongly in } L^2(Q_T), \tag{5.22}$$

$$\gamma(t,\sigma,u_n) \longrightarrow \gamma(t,\sigma,u) \text{ strongly in } L^1(\Sigma_T). \tag{5.23}$$

Lemma 5.9. *Let u_n be the sequence defined as above. Then, we have*

$$\lim_{k\mapsto+\infty} \sup_n \int\limits_{[u_n>k]} |G_n(t,x,u_n\nabla u_n)|\, dxdt = 0.$$

Proof. From relation (5.21), we derive

$$\int\limits_{Q_T} G_n(t,x,u_n,\nabla u_n)\, T_k(u_n)\, dxdt \le \int\limits_{Q_T} T_k(u_n)\, f_n\, dxdt + \int\limits_{\Sigma_T} T_k(u_n) g_n\, d\sigma dt.$$

Then for every $0 < M < k$, we have

$$\begin{aligned} k \int\limits_{[u_n>k]} G_n(t,x,u_n\nabla u_n)\, dxdt \le k &\left(\int\limits_{Q_T\cap[u_n>M]} f_n\, dxdt + \int\limits_{\Sigma_T\cap[u_n>M]} g_n\, d\sigma dt \right) \\ &+ M \left(\int\limits_{Q_T\cap[u_n\le M]} f_n\, dxdt + \int\limits_{\Sigma_T\cap[u_n\le M]} g_n\, d\sigma dt \right). \end{aligned}$$

Hence

$$\begin{aligned} \int\limits_{[u_n>k]} G_n(t,x,u_n,\nabla u_n)\, dxdt \le &\left(\int\limits_{Q_T} f\chi_{[u_n>M]}\, dxdt + \int\limits_{\Sigma_T} g\chi_{[u_n>M]}\, d\sigma dt \right) \\ &+ \frac{M}{k} \left(\int\limits_{Q_T} f\, dxdt + \int\limits_{\Sigma_T} g\, d\sigma dt \right). \end{aligned}$$

To finally obtain the sought result, we need to show that

$$\lim_{k\mapsto+\infty}\sup_n\left(\int\limits_{Q_T} f\chi_{[u_n>M]}\,dxdt+\int\limits_{\Sigma_T} g\chi_{[u_n>M]}\,d\sigma dt\right)=0.$$

To this aim, we shall use (5.8). We get

$$|\,[u_n>M]\,|\le\frac{1}{M}\|u_n\|_{L^1(Q_T)}\le\frac{1}{M}\|w\|_{L^1(Q_T)}$$

whence

$$\lim_{M\mapsto+\infty}\sup_n|\,[u_n>M]\,|=0.$$

Since $(f,g)\in L^1(Q_T)\times L^1(\Sigma_T)$, then is equi-integrable. Hence for each $\epsilon>0$ there exists (δ_1,δ_2) such that for all mesurable $E_1\subset Q_T$ and $E_2\subset\Sigma_T$ we have

$$|E_1|<\delta_1,\quad\int\limits_{E_1} f\,dxdt\le\frac{\epsilon}{3},$$
$$|E_2|<\delta_2,\quad\int\limits_{E_2} g\,d\sigma dt\le\frac{\epsilon}{3}.$$

According to the previous result, we obtain that for each $\epsilon>0$, there exists M_ϵ such that for all $M\ge M_\epsilon$

$$\sup_n\left(\int\limits_{Q_T} f\chi_{[u_n>M]}\,dxdt+\int\limits_{\Sigma_T} g\chi_{[u_n>M]}\,d\sigma dt\right)\le\frac{2\epsilon}{3}.$$

Choosing $M=M_\epsilon$ and letting k tend to infinity, we obtain

$$\lim_{k\mapsto+\infty}\sup_n\left(\int\limits_{[u_n>k]} G_n(t,x,u_n,\nabla u_n)\,dxdt\right)=0.$$

□

5.1.3. Strong convergence of truncations

This section takes interest in proving the strong convergence of $(T_k(u_n))$ in $L^2(0,T;H^1(\Omega))$. We have the following result:

Lemma 5.10. *Let (u_n) be the sequence defined as above. Then, we have*

$$(T_k(u_n))\to T_k(u)\ \textit{strongly in}\ L^2\left(0,T;H^1(\Omega)\right),$$

for every fixed $k>0$.

Proof. The proof will be done by following steps.

Step 1. First, let us take a smooth approximation of $T_k(u)$ denoted $T_k(u)_\nu$ that has the following properties:

$$\begin{cases} (T_k(u)_\nu)_t = \nu\left(T_k(u) - T_k(u)_\nu\right), \quad |T_k(u)_\nu| \leqslant k, \\ T_k(u)_\nu \to T_k(u) \text{ strongly in } L^2\left(0,T;H^1(\Omega)\right), \text{ as } \nu \text{ tends to infinity.} \end{cases} \tag{5.24}$$

In all what follows, we will denote by $\omega(n,\nu,h)$ all quantities (possibly different) such that

$$\lim_{h\to+\infty}\lim_{\nu\to+\infty}\lim_{n\to+\infty}\omega(n,\nu,h) = 0. \tag{5.25}$$

We take $\varphi_\lambda(u_n - T_k(u)_\nu)^-$ as a test function in (5.7) with $\varphi_\lambda(s) = se^{\lambda s^2}$, one gets

$$\begin{aligned} &\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^- \right\rangle dt + \int_{Q_T} \nabla u_n \nabla\varphi_\lambda(u_n - T_k(u)_\nu)^-\, dxdt \\ &+ \int_{\Sigma_T} \gamma(t,\sigma,u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^-\, d\sigma dt + \int_{Q_T} G_n(t,x,u_n,\nabla u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^-\, dxdt \\ &= \int_{Q_T} f_n\varphi_\lambda(u_n - T_k(u)_\nu)^-\, dxdt + \int_{\Sigma_T} g_n\varphi_\lambda(u_n - T_k(u)_\nu)^-\, d\sigma dt. \end{aligned} \tag{5.26}$$

It follows that

$$\begin{aligned} &\int_{[u_n\leq T_k(u)_\nu]} \left(\nabla u_n - \nabla(T_k(u)_\nu)\right)\nabla\left(u_n - T_k(u)_\nu\right)\varphi'_\lambda\left(u_n - T_k(u)_\nu\right)^-\, dxdt \\ &\leq -\int_{Q_T} f_n\varphi_\lambda\left(u_n - T_k(u)_\nu\right)^-\, dxdt - \int_{\Sigma_T} g_n\varphi_\lambda\left(u_n - T_k(u)_\nu\right)^-\, d\sigma dt \\ &\quad + \int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^- \right\rangle dt - \int_{Q_T} \nabla T_k(u)_\nu \nabla\varphi_\lambda\left(u_n - T_k(u)_\nu\right)^-\, dxdt \\ &\quad + \int_{\Sigma_T} \gamma(t,\sigma,u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^-\, d\sigma dt \\ &\quad + \int_{Q_T} G_n(t,x,u_n,\nabla u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^-\, dxdt. \end{aligned} \tag{5.27}$$

Now, since $(T_k(u)_\nu)$ is bounded by k, then $\varphi_\lambda(u_n - T_k(u)_\nu)^- \equiv 0$ on the set where $u_n > k$. Hence

$$\begin{aligned}&\int_{Q_T} \nabla(T_k(u)_\nu)\nabla\left(u_n - T_k(u)_\nu\right)^- \varphi'_\lambda\left(u_n - T_k(u)_\nu\right)^- dxdt\\ &= \int_{Q_T} \nabla(T_k(u)_\nu)\nabla\left(T_k(u_n) - T_k(u)_\nu\right)\varphi'_\lambda\left(u_n - T_k(u)_\nu\right)^- dxdt.\end{aligned}$$

Passing to the limit via Lebesgue's convergence theorem, we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\int_{Q_T} \nabla(T_k(u)_\nu)\nabla\left(u_n - T_k(u)_\nu\right)^- \varphi'_\lambda\left(u_n - T_k(u)_\nu\right)^- dxdt\\ &= \int_{Q_T} \nabla(T_k(u)_\nu)\nabla\left(T_k(u) - T_k(u)_\nu\right)\varphi'_\lambda\left(u - T_k(u)_\nu\right)^- dxdt.\end{aligned}$$

Recalling that $(T_k(u)_\nu) \to T_k(u)$ strongly in $L^2\left(0,T;H^1(\Omega)\right)$ and a.e. in Q_T, one gets

$$\int_{Q_T} \nabla(T_k(u)_\nu)\nabla\left(u_n - T_k(u)_\nu\right)^- \varphi'_\lambda\left(u_n - T_k(u)_\nu\right)^- dxdt = \omega(n,\nu).$$

Now, using (5.27), we obtain

$$\begin{aligned}&\int_{[u_n\le T_k(u)_\nu]} \left(\nabla u_n - \nabla(T_k(u)_\nu)\right)\nabla\left(u_n - T_k(u)_\nu\right)\varphi'_\lambda\left(u_n - T_k(u)_\nu\right)^- dxdt\\ &\le -\int_{Q_T} f_n\varphi_\lambda\left(u_n - T_k(u)_\nu\right)^- dxdt - \int_{\Sigma_T} g_n\varphi_\lambda\left(u_n - T_k(u)_\nu\right)^- d\sigma dt\\ &\quad+\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^-\right\rangle dt + \int_{\Sigma_T}\gamma(t,\sigma,u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^- d\sigma dt\\ &\quad+\int_{Q_T} G_n(t,x,u_n,\nabla u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt + \omega(n,\nu).\end{aligned} \tag{5.28}$$

Since $\varphi_\lambda(u_n - T_k(u)_\nu)^- \equiv 0$, if $u_n > k$, we also have

$$\begin{aligned}
&\int_{Q_T} |\nabla u_n|^2 \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt \\
&\le \int_{Q_T} \nabla u_n \nabla\left(u_n - T_k(u)_\nu\right) \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt \\
&\quad + \int_{Q_T} \nabla u_n \nabla T_k(u)_\nu \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt \\
&\le \int_{[u_n \le T_k(u)_\nu]} |\nabla u_n - \nabla T_k(u)_\nu|^2 \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt \\
&\quad + \int_{Q_T} \nabla T_k(u_n) \nabla T_k(u)_\nu \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt \\
&\quad + \int_{Q_T} \nabla T_k(u)_\nu \nabla\left(u_n - T_k(u)_\nu\right) \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt.
\end{aligned}$$

Recalling the convergence of $(T_k(u)_\nu)$ to $T_k(u)$ in $L^2\left(0,T;H^1(\Omega)\right)$, as well as the convergence of $(T_k(u_n))$ to $T_k(u)$ weakly in $L^2\left(0,T;H^1(\Omega)\right)$ and the fact that $\varphi_\lambda\left(u - T_k(u)\right)^- \equiv 0$. We find, as n and ν tend to infinity,

$$\begin{aligned}
&\int_{Q_T} |\nabla u_n|^2 \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt \\
&\le \int_{[u_n \le T_k(u)_\nu]} |\nabla u_n - \nabla T_k(u)_\nu|^2 \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt + \omega(n,\nu).
\end{aligned}$$

And so, (5.28) becomes

$$\begin{aligned}
&\int_{[u_n \le T_k(u)_\nu]} |\nabla u_n - \nabla T_k(u)_\nu|^2 \varphi_\lambda^{'}(u_n - T_k(u)_\nu)^- dxdt \\
&\le \int_{Q_T} \mu(k) |\nabla u_n|^2 \varphi_\lambda(u_n - T_k(u)_\nu)^- dxdt \\
&\quad + \omega(n,\nu) + \int_{\Sigma_T} \gamma(t,\sigma,u_n) \varphi_\lambda(u_n - T_k(u)_\nu)^- d\sigma dt \\
&\quad + \int_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^- \right\rangle dt \\
&\quad - \int_{Q_T} f_n \varphi_\lambda\left(u_n - T_k(u)_\nu\right)^- dxdt - \int_{\Sigma_T} g_n \varphi_\lambda\left(u_n - T_k(u)_\nu\right)^- d\sigma dt,
\end{aligned}$$

hence

$$\begin{aligned}
&\int\limits_{[u_n \le T_k(u)_\nu]} |\nabla u_n - \nabla T_k(u)_\nu|^2 [\varphi_\lambda^{'} - \mu(k)\varphi_\lambda] dxdt \\
&\le \omega(n,\nu) + \int\limits_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^- \right\rangle dt \\
&+ \int\limits_{\Sigma_T} \gamma(t,\sigma,u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^- d\sigma dt - \int\limits_{Q_T} f_n \varphi_\lambda \left(u_n - T_k(u)_\nu\right)^- dxdt \\
&- \int\limits_{\Sigma_T} g_n \varphi_\lambda \left(u_n - T_k(u)_\nu\right)^- d\sigma dt.
\end{aligned}$$

Choosing λ large enough such that $\varphi_\lambda^{'}(s) - \mu(k)\varphi_\lambda(s) \ge \alpha_0$ we obtain what follows:

$$\begin{aligned}
&\alpha_0 \int\limits_{[u_n \le T_k(u)_\nu]} |\nabla u_n - \nabla T_k(u)_\nu|^2 dxdt \\
&\le \omega(n,\nu) + \int\limits_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^- \right\rangle dt \\
&+ \int\limits_{\Sigma_T} \gamma(t,\sigma,u_n)\varphi_\lambda(u_n - T_k(u)_\nu)^- d\sigma dt - \int\limits_{Q_T} f_n \varphi_\lambda \left(u_n - T_k(u)_\nu\right)^- dxdt \\
&- \int\limits_{\Sigma_T} g_n \varphi_\lambda \left(u_n - T_k(u)_\nu\right)^- d\sigma dt.
\end{aligned} \tag{5.29}$$

Recalling the convergence (5.23), as well as the convergence of $\varphi_\lambda \left(u_n - T_k(u)_\nu\right)^-$ to $\varphi_\lambda \left(u - T_k(u)_\nu\right)^-$ as n tends to infinity, we obtain

$$\int\limits_{\Sigma_T} \gamma\left(t,\sigma,u_n\right) \varphi_\lambda \left(u_n - T_k(u)_\nu\right)^- d\sigma dt = \omega(n,\nu). \tag{5.30}$$

In the same way, one may use the convergence results of (5.4) and (5.5) to deduce that

$$\int\limits_{Q_T} f_n \varphi_\lambda \left(u_n - T_k(u)_\nu\right)^- dxdt = \omega(n,\nu), \tag{5.31}$$

$$\int\limits_{\Sigma_T} g_n \varphi_\lambda \left(u_n - T_k(u)_\nu\right)^- d\sigma dt = \omega(n,\nu). \tag{5.32}$$

By recapping (5.29), (5.30), (5.31) and (5.32), one obtains

$$\alpha_0 \int\limits_{[u_n \leq T_k(u)_\nu]} |\nabla u_n - \nabla T_k(u)_\nu|^2 dxdt \leq \omega(n,\nu) + \int\limits_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^- \right\rangle dt. \tag{5.33}$$

On the other hand, by following the same idea of [26], one has no difficulty verifying that

$$\int\limits_0^T \left\langle \frac{\partial u_n}{\partial t}, \varphi_\lambda(u_n - T_k(u)_\nu)^- \right\rangle dt \leq \omega(n,\nu,h). \tag{5.34}$$

Using (5.34), inequality (5.33) becomes

$$\int\limits_{[u_n \leq T_k(u)_\nu]} |\nabla u_n - \nabla T_k(u)_\nu|^2 dxdt \leq \omega(n,\nu).$$

Step 2. Closely following the technique of [26], we choose

$$\Psi_n = T_{2k}\left(u_n - T_h\left(u_n\right) + T_k\left(u_n\right) - T_k(u)_\nu\right),$$

as a test function in (5.7), with $h > k > 0$. We have

$$\begin{aligned} &\int\limits_0^T \left\langle \frac{\partial u_n}{\partial t}, \Psi_n \right\rangle dt + \int\limits_{Q_T} \nabla u_n \nabla \Psi_n dxdt + \int\limits_{\Sigma_T} \gamma(t,\sigma,u_n)\Psi_n d\sigma dt \\ &+ \int\limits_{Q_T} G_n(t,x,u_n,\nabla u_n)\Psi_n dxdt = \int\limits_{Q_T} f_n \Psi_n dxdt + \int\limits_{\Sigma_T} g_n \Psi_n d\sigma dt. \end{aligned} \tag{5.35}$$

Since Ψ_n is positive, one may use results of (5.11) and (5.12) to achieve that

$$\int\limits_{\Sigma_T} \gamma(t,\sigma,u_n)\Psi_n d\sigma dt + \int\limits_{Q_T} G_n(t,x,u_n,\nabla u_n)\Psi_n dxdt \geq 0. \tag{5.36}$$

On the other hand, by a simple changes in the proof of Lemma 2.1 from [26], we can prove that

$$\int\limits_0^T \left\langle \frac{\partial u_n}{\partial t}, \Psi_n \right\rangle dt \geq \omega(n,\nu,h). \tag{5.37}$$

Then by using (5.36) and (5.37), equality (5.35) becomes

$$\int\limits_{Q_T} \nabla u_n \nabla \Psi_n dxdt \leq \int\limits_{Q_T} f_n \Psi_n dxdt + \int\limits_{\Sigma_T} g_n \Psi_n d\sigma dt + \omega(n,\nu,h). \tag{5.38}$$

Now, note that $\nabla\Psi_n = 0$ if $|u_n| > h + 4k$. Let us then set $M = h + 4k$. We will start by splitting the first integral on the left hand side of (5.38) on the sets $[|u_n| > k]$ and $[|u_n| \le k]$, we then obtain for n large:

$$\begin{aligned}\int\limits_{Q_T} \nabla u_n \nabla\Psi_n dxdt &= \int\limits_{Q_T} \nabla T_M(u_n)\nabla\Psi_n dxdt \\ &\ge \int\limits_{Q_T} \nabla T_k(u_n).\nabla\left(T_k(u_n) - T_k(u)_\nu\right) - \int\limits_{[|u_n|>k]} |\nabla T_M(u_n)||\nabla T_k(u)_\nu|.\end{aligned} \tag{5.39}$$

The last term of the equation (5.39) can be dealt with in the following way:

$$\begin{aligned}&\int\limits_{[|u_n|>k]} |\nabla T_M(u_n)||\nabla T_k(u)_\nu| dxdt \\ &\le \int\limits_{Q_T} |\nabla T_M(u_n)||\nabla T_k(u)|\chi_{[|u_n|>k]} dxdt \\ &\quad + \int\limits_{[|u_n|>k]} |\nabla T_M(u_n)||\nabla T_k(u)_\nu - \nabla T_k(u)| dxdt,\end{aligned}$$

hence

$$C_1(M)\int\limits_{Q_T} |\nabla T_k(u)|\chi_{[|u_n|>k]} + C_2(M)\int\limits_{Q_T} |\nabla T_k(u)_\nu - \nabla T_k(u)| \le w(n,\nu,h), \tag{5.40}$$

where $C_1(M)$ and $C_2(M)$ are constants depending on M. This implies that

$$\int\limits_{Q_T} \nabla T_k(u_n).\nabla\left(T_k(u_n) - T_k(u)_\nu\right) dxdt \le \int\limits_{Q_T} f_n\Psi_n + \int\limits_{\Sigma_T} g_n\Psi_n + \omega(n,\nu,h). \tag{5.41}$$

The other terms of the same equation (5.38) will be dealt with in the following way:

$$\begin{aligned}\int\limits_{Q_T} f_n\Psi_n dxdt &= \int\limits_{Q} fT_{2k}\left(u - T_h(u) + T_k(u) - T_k(u)_\nu\right) dxdt + \omega(n) \\ &= \int\limits_{Q} fT_{2k}\left(u - T_h(u)\right) dxdt + \omega(n,\nu),\end{aligned} \tag{5.42}$$

hence

$$\int\limits_{Q_T} f_n\Psi_n dxdt = \omega(n,\nu,h). \tag{5.43}$$

In the same way, we deduce that

$$\int\limits_{\Sigma_T} g_n\Psi_n d\sigma dt = \omega(n,\nu,h). \tag{5.44}$$

Therefore, we can conclude that

$$\int_{Q_T} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \leq \omega(n,\nu,h) - \int_{Q_T} \nabla T_k(u).\left(\nabla T_k(u_n) - \nabla T_k(u)\right).$$

Since $(T_k(u_n))$ converges to $T_k(u)$ weakly in $L^2\left(0,T;H^1(\Omega)\right)$. Letting first n tend to infinity, then respectively ν and h, we finally obtain that $(\nabla T_k(u_n))$ converges to $\nabla T_k(u)$ strongly in $L^2(Q_T)$ for every fixed $k>0$, and by using (5.22) we conclude the proof. □

5.1.4. Passing to the limit

Let $\varphi \in \mathcal{C}^1\left(\overline{Q_T}\right)$ such that $\varphi(T,\cdot)=0$. Multiplying the first equation of (5.6) by φ, integrating over Q_T and applying integration by part formula, one has

$$\begin{aligned} &-\int_{Q_T} u_n \frac{\partial\varphi}{\partial t}\,dxdt + \int_{Q_T} \nabla u_n \nabla\varphi\,dxdt + \int_{\Sigma_T} \gamma(t,\sigma,u_n)\varphi\,d\sigma dt \\ &+\int_{Q_T} G_n(t,x,u_n,\nabla u_n)\varphi\,dxdt = \int_{Q_T} f_n\varphi\,dxdt + \int_{\Sigma_T} g_n\varphi\,d\sigma dt. \end{aligned} \tag{5.45}$$

Our aim is to pass to the limit in (5.45) as n goes to ∞. By using Remark 5.8, we have the following convergence result

$$u_n \longrightarrow u \text{ strongly in } L^1(0,T,W^{1,1}(\Omega)), \tag{5.46}$$

$$(u_n,\nabla u_n) \longrightarrow (u,\nabla u) \text{ a.e. in } Q_T, \tag{5.47}$$

$$\gamma(t,\sigma,u_n) \longrightarrow \gamma(t,\sigma,u) \text{ strongly in } L^1(\Sigma_T) \text{ and a.e. in } \Sigma_T, \tag{5.48}$$

$$G_n(t,x,u_n,\nabla u_n) \longrightarrow G(t,x,u,\nabla u) \text{ a.e. in } Q_T. \tag{5.49}$$

Finally, we need now to prove that the convergence in (5.49) holds in $L^1(Q_T)$. To this aim, we propose to use the Vitali lemma. Let us then prove that for each $\varepsilon>0$ there exists $\theta>0$ such that

$$(\mathbb{K}\subset Q_T \text{ measurable, } \operatorname{meas}(\mathbb{K})<\theta) \Longrightarrow \int_{\mathbb{K}} |G_n(t,x,u_n,\nabla u_n)|\,dxdt < \varepsilon \quad \forall n. \tag{5.50}$$

In order to prove (5.50), we divide the integral as follows:

$$\begin{aligned} \int_{\mathbb{K}} |G_n(t,x,u_n,\nabla u_n)|\,dxdt \leq &\int_{\mathbb{K}\cap[u_n>k]} |G_n(t,x,u_n,\nabla u_n)|\,dxdt \\ &+\int_{\mathbb{K}\cap[u_n\leq k]} |G_n(t,x,u_n,\nabla u_n)|\,dxdt := I_1 + I_2. \end{aligned}$$

The first integral I_1 verifies the following inequality

$$I_1 \leq \int_{[u_n>k]} G_n\left(t,x,u_n,\nabla u_n\right)dxdt,$$

Lemma 5.9 insures the existence of $k^* > 0$, such that, for all $k \geqslant k^*$, we have

$$I_1 \leq \frac{\epsilon}{3}.$$

As the second integral I_2 is concerned, assumption (5.2) implies that for all $k \geq k^*$

$$I_2 \leq \mu(k) \int\limits_{\mathbb{K}} \left(L(t,x) + |\nabla T_k(u_n)|^2 \right) dxdt.$$

On the one hand, since $L \in L^1(Q_T)$, then L is equi-integrable in $L^1(Q_T)$, hence there exists $\delta_1 > 0$, such that, if $|\mathbb{K}| \leq \delta_1$, then

$$\mu(k) \int\limits_{\mathbb{K}} L(t,x) dxdt \leq \frac{\epsilon}{3}.$$

On the other hand, using Lemma 5.10, the sequence $\left(|\nabla T_k(u_n)|^2\right)_n$ is equi-integrable in $L^1(Q_T)$. Consequently, there exists $\delta_2 > 0$ such that if $|\mathbb{K}| \leq \delta_2$, we have

$$\mu(k) \int\limits_{\mathbb{K}} |\nabla T_k(u_n)|^2 dxdt \leq \frac{\epsilon}{3}.$$

Last but not least, if we choose $\delta^* = \inf(\delta_1, \delta_2)$ and if $|\mathbb{K}| \leq \delta^*$, we obtain

$$\int\limits_{\mathbb{K}} G_n(t,x,u_n,\nabla u_n) \, dxdt \leq \varepsilon,$$

which concludes the proof of our statement.

6. APPENDIX

This section tackles the proof of some interesting results which rely strongly on parabolic equations with nonlinear boundary conditions.

Lemma 6.1. *Let (u_n) be a weak solution to the following problem:*

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta u_n = \kappa_n & \text{in } Q_T, \\ u_n(0,x) = 0 & \text{in } \Omega, \\ \frac{\partial u_n}{\partial \nu} = \tau_n & \text{on } \Sigma_T, \end{cases} \tag{6.1}$$

such that the following estimate holds:

$$\|\kappa_n\|_{L^1(Q_T)} + \|\tau_n\|_{L^1(\Sigma_T)} \leq C.$$

Then, there exists u in $L^1(0,T,W^{1,1}(\Omega))$ such that, up to a subsequence, the following convergences are satisfied:

$$\begin{aligned} u_n &\longrightarrow u \text{ strongly in } L^1(0,T,W^{1,1}(\Omega)),\\ (u_n,\nabla u_n) &\longrightarrow (u,\nabla u) \text{ a.e. in } Q_T,\\ u_n &\longrightarrow u \text{ strongly in } L^r(\Sigma_T) \quad \text{for all } 1\le r<\frac{N+2}{N}. \end{aligned}$$

Proof. The proof of this lemma is more or less classical (we refer the reader to [10, Lemma 5]; see also [9]). □

The above result will be used to deal with the compactness result for the heat parabolic equation with nonlinear boundary conditions. In the following lemma, we will establish the existence of a weak solution to (3.1) in the L^1 setting.

Lemma 6.2. *Assume that* (2.5) *holds and let* (f,g) *be two nonnegative measurable functions belonging to* $L^1(Q_T)\times L^1(\Sigma_T)$. *Let us consider the following problem:*

$$\begin{cases} \frac{\partial w}{\partial t}-\Delta w=f(t,x) & \text{in } Q_T\\ w(0,\cdot)=0 & \text{in } \Omega\\ \frac{\partial w}{\partial \nu}+\gamma(t,\sigma,w)=g(t,\sigma) & \text{on } \Sigma_T. \end{cases} \tag{6.2}$$

Then, problem (6.2) *admits a weak solution w in the following sense:*

$$w\in\mathcal{C}\left([0,T],L^1(\Omega)\right)\cap L^1\left(0,T,W^{1,1}(\Omega)\right),$$

$$\begin{aligned} &-\int_{Q_T} w\frac{\partial\varphi}{\partial t}\,dxdt+\int_{Q_T}\nabla w\nabla\varphi\,dxdt+\int_{\Sigma_T}\gamma(t,\sigma,w)\varphi\,d\sigma dt\\ &=\int_{Q_T} f\varphi\,dxdt+\int_{\Sigma_T} g\varphi\,d\sigma dt, \end{aligned}$$

for all $\varphi\in\mathcal{C}^1\left(\overline{Q_T}\right)$ *such that* $\varphi(T,\cdot)=0$.

Proof. Since $f\in L^1(Q_T)^+$ and $g\in L^1(\Sigma_T)^+$ then we can construct a pair of sequence $(f_n)\in L^2(Q_T)$ and $(g_n)\in L^2(\Sigma_T)$ such that

$$\begin{aligned} 0\le f_n\le f \quad &\text{and} \quad f_n\to f \text{ strongly in } L^1(Q_T),\\ 0\le g_n\le g \quad &\text{and} \quad g_n\to g \text{ strongly in } L^1(\Sigma_T). \end{aligned}$$

From the result of Theorem 3.2, we know the existence and uniqueness of w_n a weak solution to the following problem:

$$\begin{cases} \frac{\partial w_n}{\partial t}-\Delta w_n=f_n(t,x) & \text{in } Q_T,\\ w_n(0,\cdot)=0 & \text{in } \Omega,\\ \frac{\partial u_n}{\partial \nu}+\gamma(t,\sigma,w_n)=g_n(t,\sigma) & \text{on } \Sigma_T. \end{cases} \tag{6.3}$$

Since (f_n) and (g_n) are nonnegative functions, we derive that $w_n \geq 0$. Furthermore, by using the fact that $f_n \leq f$ and $g_n \leq g$, we get

$$\frac{\partial w_n}{\partial t} - \Delta w_n \leq f. \tag{6.4}$$

Next, we integrate the first equation of (6.3) over Q_T, one obtains

$$\int_{\Omega} w_n(T,x)\,dx + \int_{\Sigma_T} \gamma(t,\sigma,w_n)\,d\sigma dt \leq \int_{Q_T} f\,dxdt + \int_{\Sigma_T} g\,d\sigma dt. \tag{6.5}$$

Since $w_n \geq 0$, the following estimate holds:

$$\|\gamma(w_n)\|_{L^1(\Sigma_T)} \leq \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)}.$$

Then, by using the compactness result of Lemma 6.1, we derive the existence of w in $L^1(0,T,W^{1,1}(\Omega))$ such that, up to a subsequence, the following conditions are satisfied:

$$\begin{aligned} w_n &\longrightarrow u \text{ strongly in } L^1(0,T,W^{1,1}(\Omega)),\\ (w_n, \nabla w_n) &\longrightarrow (w, \nabla w) \text{ a.e. in } Q_T,\\ w_n &\longrightarrow w \text{ strongly in } L^r(\Sigma_T) \quad \text{for all } 1 \leq r < \frac{N+2}{N}. \end{aligned}$$

Furthermore, by assumption (2.5), we have

$$|\gamma(t,\sigma,r)| \leq \gamma_1(1+|r|^\theta) \text{ a.e. } (t,\sigma) \text{ in } \Sigma_T \text{ for all } r \in \mathbb{R}, \text{ and } 1 \leq \theta < \frac{N+2}{N}.$$

Hence $\gamma(t,\sigma,w_n) \to \gamma(t,\sigma,w)$ strongly in $L^1(\Sigma_T)$ and then passing to the limit in the weak formulation of (6.3) leads to

$$\begin{aligned} &-\int_{Q_T} w\frac{\partial \varphi}{\partial t}\,dxdt + \int_{Q_T} \nabla w \nabla\varphi\,dxdt + \int_{\Sigma_T} \gamma(t,\sigma,w)\varphi\,d\sigma dt\\ &= \int_{Q_T} f\varphi\,dxdt + \int_{\Sigma_T} g\varphi\,d\sigma dt, \end{aligned}$$

for all $\varphi \in \mathcal{C}^1\left(\overline{Q}_T\right)$ such that $\varphi(T,\cdot) = 0$. Hence w solves problem (6.1), which concludes the proof. □

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REFERENCES

[1] H. Alaa, N.E. Alaa, A. Charkaoui, *Time periodic solutions for strongly nonlinear parabolic systems with $p(x)$-growth conditions*, J. Elliptic Parabol. Equ. **7** (2021), 815–839.

[2] H. Alaa, M. El Ghabi, A. Charkaoui, *Semilinear periodic parabolic problem with discontinuous coefficients: Mathematical analysis and numerical simulation*, Filomat **37** (2023), no. 7, 2151–2164.

[3] N.E. Alaa, F. Aqel, L. Taourirte, *On singular quasilinear elliptic equations with data measures*, Adv. Nonlinear Anal. **10** (2021), 1284–1300.

[4] N.E. Alaa, A. Charkaoui, M. El Ghabi, M. El Hathout, *Integral solution for a parabolic equation driven by the $p(x)$-Laplacian operator with nonlinear boundary conditions and L^1 data*, Mediterr. J. Math. **20** (2023), Article no. 244.

[5] H. Amann, *Nonlinear elliptic equations with nonlinear boundary conditions*, North-Holland Mathematics Studies **21** (1976), 43–63.

[6] P. Amster, *Topological methods for a nonlinear elliptic system with nonlinear boundary conditions*, Int. J. Dyn. Syst. Differ. Equ. **2** (2009), no. 1–2, 56–65.

[7] P. Amster, *Multiple solutions for an elliptic system with indefinite Robin boundary conditions*, Adv. Nonlinear Anal. **8** (2017), no. 1, 603–614.

[8] F. Andreu, J.M. Mazón, S. Segura de León, J. Toledo, *Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions*, Adv. Math. Sci. Appl. **7** (1997), 183–213.

[9] P. Baras, M. Pierre, *Problèmes paraboliques semi-lineaires avec donnees measures*, Appl. Anal. **18** (1984), no. 1–2, 111–149.

[10] D. Bothe, M. Pierre, *Quasi-steady-state approximation for a reaction–diffusion system with fast intermediate*, J. Math. Anal. Appl. **368** (2010), no. 1, 120–132.

[11] A. Cabada, R.L. Pouso, *Existence results for the problem $(\phi(u)) = f(t, u, \nabla u)$ with nonlinear boundary conditions*, Nonlinear Anal. **35** (1999), no. 2, 221–231.

[12] A. Charkaoui, N.E. Alaa, *Weak periodic solution for semilinear parabolic problem with singular nonlinearities and L^1 data*, Mediterr. J. Math. **17** (2020), Article no. 108.

[13] A. Charkaoui, N.E. Alaa, *Existence and uniqueness of renormalized periodic solution to a nonlinear parabolic problem with variable exponent and L^1 data*, J. Math. Anal. Appl. **506** (2022), no. 2, Article no. 125674.

[14] A. Charkaoui, N.E. Alaa, *An L^1-theory for a nonlinear temporal periodic problem involving $p(x)$-growth structure with a strong dependence on gradients*, J. Evol. Equ. **23** (2023), Article no. 73.

[15] A. Charkaoui, H. Fahim, N.E. Alaa, *Nonlinear parabolic equation having nonstandard growth condition with respect to the gradient and variable exponent*, Opuscula Math. **41** (2021), no. 1, 25–53.

[16] F.C.Ş. Cîrstea, V. Rădulescu, *Existence and non-existence results for a quasilinear problem with nonlinear boundary condition*, J. Math. Anal. Appl. **244** (2000), no. 1, 169–183.

[17] J. Droniou, *Quelques résultats sur les espaces de Sobolev*, Working paper or preprint, (2001).

[18] A. Elaassri, K. Lamrini Uahabi, A. Charkaoui, N.E. Alaa, S. Mesbahi, *Existence of weak periodic solution for quasilinear parabolic problem with nonlinear boundary conditions*, An. Univ. Craiova Ser. Mat. Inform. **46** (2019), 1–13.

[19] G.M. Figueiredo, V.D. Rădulescu, *Nonhomogeneous equations with critical exponential growth and lack of compactness*, Opuscula Math. **40** (2020), no. 1, 71–92.

[20] M. Fila, J. Lankeit, *Lack of smoothing for bounded solutions of a semilinear parabolic equation*, Adv. Nonlinear Anal. **9** (2020), no. 1, 1437–1452.

[21] J.L. Lions, *Quelques méthodes de résolution de problèmes aux limites non linéaires*, Dunod, 1969.

[22] C.B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer Science & Business Media, 2009.

[23] M. Nakao, *Remarks on global solutions to the initial-boundary value problem for quasilinear degenerate parabolic equations with a nonlinear source term*, Opuscula Math. **39** (2019), no. 3, 395–414.

[24] M. Niezgódka, I. Pawłow, *A generalized Stefan problem in several space variables*, Appl. Math. Optim. **9** (1982), 193–224.

[25] N. Papageorgiou, V. Rădulescu, D. Repovš, *Nonlinear Analysis – Theory and Methods*, Springer Monographs in Mathematics, 2019.

[26] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncations*, Ann. Mat. Pura Appl. (4) **177** (1999), 143–172.

[27] A. Prignet, *Existence and uniqueness of "entropy" solutions of parabolic problems with L^1 data*, Nonlinear Anal. **28** (1997), 1943–1954.

[28] V. Rădulescu, *Nonlinear elliptic equations with variable exponent: old and new*, Nonlinear Anal. **121** (2015), 336–369.

[29] V. Rădulescu, *Isotropic and anisotropic double-phase problems: old and new*, Opuscula Math. **39** (2019), 259–279.

[30] J.F. Rodrigues, *The Stefan problem revisited*, [in:] *Mathematical Models for Phase Change Problems*, Basel, Birkhäuser Basel, 1989, 129–190.

[31] J.F. Rodrigues, *Variational methods in the Stefan problem*, [in:] A. Visintin (ed.), *Phase Transitions and Hysteresis*, Springer, Berlin, 1994, 147–212.

[32] T. Roubíček, *Nonlinear heat equation with L^1-data*, NoDEA Nonlinear Differential Equations Appl. **5** (1998), 517–527.

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