A Kinetostatics-Based Study of Uniqueness of Reactions and Drives in Robotics

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Marcin Pękal, Janusz Frączek

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Abstract:
Reaction and driving forces may be non-unique in many robotic systems. This may pose a problem during robot design or its control synthesis. Hence, it is useful to detect which reaction or actuation forces are non-unique. Previously developed methods are designed for reactions uniqueness analysis only. These methods studied the constraint Jacobian matrix. The kinetostatics-based approach, presented in this paper, enables the simultaneous study of reactions and driving forces uniqueness. It allows the application of the criteria derived from the concepts of linear algebra, e.g. direct sum or nullspace. In this paper only the nullspace method is presented. Moreover, in order to illustrate the approach, five examples are provided.

Keywords: *kinetostatics, nullspace method, uniqueness analysis*

1. Introduction

Redundant systems are commonly used in robotics. In this field, redundancy is often considered from the task point of view. This approach defines redundant systems as structures which have more degrees of freedom (DOFs) than are needed to perform a specific task [2, 25]. Some interesting examples of redundant manipulators are shown in [2], e.g. human-arm-like manipulators, DLR lightweight robot or – of particular interests – hyperredundant manipulators. Apparently, in such cases, reaction and drive uniqueness analysis may be very useful. However, redundancy in robotics may be defined in another ways [3]. In our article, a more general – structural – approach is used. In the present paper we treat all considered mechanisms as constrained multibody systems, and the system is regarded as redundant (overconstrained/redundantly constrained) if it has at least one redundant constraint. This approach is used, e.g. in [4, 7, 20]. It is worth noting that task and structural approaches may be equivalent in some cases.

Mechanical systems (including the redundant ones) may be composed of rigid or flexible bodies. This paper is devoted to rigid body systems only. Rigid body assumption, commonly adopted in analysis of robotic systems, exhibits certain limitations. Joint reaction forces in some robots treated as mechanical systems of rigid bodies cannot be uniquely determined by standard methods of dynamic or kinetostatic analysis. This feature of redundant systems results entirely from the structure of such mechanisms, and thus does not depend on coordinates describing the considered system [7, 19–23]. Moreover, it should be pointed out that redundantly constrained multibody systems are also problematic in modelling, i.e. special approaches, invulnerable to Jacobian matrix rank deficiency, must be adopted – see, e.g. [11, 12].

The problem of non-uniqueness of reactions in mechanisms (modeled as rigid multibody systems), e.g. in robotic manipulators, is an important but often ignored issue. Reaction non-uniqueness may be the reason for incorrect results received from simulations. To obtain the correct results in such cases, the considered system should be modeled as deformable [7, 19–23]. Unfortunately, an analysis of flexible systems involves much larger modelling effort and higher computational cost [7, 19–22].

However, some joint reactions may be uniquely determined despite the non-uniqueness of the global reaction solution [7, 19–23]. There are methods which allow to determine unique reactions (if such exist) in the redundant robots. These methods use two concepts of the linear algebra – direct sum [7, 19–23] and nullspace [4,5]. Such methods analyze Jacobian matrix of constraints. Moreover, they are limited to systems described in absolute (Cartesian) or natural coordinates, because for such coordinates, the Jacobian matrix describes all the joint constraints simultaneously, and consequently, all the joint reactions. Note that this paper is devoted to presentation of a method which uses nullspace approach and is based on a kinetostatics formulation.

In this paper, the related issue is also discussed – the analysis of uniqueness of driving forces. The propulsion non-uniqueness is usually introduced intentionally, e.g. in order to eliminate gear backlash and clearances [18], in order to improve the performance of the system [24] or in order to reduce torques acting in kinematic joints [8]. In such cases, it is usually known in advance, which driving forces are non-unique. Moreover, it is worth to point out that, using the presented method, driving force uniqueness problem may be studied together with reaction uniqueness test.

As mentioned before, the considered method is based on kinetostatics. It uses a free-body diagram (FBD) [1]. Such approach allows to analyze rigid systems described in any set of coordinates (which was not possible when the previous approach to reaction analysis was used). It is worth noting that the kinetostatics-based method was considered in a series of conference publications [10, 13, 14]. This particular article is an extension of the 14th National Conference on Robotics paper [10].

In order to illustrate the method, five examples of rigid redundant robotic systems are considered: three cases of a gripper [7, 9, 19] (without actuation, actuated and overactuated), a redundant manipulator [9] and an overactuated redundant manipulator. Note that the presented systems are not in singular positions, and the friction is neglected.

The structure of this article is as follows. Section 2 presents the kinetostatic method, section 3 formulates the uniqueness criterion based on nullspace, section 4 shows the practical examples, and section 5 contains conclusions.

2. Kinetostatics

Starting point for considerations is the formulation of kinetostatics equations. In order to perform this task, the considered system is virtually decomposed into a set of unconnected bodies. Then, active forces (actuation and external loads) and passive forces (joint reactions) acting on all the bodies are introduced. This produces a free-body diagram (FBD) of the system [1]. Subsequently, a set of equilibrium equations for all the bodies is written. Eventually, the set of m equilibrium equations is obtained (where $m = 3p$ for planar systems or $m = 6p$ in spatial cases, and p is a total number of the bodies in the considered system). For the body i, these equations have the following form

$$\begin{cases} \mathbf{F}_{bi} + \mathbf{F}_i + \sum\limits_k \sum\limits_j (\mathbf{S}_{jik} + \mathbf{F}_{djik}) = \mathbf{0} \\ \sum\limits_k \sum\limits_j [\tilde{\mathbf{r}}_k (\mathbf{S}_{jik} + \mathbf{F}_{djik}) + \mathbf{M}_{jik} + \mathbf{M}_{djik}] + \\ \qquad + \tilde{\mathbf{r}}_{Ci}(\mathbf{F}_{bi} + \mathbf{F}_i) + \mathbf{M}_{bi} + \mathbf{M}_i = \mathbf{0}, \end{cases} \tag{1}$$

where the first equation of this set is the equation of forces equilibrium, while the second is the equation of torques equilibrium. Moreover, $i \in \{1, 2, \ldots, p\}$ is an index specifying the body, $j \in \{0, 1, \ldots, p\}$: $j \neq i$ is an index describing the remaining bodies (where the base of the system is taken into account and it is denoted 0), k is an index depicting the joint, $\mathbf{S}_{jik}$ and $\mathbf{M}_{jik}$ are the reaction force and reaction torque (that body j exerts on body i in the joint k), respectively. $\mathbf{F}_{bi}$ and $\mathbf{M}_{bi}$ are the inertia forces of the body i (force and torque, respectively), $\mathbf{F}_{djik}$ is a driving force, $\mathbf{M}_{djik}$ is a driving torque, $\mathbf{F}_i$ is a vector containing the remaining external forces reduced to the center of mass of the body i, $\mathbf{M}_i$ is a sum of the other external torques acting on the body i, while $\tilde{\mathbf{r}}_k$ and $\tilde{\mathbf{r}}_{Ci}$ are skew-symmetric matrices associated with the position vectors of the joint k ($\mathbf{r}_k$) and the center of mass of the body i ($\mathbf{r}_{Ci}$), respectively. Note that the skew-symmetric matrix (for any vector $\mathbf{r} = [r_x \; r_y \; r_z]^T$) is defined as

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}. \tag{2}$$

It should be pointed out that the equilibrium equations contain unknown reaction forces (which are responsible for the effect of the constraints) and driving forces. Note that the uniqueness of these two components will be studied. Subsequently, the equilibrium equations may be rewritten in the following form

$$\mathbf{A}_{m \times n} \mathbf{x}_{n \times 1} = \mathbf{b}_{m \times 1}, \tag{3}$$

where column vector $\mathbf{x}_{n \times 1}$ contains unknown reactions and driving forces ($\mathbf{S}_{jik}$, $\mathbf{M}_{jik}$, $\mathbf{F}_{djik}$ and $\mathbf{M}_{djik}$), $\mathbf{A}_{m \times n}$ is a coefficient matrix containing the geometry-related quantities ($\mathbf{r}_k$ and $\mathbf{r}_{Ci}$), and $\mathbf{b}_{m \times 1}$ includes the remaining forces. Moreover, system of equations (3) takes into account that $\mathbf{S}_{jik} = -\mathbf{S}_{ijk}$, $\mathbf{M}_{jik} = -\mathbf{M}_{ijk}$, $\mathbf{F}_{djik} = -\mathbf{F}_{dijk}$ and $\mathbf{M}_{djik} = -\mathbf{M}_{dijk}$. It is worth noting that the uniqueness of selected components of vector $\mathbf{x}$ will be determined by examining matrix $\mathbf{A}$ and its submatrices. Therefore, there is no need to compute vector $\mathbf{b}$.

3. Nullspace Method

To verify whether the studied component (or a set of components) of vector $\mathbf{x}$ is uniquely determined, the following procedure may be performed.

1) Determine the nullspace basis of matrix $\mathbf{A}_{m \times n}$. This step leads to a nullspace matrix $\mathbf{N}_{n \times (n-r)}$ which contains the set of independent vectors spanning the nullspace [15, 16]. Note that matrix $\mathbf{N}$ may be obtained using, e.g. Gauss-Jordan Elimination or Singular Value Decomposition (SVD) [16]. Moreover, it may be pointed out, that in well-known MATLAB® environment, the nullspace basis may be computed using function *null*, which uses SVD [17]. Note also that if the nullspace contains only zero vector (nullspace matrix $\mathbf{N}$ is empty), then the solution of the linear equation is unique. Otherwise, nonzero rows of matrix $\mathbf{N}$ indicate the existence of non-unique reactions or drives. It is useful to point out that empty nullspace matrix occurs in the case of non-redundant system with unique drives. For such systems it is not necessary to perform the uniqueness analysis presented in this paper, because all its reactions and drives are unique.

2) Select a subset of unknowns $S = \{x_\eta\}$, $\eta \in \{1, \ldots, n\}$ for the uniqueness test. Note that this subset will be named *studied element* further. For the studied element, suitable submatrices are specified, i.e. vector $\mathbf{x}_S$ containing components x_η, submatrix $\mathbf{A}_S$ corresponding to $\mathbf{x}_S$ (and formed from the columns of matrix $\mathbf{A}$) and submatrix $\mathbf{N}_S$ created analogously, but from the rows of nullspace matrix $\mathbf{N}$.

3) Check an orthogonal condition in the form

$$\mathbf{A}_S \mathbf{N}_S = \mathbf{0}. \tag{4}$$

If this condition is fulfilled, a linear combination $\mathbf{A}_S \mathbf{x}_S = \mathbf{b}_S$ is uniquely determined (which cannot be transferred directly to the uniqueness of $\mathbf{x}_S$).

4) Examine the rank of submatrix $\mathbf{A}_S$. If this submatrix has full rank, then $\mathbf{x}_S$ may be uniquely determined (because $\mathbf{A}_S \mathbf{x}_S = \mathbf{b}_S$ has exactly one solution).

In the examples described below, the rank of matrix $\mathbf{A}_S$ is determined, however usually it is no need to designate its rank in fact. Note that in the most of quite common cases it can be proved, that the examined submatrix $\mathbf{A}_S$ will have full rank. In the examples considered in this paper, appropriate selection of the studied components of $\mathbf{x}_S$ (which are linearly independent) always causes the full rank of submatrix $\mathbf{A}_S$. Hence, the uniqueness of $\mathbf{A}_S\mathbf{x}_S$ implies the uniqueness of vector $\mathbf{x}_S$.

4. Examples

In order to verify our method, five examples of robotic systems are provided: three introductory examples of a planar gripper [7, 9, 19], a redundant manipulator [9] and an overactuated redundant manipulator. Firstly, for each example, figures presenting structure and free-body diagram (FBD) are shown, followed by a brief description of the system and the configuration, in which the uniqueness is analyzed. Subsequently, vector **x**, coefficient matrix **A** and the structure of nullspace matrix **N** are shown and discussed. Presentation of the obtained results concludes each example.

4.1. Gripper without Actuation

The first of the considered examples is a planar gripper, similar to the mechanism previously considered in [7, 9, 19], i.e. its structural diagram is the same. The kinematic scheme of this system is presented in Fig. 1, and its FBD is shown in Fig. 2. The considered mechanism consists of four rigid bodies connected by six joints. Note that three of the joints are revolute, and the remaining three kinematic pairs are translational. Moreover, the system has only one degree of freedom (DOF), which is not actuated in this case. This example is considered in order to show that our method may be applied when only reaction uniqueness analysis is performed. Note that it is analogous to the previous method of reaction uniqueness analysis, based on the study of constraint Jacobian matrix (see, e.g. [7, 19]).

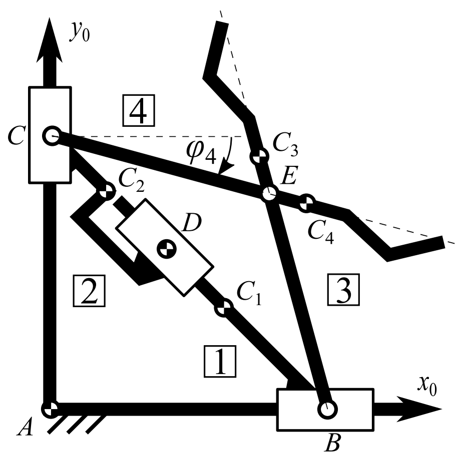

Fig. 1. Kinematic scheme of the gripper

It is assumed that the system is in the position described by $\varphi_4 = -\frac{\pi}{6}\ rad$ (see Fig. 1).

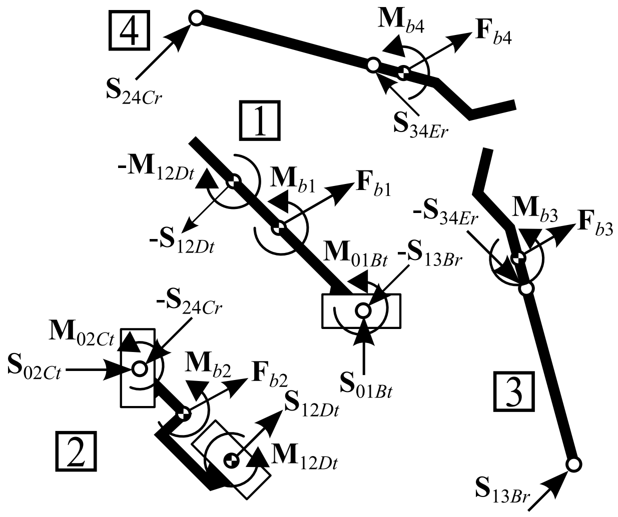

Fig. 2. Free-body diagram of the gripper without actuation

This is the first example, hence the algorithm will be presented in detail, i.e. step by step.

After creating the FBD (presented in Fig. 2), the equations of kinetostatic equilibrium may be written as (see eq. (1)):

- for body 1

$$\begin{cases} \begin{bmatrix}0\\1\end{bmatrix} S_{01Bt} - \begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \mathbf{S}_{13Br} + \mathbf{F}_{b1} = \mathbf{0} \\ \tilde{\mathbf{r}}_{Bt}\begin{bmatrix}0\\1\end{bmatrix} S_{01Bt} - \tilde{\mathbf{r}}_{Dt}\begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \tilde{\mathbf{r}}_{Br}\mathbf{S}_{13Br} + \\ \qquad + M_{01Bt} - M_{12Dt} + \tilde{\mathbf{r}}_{C1}\mathbf{F}_{b1} + M_{b1} = 0 \end{cases} \tag{5}$$

- for body 2

$$\begin{cases} \begin{bmatrix}1\\0\end{bmatrix} S_{02Ct} + \begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \mathbf{S}_{24Cr} + \mathbf{F}_{b2} = \mathbf{0} \\ \tilde{\mathbf{r}}_{Ct}\begin{bmatrix}1\\0\end{bmatrix} S_{02Ct} + \tilde{\mathbf{r}}_{Dt}\begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \tilde{\mathbf{r}}_{Cr}\mathbf{S}_{24Cr} + \\ \qquad + M_{02Ct} + M_{12Dt} + \tilde{\mathbf{r}}_{C2}\mathbf{F}_{b2} + M_{b2} = 0 \end{cases} \tag{6}$$

- for body 3

$$\begin{cases} \mathbf{S}_{13Br} - \mathbf{S}_{34Er} + \mathbf{F}_{b3} = \mathbf{0} \\ \tilde{\mathbf{r}}_{Br}\mathbf{S}_{13Br} - \tilde{\mathbf{r}}_{Er}\mathbf{S}_{34Er} + \tilde{\mathbf{r}}_{C3}\mathbf{F}_{b3} + M_{b3} = 0 \end{cases} \tag{7}$$

- for body 4

$$\begin{cases} \mathbf{S}_{24Cr} + \mathbf{S}_{34Er} + \mathbf{F}_{b4} = \mathbf{0} \\ \tilde{\mathbf{r}}_{Cr}\mathbf{S}_{24Cr} + \tilde{\mathbf{r}}_{Er}\mathbf{S}_{34Er} + \tilde{\mathbf{r}}_{C4}\mathbf{F}_{b4} + M_{b4} = 0 \end{cases} \tag{8}$$

where S_{01Bt}, S_{02Ct} and S_{12Dt} are translational joint reaction values, which are perpendicular to axes of the joints. Moreover, index k is written with two characters. The first of them represents the position point of the joint, while the second means its type. For example, $\mathbf{S}_{34Er}$ means the reaction force acting from body 3 to 4 in a joint located at a point E, which is revolute (note that the following abbreviations are used in order to specify the type of the joint: r – revolute joint, t – translational joint, c – cylindrical joint). Auxiliary vectors: $[0\ 1]^T$, $[1\ 1]^T$ and $[1\ 0]^T$ are used in order to define directions of the appropriate reaction forces. Moreover, in the case of planar systems, skew-symmetric matrix is replaced by a row vector defined as $\tilde{\mathbf{r}}_i = [-r_{iy}\ r_{ix}]$, where $\mathbf{r}_i = [r_{ix}\ r_{iy}]^T$.

These equations should be written in the matrix form specified by eq. (3). Hence, it is useful to assemble them in one set of equations as

$$\begin{cases} \begin{bmatrix}0\\1\end{bmatrix} S_{01Bt} - \begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \mathbf{S}_{13Br} = -\mathbf{F}_{b1} \\ \begin{bmatrix}1\\0\end{bmatrix} S_{02Ct} + \begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \mathbf{S}_{24Cr} = -\mathbf{F}_{b2} \\ \mathbf{S}_{13Br} - \mathbf{S}_{34Er} = -\mathbf{F}_{b3} \\ \mathbf{S}_{24Cr} + \mathbf{S}_{34Er} = -\mathbf{F}_{b4} \\ \tilde{\mathbf{r}}_{Bt} \begin{bmatrix}0\\1\end{bmatrix} S_{01Bt} - \tilde{\mathbf{r}}_{Dt} \begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \tilde{\mathbf{r}}_{Br}\mathbf{S}_{13Br} + M_{01Bt} + \\ \qquad - M_{12Dt} = -\tilde{\mathbf{r}}_{C1}\mathbf{F}_{b1} - M_{b1} \\ \tilde{\mathbf{r}}_{Ct} \begin{bmatrix}1\\0\end{bmatrix} S_{02Ct} + \tilde{\mathbf{r}}_{Dt} \begin{bmatrix}1\\1\end{bmatrix} S_{12Dt} - \tilde{\mathbf{r}}_{Cr}\mathbf{S}_{24Cr} + M_{02Ct} + \\ \qquad + M_{12Dt} = -\tilde{\mathbf{r}}_{C2}\mathbf{F}_{b2} - M_{b2} \\ \tilde{\mathbf{r}}_{Br}\mathbf{S}_{13Br} - \tilde{\mathbf{r}}_{Er}\mathbf{S}_{34Er} = -\tilde{\mathbf{r}}_{C3}\mathbf{F}_{b3} - M_{b3} \\ \tilde{\mathbf{r}}_{Cr}\mathbf{S}_{24Cr} + \tilde{\mathbf{r}}_{Er}\mathbf{S}_{34Er} = -\tilde{\mathbf{r}}_{C4}\mathbf{F}_{b4} - M_{b4} \end{cases} . \tag{9}$$

Note that these equations are arranged such that on the top, there are equations of forces equilibrium, while on the bottom – equations of torques equilibrium. Obviously, the order of these equations may be arbitrary. Moreover, on the left-hand side of the set, there are components containing unknowns, while on the right-hand side, there are the remaining components.

These equations may be written now in the matrix form (3). In the further considerations, only vector of unknowns **x** and coefficient matrix **A** (corresponding to **x**) are used (and consequently, right-hand side vector **b** is omitted). In this example, vector of unknowns **x** has the following form

$$\mathbf{x}_{12\times1} = \big[S_{01Bt}\ S_{02Ct}\ S_{12Dt}\ \mathbf{S}^T_{13Br}\ \mathbf{S}^T_{24Cr} \\ \mathbf{S}^T_{34Er}\ M_{01Bt}\ M_{02Ct}\ M_{12Dt}\big]^T, \tag{10}$$

while coefficient matrix $\mathbf{A}_{12\times12}$ is defined as

$$\mathbf{A}=\begin{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} & \mathbf{0}_{2\times1} & -\begin{bmatrix}1\\1\end{bmatrix} & -\mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{2\times1} & \begin{bmatrix}1\\0\end{bmatrix} & \begin{bmatrix}1\\1\end{bmatrix} & \mathbf{0}_{2\times2} & -\mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{I}_{2\times2} & \mathbf{0}_{2\times2} & -\mathbf{I}_{2\times2} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} \\ \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times2} & \mathbf{I}_{2\times2} & \mathbf{I}_{2\times2} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} & \mathbf{0}_{2\times1} \\ \tilde{\mathbf{r}}_{Bt}\begin{bmatrix}0\\1\end{bmatrix} & 0 & -\tilde{\mathbf{r}}_{Dt}\begin{bmatrix}1\\1\end{bmatrix} & -\tilde{\mathbf{r}}_{Br} & \mathbf{0}_{1\times2} & \mathbf{0}_{1\times2} & 1 & 0 & -1 \\ 0 & \tilde{\mathbf{r}}_{Ct}\begin{bmatrix}1\\0\end{bmatrix} & \tilde{\mathbf{r}}_{Dt}\begin{bmatrix}1\\1\end{bmatrix} & \mathbf{0}_{1\times2} & -\tilde{\mathbf{r}}_{Cr} & \mathbf{0}_{1\times2} & 0 & 1 & 1 \\ 0 & 0 & 0 & \tilde{\mathbf{r}}_{Br} & \mathbf{0}_{1\times2} & -\tilde{\mathbf{r}}_{Er} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0}_{1\times2} & \tilde{\mathbf{r}}_{Cr} & \tilde{\mathbf{r}}_{Er} & 0 & 0 & 0 \end{bmatrix}, \tag{11}$$

where $\mathbf{0}_{i\times j}$ is a zero matrix of size $i \times j$, and $\mathbf{I}_{2\times2}$ is an identity matrix of size 2×2. That matrix has rank $r(\mathbf{A}) = 11$.

The nullspace matrix $\mathbf{N}_{12\times1}$ (which represents the nullspace basis of matrix **A**) has the following structure

$$\mathbf{N} = [\mathbf{0}_{1\times9}\quad \bullet \quad \bullet \quad \bullet]^T, \tag{12}$$

where • is introduced to denote non-zero elements of matrix **N**. Note that the structure of nullspace matrix **N** is given only, because its values may be different depending on an algorithm used for its calculation.

Now, it is necessary to select studied elements. In Tab. 1 the studied elements, their components and the columns of coefficient matrix **A** used to create submatrices $\mathbf{A}_S$ are presented. Hence, it is possible to create the suitable submatrices $\mathbf{A}_S$ and $\mathbf{N}_S$, and check orthogonality condition (4). In Tab. 2 the obtained results are presented. Note that, in the column 'Result', two abbreviations are used: 'U' informing that the studied element is unique and 'N' indicating non-unique element. As mentioned earlier, column 'Rank of $\mathbf{A}_S$' was added only to complete the presentation, because the method of analysis guarantees full rank of submatrix $\mathbf{A}_S$, so the uniqueness of linear combination $\mathbf{A}_S\mathbf{x}_S$ implies uniqueness of vector $\mathbf{x}_S$ (describing studied element).

Tab. 1. Studied elements of the gripper without actuation

Studied element	Elements of **x**	Columns forming $\mathbf{A}_S$
Reaction Bt	S_{01Bt}, M_{01Bt}	1, 10
Reaction Ct	S_{02Ct}, M_{02Ct}	2, 11
Reaction Dt	S_{12Dt}, M_{12Dt}	3, 12
Reaction Br	$\mathbf{S}_{13Br}$	4–5
Reaction Cr	$\mathbf{S}_{24Cr}$	6–7
Reaction Er	$\mathbf{S}_{34Er}$	8–9

Tab. 2. Results of the analysis of the gripper without actuation

Studied element	Criterion (4) value	Rank of $\mathbf{A}_S$	Result
Reactions: Bt, Ct, Dt	$\neq \mathbf{0}$	2(full)	N
Reactions: Br, Cr, Er	$= \mathbf{0}$	2(full)	U

Note that in this example, uniqueness of reactions is not easy to guess. Eventually, it turns out that the reactions in revolute joints are unique, while in translational joints the reactions cannot be uniquely determined. Note that it is consistent with the previous publications [7, 19].

4.2. Actuated Gripper

The second example discusses an actuated planar gripper. The investigated mechanism is the same as in the previous example, however, the actuation torque is introduced into the revolute joint Cr. Hence, the kinematic scheme of this system is the same as previously, and it is shown in Fig. 1, whereas the FBD of the actuated gripper is presented in Fig. 3.

It is assumed that the system is in the same position as previously. In this example, vector of unknowns **x** can be written as (note the driving torque re-

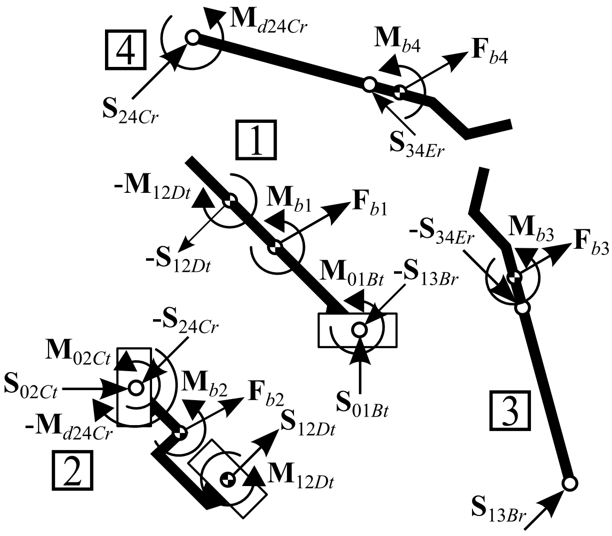

Fig. 3. Free-body diagram of the actuated gripper

presented by the last component of **x**)

$$\mathbf{x}_{13\times1} = \left[S_{01Bt}\ S_{02Ct}\ S_{12Dt}\ \mathbf{S}_{13Br}^T\ \mathbf{S}_{24Cr}^T\ \mathbf{S}_{34Er}^T\ M_{01Bt}\ M_{02Ct}\ M_{12Dt}\ M_{d24Cr}\right]^T. \quad (13)$$

For this vector, coefficient matrix $\mathbf{A}_{12\times13}$ is created. Since the first 12 elements of vector **x** are the same as in the previous example, the first 12 columns of **A** are identical to the coefficient matrix from the previous example (hence, these columns are not repeated here). The last column of the new coefficient matrix **A** has the following form

$$\mathbf{A}_{13} = [\mathbf{0}_{1\times9}\quad -1\quad 0\quad 1]^T. \quad (14)$$

Note that this matrix has rank $r(\mathbf{A}) = 12$, and null-space matrix $\mathbf{N}_{13\times1}$ corresponding to **A** has the structure

$$\mathbf{N} = [\mathbf{0}_{1\times9}\quad \bullet\quad \bullet\quad \bullet\quad 0]^T. \quad (15)$$

In Tab. 3 data analogous to those in Tab. 1 are provided, i.e. the studied elements, their components and the columns of **A** used to form submatrices $\mathbf{A}_S$. Moreover, Tab. 4 contains the results. As expected (since the system has one DOF and one drive), driving force

Tab. 3. Studied elements of the actuated gripper

Studied element	Elements of **x**	Columns forming $\mathbf{A}_S$
Reaction Bt	S_{01Bt}, M_{01Bt}	1, 10
Reaction Ct	S_{02Ct}, M_{02Ct}	2, 11
Reaction Dt	S_{12Dt}, M_{12Dt}	3, 12
Reaction Br	$\mathbf{S}_{13Br}$	4–5
Reaction Cr	$\mathbf{S}_{24Cr}$	6–7
Reaction Er	$\mathbf{S}_{34Er}$	8–9
Drive Cr	M_{d24Cr}	13

Tab. 4. Results of the actuated gripper analysis

Studied element	Criterion (4) value	Rank of $\mathbf{A}_S$	Result
Reactions: Bt, Ct, Dt	$\neq \mathbf{0}$	2(full)	N
Reactions: Br, Cr, Er	$= \mathbf{0}$	2(full)	U
Drive Cr	$= \mathbf{0}$	1(full)	U

in joint Cr is identified by the algorithm as uniquely determined. Moreover, uniqueness analysis of reactions gave the same results as in the previous example. Hence, the algorithm may be used also for actuated rigid-body mechanisms.

4.3. Overactuated Gripper

The third example presents a study of an overactuated planar gripper. The mechanism is the same as in the previous examples. Therefore, the kinematic scheme did not change and is presented in Fig. 1. An actuator is added in the translational joint Bt, which makes the 1-DOF system redundantly actuated. Note that the FBD of the gripper had to be modified (the additional force is applied to body 1), and it is shown in Fig. 4.

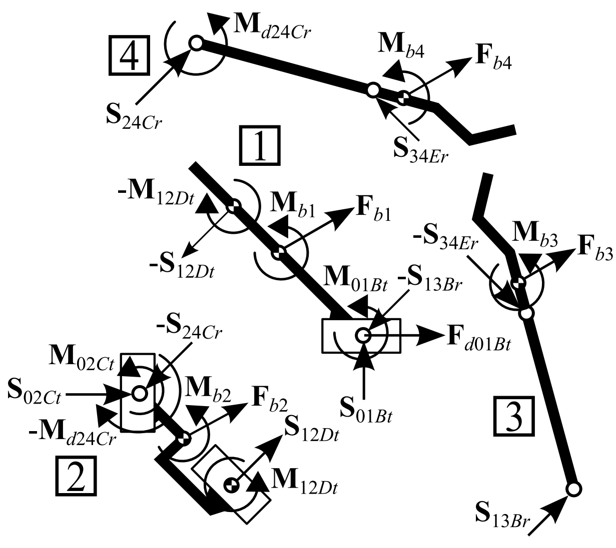

Fig. 4. Free-body diagram of the overactuated gripper

It is assumed that the system is in the same position as in the previous examples, and vector of unknowns **x** has the following form

$$\mathbf{x}_{14\times1} = \left[S_{01Bt}\ S_{02Ct}\ S_{12Dt}\ \mathbf{S}_{13Br}^T\ \mathbf{S}_{24Cr}^T\ \mathbf{S}_{34Er}^T\ M_{01Bt}\ M_{02Ct}\ M_{12Dt}\ M_{d24Cr}\ F_{d01Bt}\right]^T, \quad (16)$$

where F_{d01Bt} is a value of driving force applied in translational joint Bt (this force is parallel to the axis of the kinematic pair).

The coefficient matrix **A** is of size 12×14. Its first 13 columns are the same as the whole matrix **A** from the previous example while the 14th column may be written as

$$\mathbf{A}_{14} = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \quad \mathbf{0}_{1\times6} \quad \left(\tilde{\mathbf{r}}_{Bt} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^T \quad \mathbf{0}_{1\times3} \right]^T . \quad (17)$$

Moreover, the rank of matrix $\mathbf{A}$ is $r(\mathbf{A}) = 12$. The nullspace matrix $\mathbf{N}_{14\times2}$, corresponding to $\mathbf{A}$, has the following structure

$$\mathbf{N} = \left[\mathbf{0}_{2\times1} \quad \mathbf{N}_1\right]^T , \quad (18)$$

where $\mathbf{N}_1$ is a submatrix of size 2×13 which contains nonzero elements. Note that precise values of these matrices are not presented, since (as in the previous examples) they may be different.

Tables 5 and 6 contain the data used for the uniqueness analysis and the results, respectively. As expected, driving forces in joints Cr and Bt are non-unique. Moreover, the additional driving force caused the non-uniqueness of all the reactions, which is an interesting outcome. Note that the change in the reaction uniqueness is caused by overactuation.

Tab. 5. Studied elements of the overactuated gripper

Studied element	Elements of $\mathbf{x}$	Columns forming $\mathbf{A}_S$
Reaction Bt	S_{01Bt}, M_{01Bt}	1, 10
Reaction Ct	S_{02Ct}, M_{02Ct}	2, 11
Reaction Dt	S_{12Dt}, M_{12Dt}	3, 12
Reaction Br	$\mathbf{S}_{13Br}$	4–5
Reaction Cr	$\mathbf{S}_{24Cr}$	6–7
Reaction Er	$\mathbf{S}_{34Er}$	8–9
Drive Cr	M_{d24Cr}	13
Drive Bt	F_{d01Bt}	14

Tab. 6. Results of the overactuated gripper analysis

Studied element	Criterion (4) value	Rank of $\mathbf{A}_S$	Result
Reactions: Bt, Ct, Dt, Br, Cr, Er	$\neq \mathbf{0}$	2(full)	N
Drives: Cr, Bt	$\neq \mathbf{0}$	1(full)	N

4.4. Redundant Manipulator

To show that the kinetostatic method is also applicable to spatial systems, a redundant manipulator (taken from [9]) is examined. Kinematic scheme of this system and its FBD are shown in Figs. 5 and 6, respectively. The mechanism consists of seven bodies connected by nine joints (seven revolute and two cylindrical). The manipulator has three DOFs.

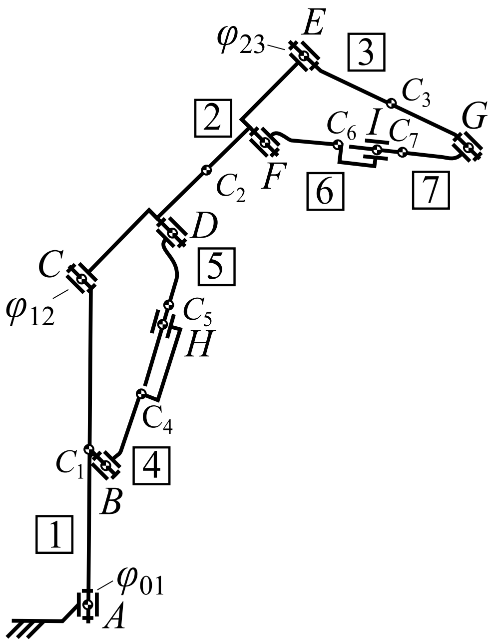

Fig. 5. Kinematic scheme of the redundant manipulator

The study of uniqueness of reaction and driving forces is conducted in the position, where $\mathbf{q} = [\varphi_{01}\ \varphi_{12}\ \varphi_{23}]^T = \left[0 \quad \frac{3\pi}{4} \quad \frac{5\pi}{6}\right]^T rad$.

Assume that z is the axis of rotation of the kinematic pair. Hence, it is possible to make the use of facts that for a revolute joint $\mathbf{M}^k_{dijk} = [0\ 0\ M^k_{dijkz}]^T$, for a cylindrical joint $\mathbf{S}^k_{ijk} = [S^k_{ijkx}\ S^k_{ijky}\ 0]^T$ and $\mathbf{F}^k_{dijk} = [0\ 0\ F^k_{dijkz}]^T$, while for both types of the joints (revolute and cylindrical) $\mathbf{M}^k_{ijk} = [M^k_{ijkx}\ M^k_{ijky}\ 0]^T$. As a result, vector of unknowns $\mathbf{x}$ has the following form

$$\mathbf{x}_{46\times1} = \left[\mathbf{S}^T_{01Ar}\ M^{Ar}_{01Arx}\ M^{Ar}_{01Ary} \middle| \ \mathbf{S}^T_{14Br}\ M^{Br}_{14Brx}\ M^{Br}_{14Bry} \middle| \mathbf{S}^T_{12Cr}\ M^{Cr}_{12Crx}\ M^{Cr}_{12Cry} \middle| \mathbf{S}^T_{25Dr}\ M^{Dr}_{25Drx}\ M^{Dr}_{25Dry} \middle| \mathbf{S}^T_{23Er}\ M^{Er}_{23Erx}\ M^{Er}_{23Ery} \middle| \mathbf{S}^T_{26Fr}\ M^{Fr}_{26Frx}\ M^{Fr}_{26Fry} \middle| \mathbf{S}^T_{37Gr}\ M^{Gr}_{37Grx}\ M^{Gr}_{37Gry} \middle| S^{Hc}_{45Hcx}\ S^{Hc}_{45Hcy}\ M^{Hc}_{45Hcx}\ M^{Hc}_{45Hcy} \middle| S^{Ic}_{67Icx}\ S^{Ic}_{67Icy}\ M^{Ic}_{67Icx}\ M^{Ic}_{67Icy} \middle| M^{Ar}_{d01Arz} \middle| F^{Hc}_{d45Hcz} \middle| F^{Ic}_{d67Icz}\right]^T , \quad (19)$$

where | is a separator introduced to improve readability.

Coefficient matrix $\mathbf{A}_{42\times46}$ has rank $r(\mathbf{A}) = 42$. Because of the large size of this matrix, only the rows for body 1 are presented here. The rows corresponding to the equations of equilibrium of forces may be written as

$$\mathbf{A}^{1F}_{3\times46} = \left[\mathbf{I}_{3\times3} \quad \mathbf{0}_{3\times2} \quad \text{-}\mathbf{I}_{3\times3} \quad \mathbf{0}_{3\times2} \quad \text{-}\mathbf{I}_{3\times3} \quad \mathbf{0}_{3\times33}\right] , \quad (20)$$

where $\mathbf{I}_{3\times3}$ is an identity matrix of size 3×3. The rows corresponding to the equations of equilibrium of torques have the form

$$\mathbf{A}^{1M}_{3\times46} = \left[\tilde{\mathbf{r}}_{Ar} \quad \mathbf{R}^0_{Ar}\mathbf{z} \quad \text{-}\tilde{\mathbf{r}}_{Br} \quad \text{-}\mathbf{R}^0_{Br}\mathbf{z} \quad \text{-}\tilde{\mathbf{r}}_{Cr} \quad \text{-}\mathbf{R}^0_{Cr}\mathbf{z} \quad \mathbf{0}_{3\times28} \quad \mathbf{R}^0_{Ar}\mathbf{w} \quad \mathbf{0}_{3\times2}\right] , \quad (21)$$

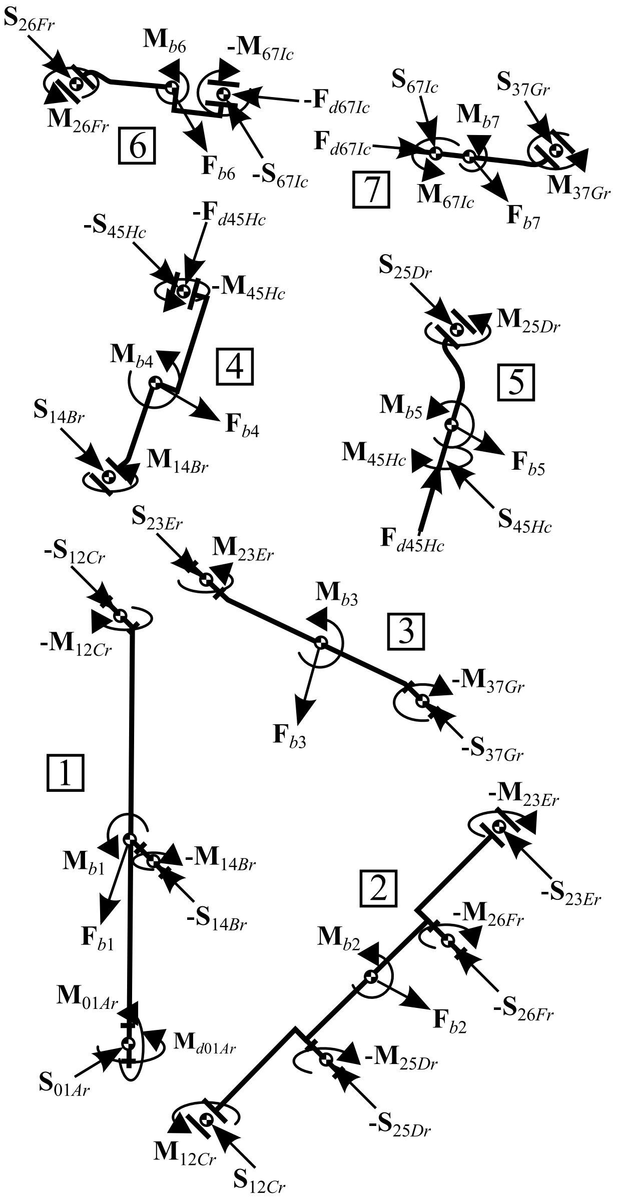

Fig. 6. Free-body diagram of the redundant manipulator

where $\mathbf{z} = \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}_{3\times2}$ is a matrix composed of versors $\mathbf{u} = [1\ 0\ 0]^T$ and $\mathbf{v} = [0\ 1\ 0]^T$, $\mathbf{R}_k^0$ is a rotation matrix that transforms the coordinates form the global to the local coordinate system associated with joint k [6]. Note that the remaining rows of matrix **A** are created analogously, by writing the equilibrium equations for all the other bodies. Subsequently, the corresponding nullspace matrix **N** has size 46×4. This matrix has the following structure

$$\mathbf{N} = [\mathbf{0}_{4\times6}\ \mathbf{N}_1\ \mathbf{0}_{4\times1}\ \mathbf{N}_2\ \mathbf{0}_{4\times1}\ \mathbf{N}_3\ \mathbf{0}_{4\times1}\ \mathbf{N}_4\ \mathbf{0}_{4\times1}\ \mathbf{N}_5\ \mathbf{0}_{4\times1}\ \mathbf{N}_6\ \mathbf{0}_{4\times1}\ \mathbf{N}_7\ \mathbf{0}_{4\times1}\ \mathbf{N}_8\ \mathbf{0}_{4\times1}\ \mathbf{N}_9\ \mathbf{0}_{4\times1}\ \mathbf{N}_{10}\ \mathbf{0}_{4\times1}\ \mathbf{N}_{11}\ \mathbf{0}_{4\times1}\ \mathbf{N}_{12}\ \mathbf{0}_{4\times2}\ \mathbf{N}_{13}\ \mathbf{0}_{4\times2}\ \mathbf{N}_{14}\ \mathbf{0}_{4\times3}]^T, \quad (22)$$

where $\mathbf{N}_i$, $i = 1, 2, \ldots 14$ are submatrices containing nonzero elements (analogously to the previous example). Moreover, these matrices are of sizes: $(\mathbf{N}_1)_{4\times1}$, $(\mathbf{N}_2)_{4\times2}$, $(\mathbf{N}_3)_{4\times1}$, $(\mathbf{N}_4)_{4\times2}$, $(\mathbf{N}_5)_{4\times1}$, $(\mathbf{N}_6)_{4\times2}$, $(\mathbf{N}_7)_{4\times1}$, $(\mathbf{N}_8)_{4\times2}$, $(\mathbf{N}_9)_{4\times1}$, $(\mathbf{N}_{10})_{4\times2}$, $(\mathbf{N}_{11})_{4\times1}$, $(\mathbf{N}_{12})_{4\times3}$, $(\mathbf{N}_{13})_{4\times2}$, $(\mathbf{N}_{14})_{4\times1}$.

Tab. 7 shows studied elements, their components and columns used to create submatrices $\mathbf{A}_S$. The results of the procedure are presented in Tab. 8. Note that these outcomes are consistent with intuition, i.e. the uniqueness of the driving forces results from their obvious linear independence, while the uniqueness of reaction in the revolute joint located at point A results from the fact that it is a total reaction between the base and the manipulator. Hence, it must be also uniquely determined.

Tab. 7. Studied elements of the redundant manipulator

Studied element	Elements of **x**	Columns forming $\mathbf{A}_S$
Reaction Ar	$\mathbf{S}_{01Ar}$, M_{01Arx}^{Ar}, M_{01Ary}^{Ar}	1–5
Reaction Br	$\mathbf{S}_{14Br}$, M_{14Brx}^{Br}, M_{14Bry}^{Br}	6–10
Reaction Cr	$\mathbf{S}_{12Cr}$, M_{12Crx}^{Cr}, M_{12Cry}^{Cr}	11–15
Reaction Dr	$\mathbf{S}_{25Dr}$, M_{25Drx}^{Dr}, M_{25Dry}^{Dr}	16–20
Reaction Er	$\mathbf{S}_{23Er}$, M_{23Erx}^{Er}, M_{23Ery}^{Er}	21–25
Reaction Fr	$\mathbf{S}_{26Fr}$, M_{26Frx}^{Fr}, M_{26Fry}^{Fr}	26–30
Reaction Gr	$\mathbf{S}_{37Gr}$, M_{37Grx}^{Gr}, M_{37Gry}^{Gr}	31–35
Reaction Hc	S_{45Hcx}^{Hc}, S_{45Hcy}^{Hc}, M_{45Hcx}^{Hc}, M_{45Hcy}^{Hc}	36–39
Reaction Ic	S_{67Icx}^{Ic}, S_{67Icy}^{Ic}, M_{67Icx}^{Ic}, M_{67Icy}^{Ic}	40–43
Drive Ar	M_{d01Arz}^{Ar}	44
Drive Hc	F_{d45Hcz}^{Hc}	45
Drive Ic	F_{d67Icz}^{Ic}	46

Tab. 8. Results of the redundant manipulator analysis

Studied element	Criterion (4) value	Rank of $\mathbf{A}_S$	Result
Reaction Ar	$= \mathbf{0}$	5(full)	U
Reactions: Br, Cr, Dr, Er, Fr, Gr	$\neq \mathbf{0}$	5(full)	N
Reactions: Hc, Ic	$\neq \mathbf{0}$	4(full)	N
Drives: Ar, Hc, Ic	$= \mathbf{0}$	1(full)	U

4.5. Overactuated Redundant Manipulator

The last example discusses an overactuated redundant manipulator. Kinematic scheme of the manipulator is presented in Fig. 7, and its FBD is shown in Fig. 8. The considered system was created by adding a supplementary actuator (consisting of bodies 8 and 9) to the redundant manipulator examined in the previous example. Hence, the redundancy of drives is introduced, since the two actuators (4–5 and 8–9) are parallel to each other.

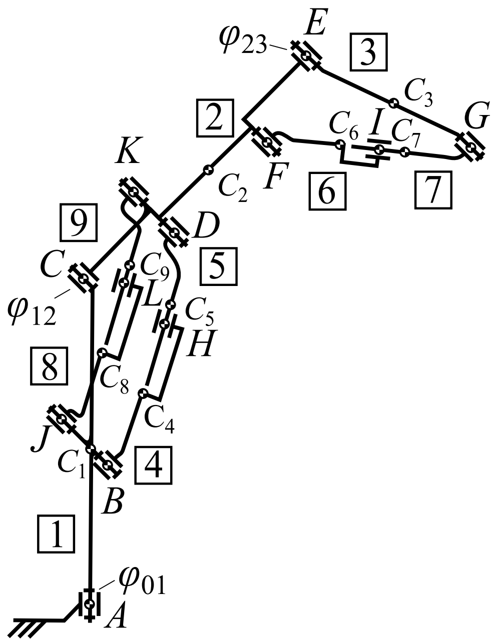

Fig. 7. Kinematic scheme of the overactuated redundant manipulator

The uniqueness test is performed in the same position as for the manipulator from example 4.4. The vector of unknowns, **x**, has the following form

$$\mathbf{x}_{61\times1} = \big[\mathbf{S}_{01Ar}^T \; M_{01Arx}^{Ar} \; M_{01Ary}^{Ar} \,|\, \mathbf{S}_{14Br}^T \; M_{14Brx}^{Br} \; M_{14Bry}^{Br} \,|\, \mathbf{S}_{12Cr}^T \; M_{12Crx}^{Cr} \; M_{12Cry}^{Cr} \,|\, \mathbf{S}_{25Dr}^T \; M_{25Drx}^{Dr} \; M_{25Dry}^{Dr} \,|\, \mathbf{S}_{23Er}^T \; M_{23Erx}^{Er} \; M_{23Ery}^{Er} \,|\, \mathbf{S}_{26Fr}^T \; M_{26Frx}^{Fr} \; M_{26Fry}^{Fr} \,|\, \mathbf{S}_{37Gr}^T \; M_{37Grx}^{Gr} \; M_{37Gry}^{Gr} \,|\, S_{45Hcx}^{Hc} \; S_{45Hcy}^{Hc} \; M_{45Hcx}^{Hc} \; M_{45Hcy}^{Hc} \,|\, S_{67Icx}^{Ic} \; S_{67Icy}^{Ic} \; M_{67Icx}^{Ic} \; M_{67Icy}^{Ic} \,|\, M_{d01Arz}^{Ar} \,|\, F_{d45Hcz}^{Hc} \,|\, F_{d67Icz}^{Ic} \,|\, \mathbf{S}_{18Jr}^T \; M_{18Jrx}^{Jr} \; M_{18Jry}^{Jr} \,|\, \mathbf{S}_{29Kr}^T \; M_{29Krx}^{Kr} \; M_{29Kry}^{Kr} \,|\, S_{89Lcx}^{Lc} \; S_{89Lcy}^{Lc} \; M_{89Lcx}^{Lc} \; M_{89Lcy}^{Lc} \,|\, F_{d89Lcz}^{Lc}\big]^T, \quad (23)$$

wherein the first 46 elements come directly from the previous example. Coefficient matrix **A** corresponding to this vector may be created analogously to the previous example. Moreover, it has size 54×61 and a full row rank $r\,(\mathbf{A}) = 54$. Subsequently, nullspace matrix **N** of size 61×7 has been computed. It has the following structure

$$\mathbf{N} = [\mathbf{0}_{7\times5} \; \mathbf{N}_1 \; \mathbf{0}_{7\times1} \; \mathbf{N}_2 \; \mathbf{0}_{7\times1} \; \mathbf{N}_3 \; \mathbf{0}_{7\times1} \; \mathbf{N}_4 \; \mathbf{0}_{7\times1} \; \mathbf{N}_5 \; \mathbf{0}_{7\times1} \; \mathbf{N}_6 \; \mathbf{0}_{7\times1} \; \mathbf{N}_7 \; \mathbf{0}_{7\times1} \; \mathbf{N}_8 \; \mathbf{0}_{7\times1} \; \mathbf{N}_9 \; \mathbf{0}_{7\times2} \; \mathbf{N}_{10} \; \mathbf{0}_{7\times2} \; \mathbf{N}_{11} \; \mathbf{0}_{7\times1} \; \mathbf{N}_{12} \; \mathbf{0}_{7\times1} \; \mathbf{N}_{13} \; \mathbf{0}_{7\times2} \; \mathbf{N}_{14}]^T. \quad (24)$$

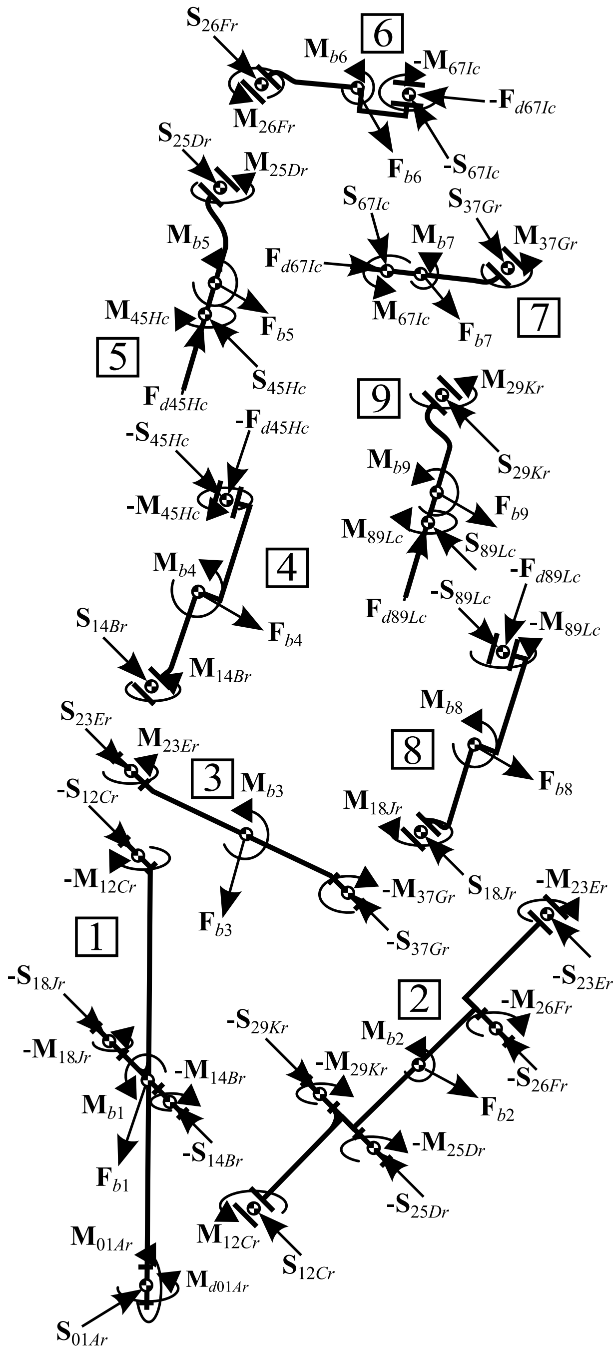

Fig. 8. Free-body diagram of the overactuated redundant manipulator

Moreover, submatrices $\mathbf{N}_i$, $i = 1, 2, \ldots 14$ (with nonzero elements) are of sizes: $(\mathbf{N}_1)_{7\times5}$, $(\mathbf{N}_2)_{7\times1}$, $(\mathbf{N}_3)_{7\times7}$, $(\mathbf{N}_4)_{7\times1}$, $(\mathbf{N}_5)_{7\times2}$, $(\mathbf{N}_6)_{7\times1}$, $(\mathbf{N}_7)_{7\times2}$, $(\mathbf{N}_8)_{7\times1}$, $(\mathbf{N}_9)_{7\times3}$, $(\mathbf{N}_{10})_{7\times2}$, $(\mathbf{N}_{11})_{7\times1}$, $(\mathbf{N}_{12})_{7\times1}$, $(\mathbf{N}_{13})_{7\times11}$, $(\mathbf{N}_{14})_{7\times2}$.

Since the first elements of vector **x** are defined as in the previous example, the information contained in Tab. 7 is also applicable here. Moreover, the analogous data for the remaining elements are given in Tab. 9. Finally, the results of the procedure are presented in Tab. 10. Note that they are consistent with intuition, i.e. the reaction in revolute joint A remained unique, and the introduction of a redundant drive resulted only in the non-uniquely determined driving forces in the parallel actuators.

Tab. 9. Additional studied elements of the overactuated redundant manipulator

Studied element	Elements of $\mathbf{x}$	Columns forming $\mathbf{A}_S$
Reaction Jr	$\mathbf{S}_{18Jr}$, M^{Jr}_{18Jrx}, M^{Jr}_{18Jry}	47–51
Reaction Kr	$\mathbf{S}_{29Kr}$, M^{Kr}_{29Krx}, M^{Kr}_{29Kry}	52–56
Reaction Lc	S^{Lc}_{89Lcx}, S^{Lc}_{89Lcy}, M^{Lc}_{89Lcx}, M^{Lc}_{89Lcy}	57–60
Drive Lc	F^{Lc}_{d89Lcz}	61

Tab. 10. Results of the overactuated redundant manipulator analysis

Studied element	Criterion (4) value	Rank of $\mathbf{A}_S$	Result
Reaction Ar	$= \mathbf{0}$	5(full)	U
Reactions: Br, Cr, Dr, Er, Fr, Gr, Jr, Kr	$\neq \mathbf{0}$	5(full)	N
Reactions: Hc, Ic, Lc	$\neq \mathbf{0}$	4(full)	N
Drives: Ar, Ic	$= \mathbf{0}$	1(full)	U
Drives: Hc, Lc	$\neq \mathbf{0}$	1(full)	N

5. Conclusions

This paper shows that the problem of non-uniqueness of joint reactions in overconstrained mechanisms should be extended by acknowledging similar problems resulting from redundant actuation. A new – kinetostatics-based – approach, combined with developed methods of nullspace analysis, was utilized to verify uniqueness of joint reactions and driving forces. The same procedure, outlined herein, may be carried out for both passive (reactions) and active (actuation) forces analysis.

The method presented in this article consists in analysis of the nullspace basis created for the coefficient matrix resulting from kinetostatics equations. This method is applicable both to planar and spatial systems. To illustrate the approach, five examples have been considered: three cases of a gripper [7,9,19], a redundant manipulator [9] and an overactuated redundant manipulator. In general, the results – with regard to joint reactions – are in accordance with the intuition and the results known from other publications. The novelty consists in taking driving forces into account.

It should be pointed out that, in example 4.3, overactuation of the gripper caused non-uniqueness of all the reactions. It is an interesting observation which demands further studies. Hence, the analysis of uniqueness of driving forces should always be performed together with the reaction uniqueness test.

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AUTHORS

Marcin Pękal* – Division of Theory of Machines and Robots, Institute of Aeronautics and Applied Mechanics, Faculty of Power and Aeronautical Engineering, Warsaw University of Technology, Nowowiejska 24, 00–665 Warsaw, Poland, e-mail: mpekal@meil.pw.edu.pl.

Janusz Frączek – Division of Theory of Machines and Robots, Institute of Aeronautics and Applied Mechanics, Faculty of Power and Aeronautical Engineering, Warsaw University of Technology, Nowowiejska 24, 00–665 Warsaw, Poland, e-mail: jfraczek@meil.pw.edu.pl.

*Corresponding author

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