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The Modelling Method of Discrete-Continuous Systems

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Abstract

The paper introduces a method of discrete-continuous systems modelling. In the proposed method a three-dimensional system is divided into finite elements in only two directions, with the third direction remaining continuous. The thus obtained discrete-continuous model is described by a set of partial differential equations. General difference equations of discrete system are obtained using the rigid finite element method. The limit of these equations leads to partial differential equations. The derived equations, expressed in matrix form, allow for the creation of a global matrix for the whole system. The equations are solved using the distributed transfer function method. Proposed approach is illustrated with the example of a simple beam fixed at both ends.

Keywords: modelling, model reduction, modal analysis, mechanical system, dynamic systems, vibration.

1. Introduction

Many different methods for modelling dynamic systems are known [1,2,5]. However, there is no universal approach which is both accurate and applicable to the wide range of dynamic systems. One of the most commonly used approaches is the finite element method, which is particularly useful in providing approximate models of the real systems. Its accuracy depends on the number of finite elements. The greater their number, the more accurate the model. However, there is an optimal division density, above which rounding errors start to seriously affect numerical calculation. The use of finite element methods for slender elements or structures is inefficient and basically ineffective, as maintaining appropriate proportions would require a very fine mesh, leading to the said rounding errors in numerical calculations. A very large number of finite elements also means creating a high-order model. Such models are not suitable for designing control systems. Additionally, the exact analytical solutions for a slender elements, such as strings, bars and beams, are already known and therefore more suitable for continuous models.

This paper proposes a hybrid method of modelling that combines the advantages of spatial discretization methods with the advantages of continuous systems modelling method. In the classical finite element method, the body is divided into all three spatial directions (Fig. 1a, 1c). In the proposed method, the same body is divided into one (Fig. 1b) or two (Fig. 1d) spatial directions, with one direction remaining continuous. Such a division results in finite elements with parameters distributed along one of the axes. Two-dimensional elements are called strips (Fig. 1b) and three-dimensional elements are called prisms (Fig. 1d). Both these elements are one-dimensional distributed systems and are therefore described by second order partial differential equations. However, these equations also have terms related to interactions between elements. Hence, the given system is described by coupled second order partial differential equations.

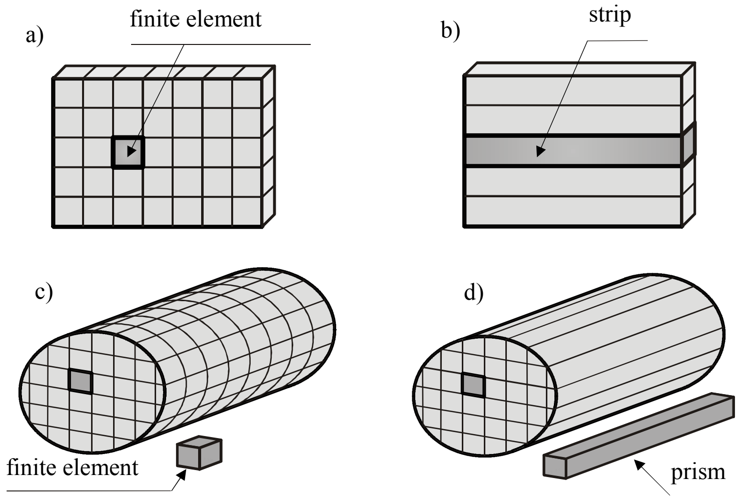

Figure 1. Spatial discretization of 2D and 3D body: a), c) conventional finite element method, b), d) proposed hybrid method

2. General model of discrete-continuous system

In order to derive a general model of the discrete-continuous system, let us consider two prisms, r and p, connected together by a layer of spring-damping elements, k, with distributed parameters (Fig. 2a). Such a discrete model is shown in Fig. 2b. Each element has 6 degrees of freedom. By applying the rigid finite element method to this discrete model, one obtains an appropriate system of ordinary differential equations for prism r. Such an FEM model may be transformed into a continuous model by letting $dx\rightarrow 0$. In this way small differences, divided by dx, become derivatives. After these transformations, the following six differential equations of the r-th prism are obtained:

$$\begin{aligned} f_{r,1}\Delta y\Delta z &= \rho\Delta y\Delta z\ddot{q}_{r,1} - E\Delta y\Delta z q''_{r,1} + \\ &+ c_{xk,1}(q_{r,1} - q_{p,1}) + c_{xk,1}(s_{r,k,3}q_{i,j,5} - s_{p,k,3}q_{p,5}) - c_{xk,1}(s_{r,k,2}q_{r,6} - s_{p,k,2}q_{p,6}) \end{aligned} \tag{1}$$

$$\begin{aligned} f_{r,2}\Delta y\Delta z &= \rho\Delta y\Delta z\ddot{q}_{r,2} - \kappa G\Delta y\Delta z q''_{r,2} + \kappa G\Delta y\Delta z q'_{r,6} \\ &+ c_{xk,2}(q_{r,2} - q_{p,2}) - c_{xk,2}(s_{r,k,3}q_{r,4} - s_{p,k,3}q_{p,4}) \end{aligned} \tag{2}$$

$$\begin{aligned} f_{r,3}\Delta y\Delta z &= \rho\Delta y\Delta z\ddot{q}_{r,3} - \kappa G\Delta y\Delta z q''_{r,3} - \kappa G\Delta y\Delta z q'_{r,5} \\ & c_{xk,3}(q_{r,3} - q_{p,3}) + c_{xk,3}(s_{r,k,2}q_{r,4} - s_{p,k,2}q_{p,4}) \end{aligned} \tag{3}$$

$$\begin{aligned}
&f_{r,4}\Delta y\Delta z=\rho I_{0x}\ddot{q}_{r,4}-GI_{0x}q''_{r,4}+\\
&c_{xk,4}(q_{r,4}-q_{p,4})+c_{xk,3}(s_{r,k,2}q_{r,3}-s_{r,k,2}q_{p,3})+c_{xk,3}(s^2_{r,k,2}q_{r,4}-s_{r,k,2}s_{p,k,2}q_{p,4})+\\
&+c_{xk,2}(s^2_{r,k,3}q_{r,4}-s_{r,k,3}s_{p,k,3}q_{p,4})+c_{xk,2}(s_{r,k,3}q_{p,2}-s_{r,k,3}q_{r,2})
\end{aligned} \tag{4}$$

$$\begin{aligned}
&f_{r,5}\Delta y\Delta z=\rho I_{yx}\ddot{q}_{r,5}-EI_{yx}q''_{r,5}+\kappa G\Delta y\Delta z q'_{r,3}+\kappa G\Delta y\Delta z q_{r,5}\\
&c_{xk,5}(q_{r,5}-q_{p,5})+c_{xk,1}s_{r,k,3}(q_{r,1}-q_{p,1})+c_{xk,1}(s^2_{r,k,3}q_{r,5}-s_{r,k,3}s_{p,k,3}q_{p,5})+\\
&-c_{xk,1}(s_{r,k,2}s_{r,k,3}q_{r,6}-s_{r,k,3}s_{p,k,2}q_{p,6})
\end{aligned} \tag{5}$$

$$\begin{aligned}
&f_{r,6}\Delta y\Delta z=\rho I_{zx}\ddot{q}_{r,6}-EI_{zx}q''_{r,6}-\kappa G\Delta y\Delta z q'_{r,2}+\kappa G\Delta y\Delta z q_{r,6}\\
&-c_{xk,1}s_{r,k,2}(q_{r,1}-q_{p,1})-c_{xk,1}(s_{r,k,2}s_{r,k,3}q_{r,5}-s_{r,k,2}s_{p,k,3}q_{p,5})\\
&+c_{xk,1}(s^2_{r,k,2}q_{r,6}-s_{r,k,2}s_{p,k,2}q_{p,6})
\end{aligned} \tag{6}$$

where: E – Young's modulus, G – shear modulus, $I_{\alpha\beta}$ – geometric moment of inertia of cross section area perpendicular to the β axis about α axis, Δy, Δz – elementary dimensions of finite element (Fig. 2b), κ – numerical shape factor of cross section, ρ – mass per unit volume, q_i – transverse displacements in i direction, $f_{r,i}$ – distributed general force applied at r-th element (excitation) in i direction , i=1,2,…,6, $s_{\alpha,\beta,\gamma}$ – distance between body α and distributed spring-damping element β in γ direction, $c_{x\alpha,\beta}$ – distributed stiffness coefficient of spring element α in β direction.

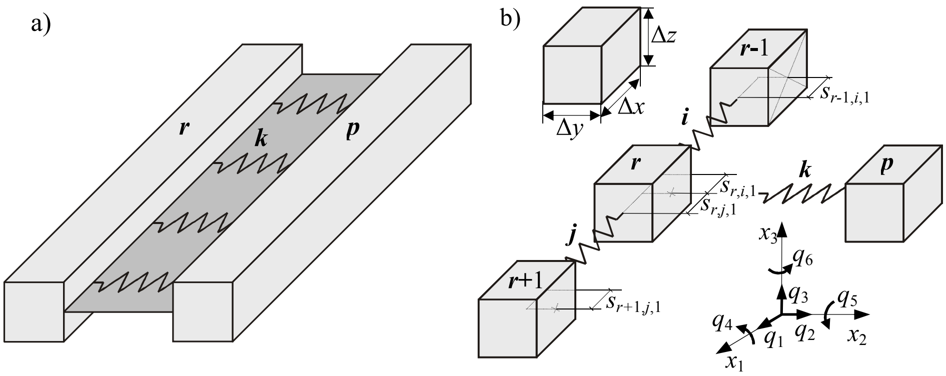

Figure 2. General model of considered system: a) discrete-continuous, b) discrete

In the same way equations for the p element can be determined. These p element equations can also be obtained from equations (1÷6) by replacing r indices with p indices and p indices with r indices. Equations (1÷6) for the r element and the corresponding equations for the p element may be written in matrix form:

$$\boldsymbol{A}_{02}\ddot{\boldsymbol{q}}(x,t)+\boldsymbol{A}_{20}\boldsymbol{q}''(x,t)+\boldsymbol{A}_{10}\boldsymbol{q}'(x,t)+\boldsymbol{A}_{00}\boldsymbol{q}(x,t)=\boldsymbol{f}(x,t) \tag{7}$$

with boundary conditions

$$\left(\boldsymbol{M}_{01}+\boldsymbol{M}_{11}\frac{\partial}{\partial x}\right)\boldsymbol{q}(0,t)=\boldsymbol{\gamma}_1(t)\,,\ \left(\boldsymbol{N}_{02}+\boldsymbol{N}_{12}\frac{\partial}{\partial x}\right)\boldsymbol{q}(l,t)=\boldsymbol{\gamma}_2(t)\,,$$

where:

$\boldsymbol{A}_{02}=diag\,(\boldsymbol{A}_{02r},\boldsymbol{A}_{02p})$, $\boldsymbol{A}_{02r}=diag\,(\rho\Delta y\Delta z,\rho\Delta y\Delta z,\rho\Delta y\Delta z,\rho I_{0x},\rho I_{yx},\rho I_{zx})$,

$\boldsymbol{A}_{02p}=diag\,(\rho\Delta y\Delta z,\rho\Delta y\Delta z,\rho\Delta y\Delta z,\rho I_{0x},\rho I_{yx},\rho I_{zx})$,

$\boldsymbol{A}_{20}=diag\,(\boldsymbol{A}_{20r},\boldsymbol{A}_{20p})$, $\boldsymbol{A}_{20r}=diag\,(-E\Delta y\Delta z,-\kappa G\Delta y\Delta z,-\kappa G\Delta y\Delta z,-GI_{0x},-EI_{yx},-EI_{zx})$,

$\boldsymbol{A}_{20p}=diag\,(-E\Delta y\Delta z,-\kappa G\Delta y\Delta z,-\kappa G\Delta y\Delta z,-GI_{0x},-EI_{yx},-EI_{zx})$, $\boldsymbol{A}_{10}=diag(\boldsymbol{A}_{10r},\boldsymbol{A}_{10p})$,

$$\boldsymbol{A}_{10r}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&\kappa G\Delta y\Delta z\\0&0&0&0&-\kappa G\Delta y\Delta z&0\\0&0&0&0&0&0\\0&0&\kappa G\Delta y\Delta z&0&0&0\\0&-\kappa G\Delta y\Delta z&0&0&0&0\end{bmatrix},\ \boldsymbol{A}_{00}=\left[\begin{array}{c:c}\boldsymbol{K}_{rrk}&\boldsymbol{K}_{rpk}\\\hdashline\boldsymbol{K}_{prk}&\boldsymbol{K}_{ppk}\end{array}\right],$$

$$\boldsymbol{K}_{rrk}=\begin{bmatrix}c_{xk1}&0&0&0&c_{xk1}s_{rk3}&-c_{xk1}s_{rk2}\\0&c_{xk2}&0&-c_{xk2}s_{rk3}&0&0\\0&0&c_{xk3}&c_{xk3}s_{rk2}&0&0\\0&-c_{xk2}s_{rk3}&c_{xk3}s_{rk2}&c_{xk4}+c_{xk2}s_{rk3}^2+c_{xk3}s_{rk2}^2&0&0\\c_{xk1}s_{rk3}&0&0&0&c_{xk5}+c_{xk1}s_{rk3}^2&-c_{xk1}s_{rk2}s_{rk3}\\-c_{xk1}s_{rk2}&0&0&0&-c_{xk1}s_{rk2}s_{rk3}&c_{xk6}+c_{xk1}s_{rk2}^2\end{bmatrix}$$

$$\boldsymbol{K}_{rpk}=\begin{bmatrix}-c_{xk1}&0&0&0&-c_{xk1}s_{pk3}&c_{xk1}s_{pk2}\\0&-c_{xk2}&0&c_{xk2}s_{pk3}&0&0\\0&0&-c_{xk3}&-c_{xk3}s_{pk2}&0&0\\0&c_{xk2}s_{rk3}&-c_{xk3}s_{rk2}&-c_{xk4}-c_{xk2}s_{rk3}s_{pk3}-c_{xk3}s_{rk2}s_{pk2}&0&0\\-c_{xk1}s_{rk3}&0&0&0&-c_{xk5}-c_{xk1}s_{rk3}s_{pk3}&c_{xk1}s_{rk3}s_{pk2}\\c_{xk1}s_{rk2}&0&0&0&c_{xk1}s_{rk2}s_{pk3}&-c_{xk6}-c_{xk1}s_{rk2}s_{pk2}\end{bmatrix}$$

$\boldsymbol{K}_{prk}=\boldsymbol{K}_{rpk}^T$, matrices $\boldsymbol{K}_{ppk}$ and $\boldsymbol{A}_{10p}$ are obtained from $\boldsymbol{K}_{rrk}$ and $\boldsymbol{A}_{10r}$ by replacing indices r with p.

A global model for the whole system is built the same way as the FEM model. Global matrices $\boldsymbol{A}_{02}$, $\boldsymbol{A}_{20}$, $\boldsymbol{A}_{10}$ include sub-matrices of each prism element, located on their main diagonal. Matrix $\boldsymbol{A}_{00}$ is formed by summing the stiffness matrices of each prism element in the global system.

The solution of these equations with appropriate boundary conditions gives semi-analytical results for the tree-dimensional structure. To solve partial differential equation (7), the distributed transfer function method was used [2,4].

The proposed approach may be applied in modelling 1D, 2D and 3D continuous systems. In the case of a 1D system, there are of course no interactions between prisms.

3. Example of method application

As a simple example, let us consider a beam fixed at both ends (Fig. 3) with the following parameters: $E = 2 \cdot 10^{11}$ [Pa], $G = 8 \cdot 10^{10}$ [Pa], $\rho = 8000$ [kg/m^3], Δy=0.15 [m], Δz=0.15 [m], l=1 [m], $\kappa = 1.2$.

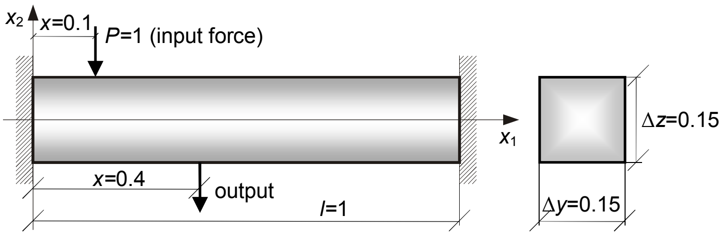

Figure 3. Fixed beam

The beam is divided into four prisms (Fig. 4) and four distributed spring elements. Each prism has three degrees of freedom – displacement along x_1 and x_2 axes and rotation angle around x_3 axis.

For this example the frequency responses of the proposed model are compared with those of Euler and Timoshenko beam models (Fig. 5).

The beam frequency responses (Fig. 5) are obtained for the unit step force input signal acting at beam point x=0.1 [m] (Fig. 3) and the displacement output signal is observed at the x=0.4 [m] point.

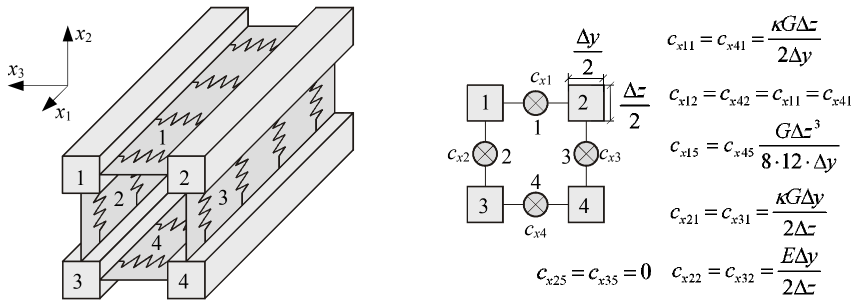

Figure 4. Discrete model of beam: a) general scheme, b) equivalent scheme

The characteristic in Fig. 5 shows that the first two frequencies of the proposed model and that of the Timoshenko beam model are very similar. This proves that the proposed model is correct. The later trend shows that the frequencies in the proposed model are even more to the left than in Timoshenko's model. The characteristic of the Euler beam model differs significantly from the other two. This is because the Euler beam model does not take into account the effect of shear deformation and is therefore less accurate. Timoshenko included shear deformation to produce a more accurate model than Euler, with a frequency trend more to the left. The beam model proposed in this paper is closer to Timoshenko's model but the subsequent frequency trend is even more to the left. In the future, these results will be verified and compared with a corresponding FEM model.

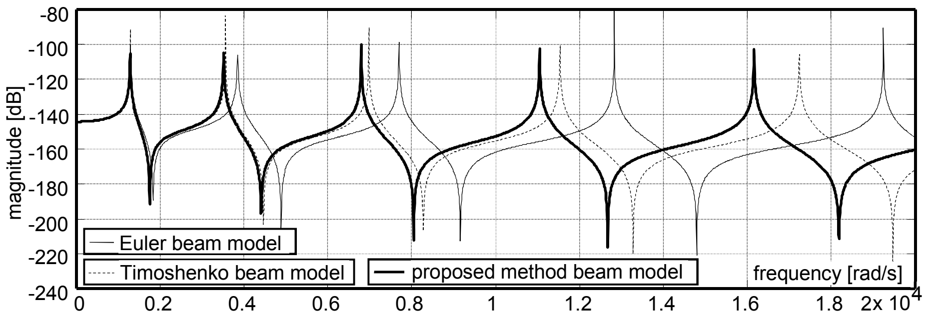

Figure 5. Frequency characteristics

4. Conclusions

This paper has presented a discrete-continuous modelling method. For the proposed method, general partial differential equations were derived. These equations were next written in a formalized matrix form that is very easily applied in computer algorithms. A beam fixed at both ends was used to illustrate the general concept. The obtained numerical calculation results show that the proposed method is efficient and applicable to discrete-continuous dynamic system modelling.

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