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ON THE EIGENVALUES OF A 2×2 BLOCK OPERATOR MATRIX

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Abstract. A 2×2 block operator matrix $\mathbf{H}$ acting in the direct sum of one- and two-particle subspaces of a Fock space is considered. The existence of infinitely many negative eigenvalues of H_{22} (the second diagonal entry of $\mathbf{H}$) is proved for the case where both of the associated Friedrichs models have a zero energy resonance. For the number $N(z)$ of eigenvalues of H_{22} lying below $z<0$, the following asymptotics is found

$$\lim_{z\to -0} N(z)|\log|z||^{-1} = \mathcal{U}_0 \quad (0<\mathcal{U}_0<\infty).$$

Under some natural conditions the infiniteness of the number of eigenvalues located respectively inside, in the gap, and below the bottom of the essential spectrum of $\mathbf{H}$ is proved.

Keywords: block operator matrix, Fock space, discrete and essential spectra, Birman-Schwinger principle, the Efimov effect, discrete spectrum asymptotics, embedded eigenvalues.

Mathematics Subject Classification: 81Q10, 35P20, 47N50.

1. INTRODUCTION

The number of eigenvalues of Hamiltonians (block operator matrices) on a Fock space is one of the most actively studied objects in operator theory, in many problems in mathematical physics and other related domains. An important problem in the spectral analysis of these operators is to find out whether the set of eigenvalues located inside, in the gap or in below the bottom of the essential spectrum is infinite. The latter result is the remarkable phenomenon known as the *Efimov effect* in the spectral theory of the three-particle Schrödinger operators. This property was discovered by V. Efimov [7] and has been the subject of many papers [4, 6, 18, 23, 24, 26]. The first mathematical proof of the existence of this effect was given by D. Yafaev [26], and

A. Sobolev [23] established the asymptotics of the number of eigenvalues near the threshold of the essential spectrum.

Perturbation problems for operators with embedded eigenvalues are generally challenging since the embedded eigenvalues cannot be separated from the rest of the spectrum. Embedded eigenvalues occur in many applications arising in physics. In quantum mechanics, for instance, eigenvalues of the energy operator correspond to energy bound states that can be attained by the underlying physical system. If such an eigenvalue is embedded in the continuous spectrum, it is of fundamental importance to determine whether it, and therefore the corresponding bound state, persists after perturbing the potential. Many works have been devoted to the study of embedded eigenvalues of Schrödinger operators (see, for example [1, 5, 17, 22]). In the paper [15], it is shown that the embedded eigenvalues of the three-particle Schrödinger operator on a one-dimensional lattice is infinite in the case where the masses of two particles are infinite.

It is remarkable that the above mentioned operators describe the systems with a conserved finite number of particles in continuous space or on a lattice. However, in both cases, there exist problems with a non-conserved number of particles that are more interesting in a certain sense. Such problems occur in statistical physics [11, 12], solid state physics [13] and the theory of quantum fields [8]. Systems with a non-conserved finite number of particles in continuous space were considered in [12,27]. Usually the Hamiltonians describing such systems in both cases can be expressed as block operator matrices.

In the present paper we consider the 2×2 block operator matrix $\mathbf{H}$ acting in the direct sum of one- and two-particle subspaces of a Fock space. The main aim of this paper is to give a thorough mathematical treatment of the spectral properties of $\mathbf{H}$ with emphasis on the infiniteness of the number of eigenvalues embedded in its essential spectrum.

Let us briefly set up the problem. Denote by $\mathbb{T}^3$ the three-dimensional torus (the cube $(-\pi, \pi]^3$ with appropriately identified sides) and by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_1 := L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 := L_2((\mathbb{T}^3)^2)$, that is, $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. The Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are one-particle and two-particle subspaces of the Fock space $\mathcal{F}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively.

We consider the block operator matrix $\mathbf{H}$ acting in the Hilbert space $\mathcal{H}$ given by

$$\mathbf{H} := \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{pmatrix}$$

with the entries $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i \leq j$, $i, j = 1, 2$:

$$(H_{11} f_1)(p) = u(p) f_1(p), \quad (H_{12} f_2)(p) = \int\limits_{\mathbb{T}^3} v(s) f_2(p, s) ds,$$

$$(H_{22} f_2)(p, q) = w(p, q) f_2(p, q) - \mu_1 \int\limits_{\mathbb{T}^3} f_2(p, s) ds - \mu_2 \int\limits_{\mathbb{T}^3} f_2(s, q) ds,$$

where H_{12}^* denotes the adjoint operator to H_{12} and $f_i \in \mathcal{H}_i$, $i = 1, 2$.

Here μ_α, $\alpha = 1, 2$, are positive real numbers, $u(\cdot)$ and $v(\cdot)$ are real-valued continuous functions on $\mathbb{T}^3$ and the function $w(\cdot, \cdot)$ has the form

$$w(p, q) := l_1\varepsilon(p) + l_2\varepsilon(q) + l_3\varepsilon(p + q)$$

with $l_i > 0$, $i = 1, 2, 3$, and

$$\varepsilon(p) := \sum_{i=1}^{3}(1 - \cos(2p^{(i)})), \quad p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3. \tag{1.1}$$

Under these assumptions the operator $\mathbf{H}$ is bounded and self-adjoint.

We remark that the operators H_{12} and H_{12}^* are called annihilation and creation operators [8], respectively. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

Notice that the operator H_{22} is a model operator associated with a system of three particles on $\mathbb{Z}^3$, where the role of the two-particle discrete Schrödinger operators is played by a family of Friedrichs models with parameters $h_{\mu_\alpha}(p)$, $\mu_\alpha > 0$, $\alpha = 1, 2$, $p \in \mathbb{T}^3$. Under some smoothness assumptions:

(i) we describe the location and structure of the essential spectrum of $\mathbf{H}$;
(ii) we find a value μ_α^0 of the parameter μ_α that for $\mu_\alpha = \mu_\alpha^0$, $\alpha = 1, 2$ the operator H_{22} has infinitely many negative eigenvalues accumulating at zero (Efimov's effect). Moreover, we show that for the number $N(z)$ of eigenvalues of H_{22} lying below $z < 0 = \min \sigma_{\text{ess}}(H_{22})$, the limit $\lim_{z \to -0} N(z) |\log |z||^{-1} = \mathcal{U}_0$ exists for some $\mathcal{U}_0 \in (0; \infty)$;
(iii) we find conditions which guarantee the infiniteness of the number of eigenvalues located inside, in the gap, and below the bottom of the essential spectrum of $\mathbf{H}$, respectively.

We note that such type of operator matrices were considered in [16, 19, 21, 25] where only its essential spectrum was investigated.

Now we are going to explain the importance of the problem and the meaning of the dispersion function. In the physical literature, the function $\varepsilon(\cdot)$ given by the Fourier series

$$\varepsilon(p) = \sum_{s \in \mathbb{Z}^3} \widehat{\varepsilon}(s) e^{i(p,s)}, \quad p \in \mathbb{T}^3$$

being a real-valued function on $\mathbb{T}^3$, is called the *dispersion function of normal modes* associated with the free particle. Note that the Fourier coefficients of the function $\varepsilon(\cdot)$ differ from the coefficients $\widehat{\varepsilon}(\cdot)$ by the factor $(2\pi)^{3/2}$. Here

$$(p, s) := p^{(1)}s^{(1)} + p^{(2)}s^{(2)} + p^{(3)}s^{(3)}, \; p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3, \; s = (s^{(1)}, s^{(1)}, s^{(1)}) \in \mathbb{Z}^3,$$

and the series $\sum_{s\in\mathbb{Z}^3}\widehat{\varepsilon}(s)$ is assumed to be absolutely convergent. It is known that if the dispersion function $\varepsilon(\cdot)$ is conditionally negative definite, then $\varepsilon(\cdot)$ admits a (Levy-Khinchin) representation

$$\varepsilon(p)=\varepsilon(0)+\sum_{s\in\mathbb{Z}^3}\widehat{\varepsilon}(s)(e^{i(p,s)}-1),\quad p\in\mathbb{T}^3,$$

which is equivalent to the requirement that the Fourier coefficients $\widehat{\varepsilon}(s)$ with $s\neq 0$ are non-positive.

If the (Fourier) coefficients $\widehat{\varepsilon}(s)$ are defined by

$$\widehat{\varepsilon}(s)=\begin{cases}3, & s=0,\\ -1/2, & |s|=1,\\ 0, & \text{otherwise},\end{cases}$$

then the corresponding dispersion function

$$\varepsilon(p)=\sum_{i=1}^{3}(1-\cos p^{(i)}) \tag{1.2}$$

is a conditionally negative definite function and it has a unique non-degenerate minimum. We recall that threshold analysis for the operators $h_{\mu_\alpha}(p)$, $\alpha=1,2$, with dispersion function (1.2) are studied in [2], where the existence of Efimov's effect for H_{22} was proven and the corresponding asymptotics of the discrete spectrum was obtained. What happens if the function $\varepsilon(\cdot)$ has non-degenerate minima at several points? In order to justify the importance of this question we consider the Fourier coefficients $\widehat{\varepsilon}(s)$ defined by

$$\widehat{\varepsilon}(s)=\begin{cases}3, & s=0,\\ -1/2, & s\in\{(\pm 2,0,0),(0,\pm 2,0),(0,0,\pm 2)\},\\ 0, & \text{otherwise}.\end{cases}$$

Then the corresponding dispersion function $\varepsilon(\cdot)$ is of the form (1.1) with the non-degenerate minima at 8 different points of $\mathbb{T}^3$. We show that the asymptotics of the discrete spectrum of $\mathbf{H}$ with respect to the dispersion functions (1.1) and (1.2) does not change.

The organization of the present paper is as follows. Section 1 is an introduction to the whole work. In Section 2, the main results of the paper are formulated. In Section 3, we discuss some results concerning threshold analysis of families of Friedrichs models $h_{\mu_\alpha}(p)$. In Section 4, we describe the location and structure of the essential spectrum of $\mathbf{H}$. In Section 5, first we give a realization of the Birman-Schwinger principle and then we obtain an asymptotic formula for the number of negative eigenvalues of H_{22}. In Section 6, we prove the infiniteness of the number of eigenvalues of $\mathbf{H}$ lying inside (in the gap, below the bottom) of its essential spectrum. At the end we show non-emptiness of the class of functions $u(\cdot)$ and $v(\cdot)$ satisfying the conditions of the main results of the present paper.

2. NOTATIONS AND MAIN RESULTS

Throughout the paper we adopt the following conventions. Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of all positive integers, integers, real and complex numbers, respectively. The subscripts α and β always are equal to 1 or 2 and $\alpha \neq \beta$. We denote by $L_2(\Omega)$ the Hilbert space of square integrable (complex) functions defined on a measurable set $\Omega \subset \mathbb{R}^n$, by $L_2^{(m)}(\Omega)$ the Hilbert space of m-component vector functions $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_k \in L_2(\Omega)$, $k = 1, \ldots, m$, and by $\mathrm{diag}\{B_1, \ldots, B_m\}$ the $m \times m$ diagonal matrix with operators $B_1, \ldots, B_m$ as diagonal entries. In what follows we deal with operators in various spaces of vector-valued functions. They will be denoted by bold letters and will be written in matrix form. We denote by $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$ and $\sigma_{\mathrm{disc}}(\cdot)$, respectively, the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator.

Set $w_1(p,q) := w(p,q)$, $w_2(p,q) := w(q,p)$ and $\mathcal{H}_0 := \mathbb{C}$.

To study the spectral properties of the operator $\mathbf{H}$ we introduce the following two families of bounded self-adjoint operators (Friedrichs models), acting in $\mathcal{H}_0 \oplus \mathcal{H}_1$ and $\mathcal{H}_1$, by

$$\mathbf{h}_{\mu_1}(p) := \begin{pmatrix} h_{00}(p) & h_{01} \\ h_{01}^* & h_{11}(p) \end{pmatrix} \quad \text{and} \quad h_{\mu_2}(p) := h_2^0(p) - \mu_2 v,$$

respectively, where

$$h_{00}(p) f_0 = u(p) f_0, \quad h_{01} f_1 = \int_{\mathbb{T}^3} v(s) f_1(s) ds,$$

$$h_{11}(p) = h_1^0(p) - \mu_1 v, \quad (v f_1)(q) = \int_{\mathbb{T}^3} f_1(s) ds,$$

$$(h_\alpha^0(p) f_1)(q) = w_\alpha(p,q) f_1(q), \quad \alpha = 1, 2.$$

The following theorem describes the location of the essential spectrum of the operator $\mathbf{H}$ by the spectrum of the families $\mathbf{h}_{\mu_1}(p)$ and $h_{\mu_2}(p)$.

Theorem 2.1. *The essential spectrum of* $\mathbf{H}$ *satisfies*

$$\sigma_{\mathrm{ess}}(\mathbf{H}) = \bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(\mathbf{h}_{\mu_1}(p)) \cup \bigcup_{p \in \mathbb{T}^3} \sigma_{\mathrm{disc}}(h_{\mu_2}(p)) \cup [0; M], \quad M := \frac{9}{2}(l_1 + l_2 + l_3). \tag{2.1}$$

Moreover, the set $\sigma_{\mathrm{ess}}(\mathbf{H})$ *is a union of at most five intervals.*

Throughout this paper we assume the following additional assumption that the real-valued continuous function $v(\cdot)$ satisfies the condition

$$\int_{\mathbb{T}^3} v(s) g(p,s) ds = 0 \tag{2.2}$$

for any function $g \in L_2((\mathbb{T}^3)^2)$, which is considered periodical on each variable with period π.

Note that the functions

$$v(p) = \sum_{i=1}^{3} c_i \cos p^{(i)}$$

and

$$v(p) = \sum_{i=1}^{3} c_i \cos p^{(i)} \cos(2p^{(i)}),$$

where c_i, $i = 1, 2, 3$ are arbitrary real numbers, satisfy the condition (2.2). Indeed, for $v(p) = \sum_{i=1}^{3} c_i \cos p^{(i)}$, we have

$$\int\limits_{\mathbb{T}^3} v(s)g(p,s)ds = \int\limits_{\mathbb{T}^3} v(s+\bar{\pi})g(p,s+\bar{\pi})ds = -\int\limits_{\mathbb{T}^3} v(s)g(p,s)ds, \quad \bar{\pi} = (\pi,\pi,\pi),$$

which yields the equality (2.2).

Under the condition (2.2) the discrete spectrum of $\mathbf{h}_{\mu_1}(p)$ coincides (see Lemma 3.1 below) with the union of discrete spectra of the operators

$$h_{\mu_1}(p) := h_{11}(p) \quad \text{and} \quad \mathbf{h}(p) := \begin{pmatrix} h_{00}(p) & h_{01} \\ h_{01}^* & h_1^0(p) \end{pmatrix}.$$

It follows from the definition of the operators $h_{\mu_1}(p)$ and $\mathbf{h}(p)$ that their structure is simpler than that of $\mathbf{h}_{\mu_1}(p)$, and the equality (2.1) can be rewritten as

$$\sigma_{\text{ess}}(\mathbf{H}) = \bigcup_{p\in\mathbb{T}^3} \sigma_{\text{disc}}(h_{\mu_1}(p)) \cup \bigcup_{p\in\mathbb{T}^3} \sigma_{\text{disc}}(h_{\mu_2}(p)) \cup \bigcup_{p\in\mathbb{T}^3} \sigma_{\text{disc}}(\mathbf{h}(p)) \cup [0; M].$$

Let

$$m_\alpha(p) := \min_{q\in\mathbb{T}^3} w_\alpha(p,q), \quad M_\alpha(p) := \max_{q\in\mathbb{T}^3} w_\alpha(p,q).$$

For any fixed $p \in \mathbb{T}^3$ and $\mu_\alpha > 0$ we define the functions

$$\Delta(p\,; z) := u(p) - z - \int\limits_{\mathbb{T}^3} \frac{v(s)^2 ds}{w_1(p,s) - z}, \quad z \in \mathbb{C} \setminus [m_1(p); M_1(p)],$$

$$\Delta_{\mu_\alpha}(p\,; z) := 1 - \mu_\alpha \int\limits_{\mathbb{T}^3} \frac{ds}{w_\alpha(p,s) - z}, \quad z \in \mathbb{C} \setminus [m_\alpha(p); M_\alpha(p)].$$

These functions are the Fredholm determinants associated with the operators $\mathbf{h}(p)$ and $h_{\mu_\alpha}(p)$, respectively.

We introduce the following points of $\mathbb{T}^3$:

$$p_1 := (0,0,0), \quad p_2 := (\pi,0,0), \quad p_3 := (0,\pi,0), \quad p_4 := (0,0,\pi),$$

$$p_5 := (\pi,\pi,0), \quad p_6 := (\pi,0,\pi), \quad p_7 := (0,\pi,\pi), \quad p_8 := (\pi,\pi,\pi).$$

It is easy to verify that the function $w(\cdot,\cdot)$ (and hence the functions $w_\alpha(\cdot,\cdot)$, $\alpha = 1,2)$ has non-degenerate minimum at the points $(p_i, p_j) \in (\mathbb{T}^3)^2$, $i, j = \overline{1,8}$; where $\overline{1,n} = 1, \ldots, n$. Therefore, for any $p \in \mathbb{T}^3$ the integral

$$\int\limits_{\mathbb{T}^3} \frac{v(s)^2 ds}{w_1(p,s)}$$

is finite.

The Lebesgue dominated convergence theorem yields

$$\Delta(p_i\,;0) = \lim_{p \to p_i} \Delta(p_i\,;0), \quad i = \overline{1,8},$$

and hence the function $\Delta(\cdot\,;0)$ is continuous on $\mathbb{T}^3$.

Let a and b be the lower and upper bounds of the set $\bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\mathbf{h}(p)) \cap (-\infty;0]$, respectively, and

$$\mu_\alpha^0 := (l_3 + l_\alpha) \left(\int\limits_{\mathbb{T}^3} \frac{ds}{\varepsilon(s)} \right)^{-1}, \quad \alpha = 1,2.$$

Since the operator $h_{\mu_\alpha^0}(p_1)$ has no negative eigenvalues (see Lemma 3.8), that is, non-negative, by Theorem 1 of [14] the operator $h_{\mu_\alpha^0}(p)$ is non-negative for all $p \in \mathbb{T}^3$. By the other side from the positivity of v it follows that the operator $h_{\mu_\alpha}(p)$ has no eigenvalues greater than M for any $\mu_\alpha > 0$ and $p \in \mathbb{T}^3$. Hence for $\mu_\alpha = \mu_\alpha^0$ we have

$$\sigma_{\rm ess}(H_{22}) = \bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(h_{\mu_1^0}(p)) \cup \bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(h_{\mu_2^0}(p)) \cup [0;M] = [0;M]. \tag{2.3}$$

Therefore, the study of the structure of the set $\sigma_{\rm ess}(\mathbf{H})$ is reduced to the study of the structure of the set $\bigcup_{p\in\mathbb{T}^3} \sigma_{\rm disc}(\mathbf{h}(p)) \cup [0;M]$, which was completely studied in [20].

The following theorem describes the structure of the part of the essential spectrum of $\mathbf{H}$ located in $(-\infty;M]$.

Theorem 2.2. *Let $\mu = \mu_\alpha^0$, $\alpha = 1,2$. Then the following assertions hold:*

(i) *if* $\min\limits_{p\in\mathbb{T}^3} \Delta(p\,;0) \geq 0$, *then* $(-\infty;M] \cap \sigma_{\rm ess}(\mathbf{H}) = [0;M]$,

(ii) *if* $\min\limits_{p\in\mathbb{T}^3} \Delta(p\,;0) < 0$, $\max\limits_{p\in\mathbb{T}^3} \Delta(p\,;0) \geq 0$, *then* $(-\infty;M] \cap \sigma_{\rm ess}(\mathbf{H}) = [a;M]$ *and* $a < 0$,

(iii) *if* $\max\limits_{p\in\mathbb{T}^3} \Delta(p\,;0) < 0$, *then* $(-\infty;M] \cap \sigma_{\rm ess}(\mathbf{H}) = [a;b] \cup [0;M]$ *and* $a < b < 0$.

Let us denote by $\tau_{\rm ess}(H_{22})$ the bottom of the essential spectrum of H_{22} and by $N(z)$ the number of eigenvalues of H_{22} lying below the point z, $z < \tau_{\rm ess}(H_{22})$.

By the equality (2.3), we have $\tau_{\rm ess}(H_{22}) = 0$ for $\mu = \mu_\alpha^0$, $\alpha = 1,2$.

The main results of the present paper are as follows.

Theorem 2.3. *Assume $\mu = \mu_\alpha^0$, $\alpha = 1, 2$. Then the operator H_{22} has infinitely many negative eigenvalues $E_1, \dots, E_n, \dots,$ such that $\lim_{n\to\infty} E_n = 0$, and the function $N(\cdot)$ obeys the relation*

$$\lim_{z\to -0} N(z)|\log|z||^{-1} = \mathcal{U}_0, \quad 0 < \mathcal{U}_0 < \infty. \tag{2.4}$$

Clearly, by equality (2.4), the infinite cardinality of the negative discrete spectrum of H_{22} follows automatically from the positivity of $\mathcal{U}_0$.

We point out that the operator H_{22} has been considered in [2], in the case where $l_i = 1$, $i = 1, 2, 3$, and the function $\varepsilon(\cdot)$ has the form (1.2). This function has a unique non-degenerate minimum at $(0, 0, 0) \in \mathbb{T}^3$. Therefore, Theorem 2.3 can be considered as a generalization of Theorem 2.4 in [2], since in our case the function $\varepsilon(\cdot)$ has non-degenerate minimum at 8 different points of $\mathbb{T}^3$ and the asymptotics (2.4) does not depend on these points.

An easy computation shows that the operator

$$(Vf_2)(p, q) = \mu_1 \int_{\mathbb{T}^3} f_2(p, s)ds + \mu_2 \int_{\mathbb{T}^3} f_2(s, q)ds, \quad f_2 \in \mathcal{H}_2,$$

is a positive operator and $\max(\sigma_{\mathrm{ess}}(H_{22})) = \max(\sigma(H_{22} + V)) = M$, and hence, it is obvious that the operator H_{22} has no eigenvalues greater than M. So, the discrete spectrum of H_{22} is always negative or empty.

For $n \in \mathbb{N}$ denote by $f_2^{(n)}$ the eigenfunction corresponding to the eigenvalue E_n of H_{22} with $\mu = \mu_\alpha^0$, $\alpha = 1, 2$.

Theorem 2.4. *Let $\mu = \mu_\alpha^0$, $\alpha = 1, 2$. Then the numbers $E_1, \dots, E_n, \dots$ are eigenvalues of $\mathbf{H}$ and the corresponding eigenfunction has the form $f^{(n)} = (0, f_2^{(n)})$, $n \in \mathbb{N}$. Moreover,*

(i) *if $\min_{p\in\mathbb{T}^3} \Delta(p\,; 0) \geq 0$, then the set $\{E_n : n \in \mathbb{N}\}$ is located on below the bottom of the essential spectrum of $\mathbf{H}$,*

(ii) *if $\min_{p\in\mathbb{T}^3} \Delta(p\,; 0) < 0$, $\max_{p\in\mathbb{T}^3} \Delta(p\,; 0) \geq 0$, then the countable (infinite) subset of $\{E_n : n \in \mathbb{N}\}$ is located in the essential spectrum of $\mathbf{H}$,*

(iii) *if $\max_{p\in\mathbb{T}^3} \Delta(p\,; 0) < 0$, then the countable (infinite) subset of $\{E_n : n \in \mathbb{N}\}$ is located in the gap of the essential spectrum of $\mathbf{H}$.*

Note that the class of functions $u(\cdot)$ and $v(\cdot)$ satisfying the conditions in Theorem 2.4 is nonempty, for the corresponding example see Section 7.

3. THRESHOLD ANALYSIS OF THE FAMILY OF FRIEDRICHS MODELS $h_{\mu_\alpha}(p)$

In this section we study some spectral properties of the family of Friedrichs models $\mathbf{h}_{\mu_1}(p)$ and $h_{\mu_\alpha}(p)$, which play a crucial role in the study of spectral properties of the operators $\mathbf{H}$ and H_{22}.

According to the Weyl theorem we have $\sigma_{\text{ess}}(\mathbf{h}_{\mu_1}(p)) = [m_1(p); M_1(p)]$.

The following lemma describes the relation between the eigenvalues of the operators $\mathbf{h}_{\mu_1}(p)$, $h_{\mu_1}(p)$ and $\mathbf{h}(p)$.

Lemma 3.1. *The number $z \in \mathbb{C} \setminus [m_1(p); M_1(p)]$ is an eigenvalue of $\mathbf{h}_{\mu_1}(p)$ if and only if the number z is an eigenvalue of at least one of the operators $h_{\mu_1}(p)$ and $\mathbf{h}(p)$.*

Proof. Suppose $(f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1$ is an eigenvector of the operator $\mathbf{h}_{\mu_1}(p)$ associated with the eigenvalue $z \in \mathbb{C} \setminus [m_1(p); M_1(p)]$. Then f_0 and f_1 satisfy the following system of equations:

$$\begin{cases} (u(p) - z) f_0 + \int\limits_{\mathbb{T}^3} v(s) f_1(s) ds = 0, \\ v(q) f_0 + (w_1(p, q) - z) f_1(q) - \mu_1 \int\limits_{\mathbb{T}^3} f_1(s) ds = 0. \end{cases} \tag{3.1}$$

Since for any $z \in \mathbb{C} \setminus [m_1(p); M_1(p)]$ and $q \in \mathbb{T}^3$ the relation $w_1(p, q) - z \neq 0$ holds for all $p \in \mathbb{T}^3$, from the second equation of (3.1) for f_1 we have

$$f_1(q) = \frac{\mu_1 C_{f_1}}{w_1(p, q) - z} - \frac{v(q) f_0}{w_1(p, q) - z}, \tag{3.2}$$

where

$$C_{f_1} = \int\limits_{\mathbb{T}^3} f_1(s) ds. \tag{3.3}$$

Substituting the expression (3.2) for f_1 into the first equation of system (3.1) and equality (3.3), and then using condition (2.2), we conclude that the system of equations (3.1) has a nontrivial solution if and only if the system of equations

$$\begin{cases} \Delta(p\,; z) f_0 = 0, \\ \Delta_{\mu_1}(p\,; z) C_{f_1} = 0 \end{cases}$$

has a nontrivial solution, i.e., if the condition $\Delta_{\mu_1}(p\,; z) \Delta(p\,; z) = 0$ is satisfied.

If we set $\mu_1 = 0$ in above analysis, then $\mathbf{h}_{\mu_1}(p) = \mathbf{h}(p)$; in this case the number $z \in \mathbb{C} \setminus [m_1(p); M_1(p)]$ is an eigenvalue of $\mathbf{h}(p)$ if and only if $\Delta(p\,; z) = 0$.

Similarly, putting $f_0 = 0$ in above analysis, we can assert that the number $z \in \mathbb{C} \setminus [m_1(p); M_1(p)]$ is an eigenvalue of $h_{\mu_1}(p)$ if and only if $\Delta_{\mu_1}(p\,; z) = 0$. Proof of lemma is complete. $\square$

From the proof of Lemma 3.1 we obtain the following corollary.

Corollary 3.2.

(i) *The equality $\sigma_{\text{disc}}(\mathbf{h}_{\mu_1}(p)) = \sigma_{\text{disc}}(h_{\mu_1}(p)) \cup \sigma_{\text{disc}}(\mathbf{h}(p))$ holds.*

(ii) *The number $z \in \mathbb{C} \setminus [m_1(p); M_1(p)]$ is an eigenvalue of $\mathbf{h}(p)$ if and only if $\Delta(p\,; z) = 0$.*

(iii) *The number $z \in \mathbb{C} \setminus [m_\alpha(p); M_\alpha(p)]$ is an eigenvalue of $h_{\mu_\alpha}(p)$ if and only if $\Delta_{\mu_\alpha}(p\,; z) = 0$.*

The remainder of this section will be devoted to the threshold analysis of $h_{\mu_\alpha}(p)$, $\alpha=1,2$. First we remark that $\Delta_{\mu_\alpha}(p_1\,;0)=\Delta_{\mu_\alpha}(p_i\,;0)$, $i=\overline{2,8}$. Then from the definition of μ_α^0 one can see that $\Delta_{\mu_\alpha}(p_1\,;0)=0$ if and only if $\mu=\mu_\alpha^0$.

Denote by $C(\mathbb{T}^3)$ and $L_1(\mathbb{T}^3)$ the Banach spaces of continuous and integrable functions on $\mathbb{T}^3$, respectively.

Definition 3.3. The operator $h_{\mu_\alpha}(p_1)$ is said to have a *zero energy resonance* if the number 1 is an eigenvalue of the integral operator

$$(G_{\mu_\alpha}\psi)(q)=\frac{\mu_\alpha}{l_\beta+l_3}\int\limits_{\mathbb{T}^3}\frac{\psi(t)dt}{\varepsilon(t)},\quad \psi\in C(\mathbb{T}^3),$$

and at least one (up to a normalization constant) of the associated eigenfunctions ψ satisfies the condition $\psi(p_j)\neq 0$ for some $j\in\{1,\ldots,8\}$. If 1 is not an eigenvalue of G_{μ_α}, then we say that $z=0$ is a *regular type point* for the operator $h_{\mu_\alpha}(p_1)$.

Remark 3.4. The number 1 is an eigenvalue of G_{μ_α} if and only if $\mu=\mu_\alpha^0$. Consequently, the operator $h_{\mu_\alpha}(p_1)$ has a zero energy resonance if and only if $\mu=\mu_\alpha^0$.

Remark 3.5. The operator H_{22} has infinitely many negative eigenvalues accumulating at zero, if and only if, both Friedrichs models $h_{\mu_\alpha}(p_1)$, $\alpha=1,2$, have a zero energy resonance.

We notice that in the Definition 3.3 the requirement of the presence of eigenvalue 1 of G_{μ_α} corresponds to the existence of a solution of $h_{\mu_\alpha}(p_1)f_\alpha=0$ and the condition $\psi(p_j)\neq 0$ for some $j\in\{1,\ldots,8\}$ implies that the solution f_α of this equation does not belong to $L_2(\mathbb{T}^3)$. More exactly, if $h_{\mu_\alpha}(p_1)$ has a zero energy resonance, then the function

$$f_\alpha(q)=\frac{\mu_\alpha}{(l_\beta+l_3)\varepsilon(q)} \tag{3.4}$$

satisfies $h_{\mu_\alpha}(p_1)f_\alpha=0$ and $f_\alpha\in L_1(\mathbb{T}^3)\setminus L_2(\mathbb{T}^3)$.

Indeed. The proof of the fact that the function f_α satisfies $h_{\mu_\alpha}(p_1)f_\alpha=0$ is obvious. We show that $f_\alpha\in L_1(\mathbb{T}^3)\setminus L_2(\mathbb{T}^3)$.

Henceforth, we shall denote by C_1,C_2,C_3 different positive numbers and for $\delta>0$ we set

$$U_\delta(p_i):=\{p\in\mathbb{T}^3:|p-p_i|<\delta\},\quad \mathbb{T}_\delta:=\mathbb{T}^3\setminus\bigcup_{j=1}^{8}U_\delta(p_j).$$

The definition of the function $\varepsilon(\cdot)$ implies that it has a non-degenerate zero minimum at the points $p_i\in\mathbb{T}^3$, $i=\overline{1,8}$ and hence there exist $C_1,C_2,C_3>0$ and $\delta>0$ such that

$$C_1|q-p_j|^2\leq\varepsilon(q)\leq C_2|q-p_j|^2,\quad q\in U_\delta(p_j),\quad j=\overline{1,8}, \tag{3.5}$$

$$\varepsilon(q)\geq C_3,\quad q\in\mathbb{T}_\delta. \tag{3.6}$$

Using the estimates (3.5) and (3.6) we have

$$\int\limits_{\mathbb{T}^3} |f_\alpha(t)|^2 dt \geq \frac{\mu_\alpha^2}{(l_\beta + l_3)^2} \int\limits_{U_\delta(p_1)} \frac{dt}{\varepsilon^2(t)} \geq C_2 \int\limits_{U_\delta(p_1)} \frac{dt}{|t-p_1|^4} = \infty,$$

$$\int\limits_{\mathbb{T}^3} |f_\alpha(t)| dt = \frac{\mu_\alpha}{l_\beta + l_3} \Big(\sum_{j=1}^{8} \int\limits_{U_\delta(p_j)} \frac{dt}{\varepsilon(t)} + \int\limits_{\mathbb{T}_\delta} \frac{dt}{\varepsilon(t)} \Big) \leq C_1 \sum_{j=1}^{8} \int\limits_{U_\delta(p_j)} \frac{dt}{|t-p_j|} + C_3 < \infty.$$

Therefore, $f_\alpha \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3)$.

The following Lemma plays a crucial role in the proof of Theorem 2.3, that is, asymptotics (2.4).

Lemma 3.6. *The following decomposition*

$$\Delta_{\mu_\alpha^0}(p\,; z) = \frac{8\pi^2 \mu_\alpha^0}{(l_\beta + l_3)^{3/2}} \sqrt{\frac{l_1 l_2 + l_1 l_3 + l_2 l_3}{l_\beta + l_3} |p - p_i|^2 - \frac{z}{2}} + O(|p-p_i|^2) + O(|z|)$$

holds for all $|p - p_i| \to 0$, $i = \overline{1,8}$, *and* $z \to -0$.

Proof. Let us sketch the main idea of the proof. Take a sufficiently small $\delta > 0$ such that $U_\delta(p_i) \cap U_\delta(p_j) = \emptyset$ for all $i \neq j$, $i, j = \overline{1,8}$.

Using the additivity of the integral we rewrite the function $\Delta_{\mu_\alpha^0}(\cdot\,;\cdot)$ as

$$\Delta_{\mu_\alpha^0}(p\,; z) = 1 - \mu_\alpha^0 \sum_{j=1}^{8} \int\limits_{U_\delta(p_j)} \frac{ds}{w_\alpha(p,s) - z} - \mu_\alpha^0 \int\limits_{\mathbb{T}_\delta} \frac{ds}{w_\alpha(p,s) - z}. \tag{3.7}$$

Since the function $w_\alpha(\cdot,\cdot)$ has a non-degenerate minimum at the points (p_i, p_j), $i, j = \overline{1,8}$, analysis similar to that in the proof of Lemma 3.5 in [2] shows that

$$\begin{aligned} \int\limits_{U_\delta(p_j)} \frac{ds}{w_\alpha(p,s) - z} = & \int\limits_{U_\delta(p_j)} \frac{ds}{w_\alpha(p_i,s)} \\ & - \frac{\pi^2}{(l_\beta + l_3)^{3/2}} \sqrt{\frac{l_1 l_2 + l_1 l_3 + l_2 l_3}{l_\beta + l_3} |p - p_i|^2 - \frac{z}{2}} \\ & + O(|p-p_i|^2) + O(|z|), \\ \int\limits_{\mathbb{T}_\delta} \frac{ds}{w_\alpha(p,s) - z} = & \int\limits_{\mathbb{T}_\delta} \frac{ds}{w_\alpha(p_i,s)} + O(|p-p_i|^2) + O(|z|) \end{aligned}$$

as $|p - p_i| \to 0$ for $i = \overline{1,8}$ and $z \to -0$. Substituting the last two expressions in to equality (3.7) we obtain

$$\begin{aligned} \Delta_{\mu_\alpha^0}(p\,; z) = \Delta_{\mu_\alpha^0}(p_i\,; 0) + & \frac{8\pi^2 \mu_\alpha^0}{(l_\beta + l_3)^{3/2}} \sqrt{\frac{l_1 l_2 + l_1 l_3 + l_2 l_3}{l_\beta + l_3} |p - p_i|^2 - \frac{z}{2}} \\ & + O(|p-p_i|^2) + O(|z|) \end{aligned}$$

as $|p-p_i| \to 0$ for $i=\overline{1,8}$ and $z \to -0$. Now the equality $\Delta_{\mu_\alpha^0}(p_i\,;0)=0$ completes the proof of Lemma 3.6. □

Corollary 3.7. *For some $C_1, C_2, C_3 > 0$ and $\delta > 0$ the following inequalities hold:*

(i) $C_1|p-p_i| \le \Delta_{\mu_\alpha^0}(p\,;0) \le C_2|p-p_i|$, $p \in U_\delta(p_i)$, $i=\overline{1,8}$;
(ii) $\Delta_{\mu_\alpha^0}(p\,;0) \ge C_3$, $p \in \mathbb{T}_\delta$.

Proof. The Lemma 3.6 yields assertion (i) for some positive numbers C_1, C_2. The positivity and continuity of the function $\Delta_{\mu_\alpha^0}(\cdot\,;0)$ on the compact set $\mathbb{T}_\delta$ imply the assertion (ii). □

Lemma 3.8. *The operator $h_{\mu_\alpha^0}(p_1)$ has no negative eigenvalues.*

Proof. Since the function $\Delta_{\mu_\alpha}(p_1\,;\cdot)$ is decreasing on $(-\infty;0)$, the definition of μ_α^0 implies

$$\Delta_{\mu_\alpha^0}(p_1\,;z) > \Delta_{\mu_\alpha^0}(p_1\,;0) = 0$$

for all $z<0$. By part (iii) of Corollary 3.2, it means that the operator $h_{\mu_\alpha^0}(p_1)$ has no negative eigenvalues. □

4. LOCATION AND STRUCTURE OF THE ESSENTIAL SPECTRUM OF $\mathbf{H}$

In this section we give only the main ideas of the proof of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Set

$$\Sigma := \bigcup_{p\in\mathbb{T}^3} \sigma_{\mathrm{disc}}(\mathbf{h}_{\mu_1}(p)) \cup \bigcup_{p\in\mathbb{T}^3} \sigma_{\mathrm{disc}}(h_{\mu_2}(p)) \cup [0;M].$$

The inclusion $\Sigma \subset \sigma_{\mathrm{ess}}(\mathbf{H})$ is established with the use of the well-known Weyl criterion [22].

For the proof of $\sigma_{\mathrm{ess}}(\mathbf{H}) \subset \Sigma$, for each $z \in \mathbb{C}\setminus[0;M]$, we define the 3×3 block operator matrices $\mathbf{A}(z)$ and $\mathbf{K}(z)$ acting in the Hilbert space $L_2^{(3)}(\mathbb{T}^3)$ as

$$\mathbf{A}(z) := \left(A_{ij}(z)\right)_{i,j=1}^3, \quad \mathbf{K}(z) := \left(K_{ij}(z)\right)_{i,j=1}^3,$$

where the operator $A_{ij}(z)$ is the multiplication operator by the function $\Delta_{ij}(\cdot\,;z)$:

$$\Delta_{11}(p\,;z) := \Delta(p\,;z), \quad \Delta_{21}(p\,;z) := \int_{\mathbb{T}^3} \frac{v(s)ds}{w(p,s)-z}, \quad \Delta_{12}(p\,;z) := -\mu_1\Delta_{21}(p\,;z),$$

$$\Delta_{ii}(p\,;z) := \Delta_{\mu_{i-1}}(p\,;z), \quad i=2,3,$$

$$\Delta_{ij}(p\,;z) := 0, \quad \text{otherwise},$$

and the operator $K_{ij}(z)$ is the integral operator with the kernel $K_{ij}(\cdot,\cdot;z)$:

$$\begin{aligned}
&K_{13}(p,s;z) := \frac{\mu_2 v(s)}{w(p,s)-z}, \quad K_{23}(p,s;z) := \frac{\mu_2}{w(p,s)-z},\\
&K_{31}(p,s;z) := -\frac{v(p)}{w(s,p)-z},, \quad K_{32}(p,s;z) := \frac{\mu_1}{w(s,p)-z},\\
&K_{ij}(p,s;z) := 0, \quad \text{otherwise}
\end{aligned}$$

(s is the integration variable). We note that for each $z \in \mathbb{C} \setminus [0;M]$, all entries of $\mathbf{K}(z)$ belong to the Hilbert-Schmidt class and therefore, $K(z)$ is a compact.

Using the similar arguments of [19, 25] one can prove that for each $z \in \mathbb{C} \setminus \Sigma$, the operator $\mathbf{A}(z)$ is boundedly-invertible and the number $z \in \mathbb{C} \setminus \Sigma$ is an eigenvalue of the operator $\mathbf{H}$ if and only if the number $\lambda = 1$ is an eigenvalue of the operator $\mathbf{A}^{-1}(z)\mathbf{K}(z)$. Moreover, the eigenvalues z and 1 have the same multiplicities. Then analytic Fredholm theorem (see, e.g. Theorem VI.14 in [22]) proves inclusion $\sigma_{\text{ess}}(\mathbf{H}) \subset \Sigma$.

Since the function $\Delta_{\mu_2}(p\,;\cdot)$ is a monotonically decreasing on $\mathbb{R} \setminus [m_2(p); M_2(p)]$ and $((h_{\mu_2}(p)-z)f,f) < 0$ for all $z > M_2(p)$ and $f \in L_2(\mathbb{T}^3)$, the operator $h_{\mu_2}(p)$ has no more than one eigenvalue. In [21] it was shown that for any $p \in \mathbb{T}^3$ the operator $\mathbf{h}_{\mu_1}(p)$ has no more than three eigenvalues lying outside of its essential spectrum. Then the theorem on the spectrum of decomposable operators [22] and the definition of Σ imply that the set Σ consists of no more than five bounded closed intervals. □

Proof of Theorem 2.2. First we recall that if $\mu_\alpha = \mu_\alpha^0$, then by Theorem 2.1 taking into account equality (2.3) we have

$$\sigma_{\text{ess}}(\mathbf{H}) = \bigcup_{p \in \mathbb{T}^3} \sigma_{\text{disc}}(\mathbf{h}(p)) \cup [0;M]. \tag{4.1}$$

Let $\min_{p \in \mathbb{T}^3} \Delta(p\,;0) \geq 0$. Then $\Delta(p\,;0) \geq 0$ for any $p \in \mathbb{T}^3$ and hence, by part (ii) of Corollary 3.2, for any $p \in \mathbb{T}^3$ the operator $\mathbf{h}(p)$ has no negative eigenvalues, that is,

$$\bigcup_{p \in \mathbb{T}^3} \sigma_{\text{disc}}(\mathbf{h}(p)) \cap (-\infty;0) = \emptyset.$$

Assume $\min_{p \in \mathbb{T}^3} \Delta(p\,;0) < 0$ and $\max_{p \in \mathbb{T}^3} \Delta(p\,;0) \geq 0$. Then there exist points $p', p'' \in \mathbb{T}^3$ such that

$$\min_{p \in \mathbb{T}^3} \Delta(p\,;0) = \Delta(p'\,;0) \quad \text{and} \quad \max_{p \in \mathbb{T}^3} \Delta(p\,;0) = \Delta(p''\,;0) \geq 0.$$

We introduce the following subset of $\mathbb{T}^3$:

$$G := \{p \in \mathbb{T}^3 : \Delta(p\,;0) < 0\}.$$

Then it is obvious that G is a non-empty open set and $G \neq \mathbb{T}^3$.

For any $p \in \mathbb{T}^3$ the function $\Delta(p\,;\cdot)$ is continuous and decreasing on $(-\infty;0]$, and the equality $\lim_{z\to-\infty}\Delta(p\,;z)=+\infty$ holds. Then for any $p\in G$ there exists a unique point $E(p)\in(-\infty;0)$ such that $\Delta(p\,;E(p))=0$. By part (ii) of Corollary 3.2, for any $p\in G$ the point $E(p)$ is the unique negative eigenvalue of the operator $\mathbf{h}(p)$. For any $p\in\mathbb{T}^3\setminus G$ and $z<0$ we have $\Delta(p\,;z)>\Delta(p\,;0)\geq 0$. Hence, by part (ii) of Corollary 3.2, for each $p\in\mathbb{T}^3\setminus G$ the operator $\mathbf{h}(p)$ has no negative eigenvalues.

By assumption the function $u(\cdot)$ is continuous, $v(\cdot)$ and $w(\cdot,\cdot)$ are analytic on its domains, hence the function $E: p\in G\to E(p)$ is continuous on G.

Since for any $p\in\mathbb{T}^3$ the operator $\mathbf{h}(p)$ is bounded and $\mathbb{T}^3$ is a compact set, there exists a positive number C such that $\sup\limits_{p\in\mathbb{T}^3}\|\mathbf{h}(p)\|\leq C$ and for any $p\in\mathbb{T}^3$ we have

$$\sigma(\mathbf{h}(p))\subset[-C;C]. \tag{4.2}$$

For any $q\in\partial G=\{p\in\mathbb{T}^3:\Delta(p\,;0)=0\}$ there exist $\{q_n\}\subset G$ such that $q_n\to q$ as $n\to\infty$. If we set $E^{(n)}:=E(q_n)$, then for any $n\in\mathbb{N}$ the inequality $E^{(n)}<0$ holds and from (4.2) we get $\{E^{(n)}\}\subset[-C;0)$. Without loss of generality (otherwise we would have to take a subsequence) we assume that $E^{(n)}\to E^{(0)}$ as $n\to\infty$ for some $E^{(0)}\in[-C;0]$.

From the continuity of the function $\Delta(\cdot\,;\cdot)$ in $\mathbb{T}^3\times(-\infty;0]$ and $q_n\to q$ and $E^{(n)}\to E^{(0)}$ as $n\to\infty$ it follows that

$$0=\lim_{n\to\infty}\Delta(q_n\,;E^{(n)})=\Delta(q\,;E^{(0)}).$$

Since for any $p\in\mathbb{T}^3$ the function $\Delta(p\,;\cdot)$ is decreasing in $(-\infty;0]$ and $q\in\partial G$ we see that $\Delta(q\,;E^{(0)})=0$ if and only if $E^{(0)}=0$.

Now for $q\in\partial G$ we define

$$E(q)=\lim_{q'\to q,\,q'\in G}E(q')=0.$$

Since the function $E(\cdot)$ is continuous on the compact set $G\cup\partial G$ and $E(q)=0$ for all $q\in\partial G$ we conclude that $\operatorname{Ran}(E)=[a;0]$ and $a<0$, where $\operatorname{Ran}(E)$ denotes an image of the function $E(\cdot)$.

Hence the set

$$\bigcup_{p\in\mathbb{T}^3}\sigma_{\mathrm{disc}}(\mathbf{h}(p))\cap(-\infty;0]$$

coincides with the set $\operatorname{Ran}(E)=[a;0]$. Then the equality (4.1) completes the proof of assertion (ii) of Theorem 2.2.

If $\max_{p\in\mathbb{T}^3}\Delta(p\,;0)<0$, then $G=\mathbb{T}^3$ and the above analysis leads $\operatorname{Ran}(E)=[a;b]$ with $b<0$. Theorem 2.2 is completely proved. $\square$

5. ASYMPTOTICS FOR THE NUMBER OF NEGATIVE EIGENVALUES OF H_{22}

In this section first we review the corresponding Birman-Schwinger principle for the operator H_{22} and then we derive the asymptotic relation (2.4) for the number of negative eigenvalues of H_{22}.

5.1. THE BIRMAN-SCHWINGER PRINCIPLE

For a bounded self-adjoint operator A acting in the Hilbert space $\mathcal{R}$, we define [9] the number $n(\gamma, A)$ as follows

$$n(\gamma, A) := \sup\{\dim F : (Au, u) > \gamma,\ u \in F \subset \mathcal{R},\ \|u\| = 1\}.$$

The number $n(\gamma, A)$ is equal to infinity if $\gamma < \max \sigma_{\text{ess}}(A)$; if $n(\gamma, A)$ is finite, then it is equal to the number of eigenvalues of A bigger than γ.

By the definition of $N(z)$, we have $N(z) = n(-z, -H_{22})$, $-z > -\tau_{\text{ess}}(H_{22})$.

Since the function $\Delta_{\mu_\alpha}(\cdot\,;\cdot)$ is positive on $\mathbb{T}^3 \times (-\infty; \tau_{\text{ess}}(H_{22}))$ for any $\mu_\alpha > 0$, the positive square root of $\Delta_{\mu_\alpha}(p\,; z)$ exists for any $\mu_\alpha > 0$, $p \in \mathbb{T}^3$ and $z < \tau_{\text{ess}}(H_{22})$.

In our analysis of the discrete spectrum of H_{22} the crucial role is played by the 2×2 block operator matrix $\mathbf{T}(z)$, $z < \tau_{\text{ess}}(H_{22})$ acting on $L_2^{(2)}(\mathbb{T}^3)$ with the entries

$$T_{\alpha\alpha}(z) = 0, \quad (T_{\alpha\beta}(z)\varphi_\beta)(p) = \frac{\sqrt{\mu_1\mu_2}}{\sqrt{\Delta_{\mu_\alpha}(p\,; z)}} \int_{\mathbb{T}^3} \frac{\varphi_\beta(s)ds}{\sqrt{\Delta_{\mu_\beta}(s\,; z)}(w_\alpha(p, s) - z)}.$$

The following lemma is a realization of the well-known Birman-Schwinger principle for the operator H_{22} (see [2, 3, 23]).

Lemma 5.1. *For any $z < \tau_{\text{ess}}(H_{22})$ the operator $\mathbf{T}(z)$ is compact and continuous in z and*

$$N(z) = n(1, \mathbf{T}(z)).$$

5.2. PROOF OF THEOREM 2.3

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$ and $\sigma = L_2(\mathbb{S}^2)$. As we shall see, the discrete spectrum asymptotics of the operator $\mathbf{T}(z)$ as $z \to -0$ is determined by the integral operator $\mathbf{S}_{\mathbf{r}}$, $\mathbf{r} = 1/2|\log|z||$ in $L_2((0, \mathbf{r}), \sigma^{(2)})$ with the kernel $S_{\alpha\beta}(y, t)$, $y = x - x'$, $x, x' \in (0, \mathbf{r})$, $t = \langle \xi, \eta \rangle$, $\xi, \eta \in \mathbb{S}^2$, where

$$S_{\alpha\alpha}(y, t) = 0; \quad S_{\alpha\beta}(y, t) = \frac{1}{4\pi^2} \frac{u_{\alpha\beta}}{\cosh(y + r_{\alpha\beta}) + s_{\alpha\beta} t},$$

$$u_{\alpha\beta} = u_{\beta\alpha} = \left(\frac{(l_1 + l_3)(l_2 + l_3)}{l_1 l_2 + l_1 l_3 + l_2 l_3} \right)^{1/2}, \quad r_{\alpha\beta} = \frac{1}{2} \log \frac{l_\alpha + l_3}{l_\beta + l_3},$$

$$s_{\alpha\beta} = s_{\beta\alpha} = \frac{l_3}{(l_1 + l_3)(l_2 + l_3)}, \quad \alpha, \beta = 1, 2.$$

The eigenvalue asymptotics for the operator $\mathbf{S_r}$ have been studied in detail by Sobolev [23], by employing an argument used in the calculation of the canonical distribution of Toeplitz operators.

Let us recall some results of [23] which are important in our work.

The coefficient in the asymptotics (2.4) of $N(z)$ will be expressed by means of the self-adjoint integral operator $\widehat{\mathbf{S}}(\theta)$, $\theta \in \mathbb{R}$, in the space $\sigma^{(2)}$, whose kernel is of the form

$$\widehat{S}_{\alpha\alpha}(\theta,t)=0, \quad \widehat{S}_{\alpha\beta}(\theta,t)=\frac{1}{4\pi^2}u_{\alpha\beta}e^{ir_{\alpha\beta\theta}}\frac{\sinh[\theta\arccos s_{\alpha\beta}t]}{\sqrt{1-s_{\alpha\beta}^2 t}\sinh(\pi\theta)},$$

and depends on the inner product $t=\langle\xi,\eta\rangle$ of the arguments $\xi,\eta\in\mathbb{S}^2$. For $\gamma>0$, define

$$U(\gamma):=\frac{1}{4\pi}\int\limits_{-\infty}^{+\infty} n(\gamma,\widehat{\mathbf{S}}(\theta))d\theta.$$

This function was studied in detail in [23]; it is used in the existence proof for the Efimov effect. In particular, as was proved in [23], the function $U(\cdot)$ is continuous in $\gamma>0$, and the limit

$$\lim_{\mathbf{r}\to 0}\frac{1}{2}\mathbf{r}^{-1}n(\gamma,\mathbf{S_r})=U(\gamma) \tag{5.1}$$

exists and the number $U(1)$ is positive.

Theorem 2.3 can be derived by using a perturbation argument based on the following lemma (see Lemma 4.9 in [23]). For completeness, we reproduce the proof given there.

Lemma 5.2. *Let $A(z)=A_0(z)+A_1(z)$, where $A_0(z)$ ($A_1(z)$) is compact and continuous in the strong operator topology for $z<0$ (for $z\le 0$). Assume that the limit $\lim_{z\to -0} f(z)\,n(\gamma,A_0(z))=U(\gamma)$ exists and $U(\cdot)$ is continuous in $(0;+\infty)$ for some function $f(\cdot)$, where $f(z)\to 0$ as $z\to 0$. Then the same limit exists for $A(z)$ and $\lim_{z\to -0} f(z)\,n(\gamma,A(z))=U(\gamma)$.*

Proof. Using the Weyl inequality

$$n(\gamma_1+\gamma_2,K_1+K_2)\le n(\gamma_1,K_1)+n(\gamma_2,K_2)$$

for the sum of compact operators K_1 and K_2 and for any positive numbers γ_1 and γ_2, for $\theta\in(0;1)$, we have

$$n(\gamma,A(z))\le n((1-\theta)\gamma,A_0(z))+n(\theta\gamma,A_1(z))$$

and

$$n(\gamma,A(z))\ge n((1+\theta)\gamma,A_0(z))-n(\theta\gamma,A_1(z)).$$

Since the operator $A_1(z)$ is compact and continuous in the strong operator topology in $z\le 0$, we obtain

$$U((1+\theta)\gamma)\le \lim_{z\to -0}\inf f(z)n(\gamma,A(z))\le \lim_{z\to -0}\sup f(z)n(\gamma,A(z))\le U((1-\theta)\gamma).$$

Therefore, the continuity of the function $U(\gamma)$ for $\gamma > 0$ completes the proof of Lemma 5.2. $\square$

Remark 5.3. Since the function $U(\cdot)$ is continuous with respect to γ, it follows from Lemma 5.2 that any perturbation of $A_0(z)$ treated in Lemma 5.2 (which is compact and continuous in the strong operator topology up to $z=0$) does not contribute to the asymptotic relation (2.4).

Now we are going to reduce the study of the asymptotics for the operator $\mathbf{T}(z)$ to that of the asymptotics $\mathbf{S_r}$.

Let $\mathbf{T}(\delta;|z|)$ be the 2×2 block operator matrix in $L_2^{(2)}(\mathbb{T}^3)$ whose entries are integral operators with the kernel $T_{\alpha\beta}(\delta,|z|;\cdot,\cdot)$:

$$\begin{aligned}
&T_{\alpha\alpha}(\delta,|z|;p,q)=0,\\
&T_{\alpha\beta}(\delta,|z|;p,q)\\
&=d_0\sum_{i,j=1}^{8}\frac{\chi_\delta(p-p_i)\chi_\delta(q-p_j)(m_\alpha|p-p_i|^2+\frac{|z|}{2})^{-\frac14}(m_\beta|q-p_j|^2+\frac{|z|}{2})^{-\frac14}}{(l_\alpha+l_3)|p-p_i|^2+2l_3(p-p_i,q-p_j)+(l_\beta+l_3)|q-p_j|^2+\frac{|z|}{2}},
\end{aligned}$$

where

$$d_0:=\frac{(l_1+l_3)^{3/4}(l_2+l_3)^{3/4}}{16\pi^2},\quad m_\alpha:=\frac{l_1l_2+l_1l_3+l_2l_3}{l_\beta+l_3}$$

and $\chi_\delta(\cdot)$ is the characteristic function of the domain $U_\delta(\mathbf{0})$, $\mathbf{0}=(0,0,0)\in\mathbb{T}^3$.

The operator $\mathbf{T}(\delta;|z|)$ is called a singular part of $\mathbf{T}(z)$.

Lemma 5.4. *Let $\mu=\mu_\alpha^0$. For any $z\le 0$ and small $\delta>0$ the difference $\mathbf{T}(z)-\mathbf{T}(\delta;|z|)$ belongs to the Hilbert-Schmidt class and is continuous in the strong operator topology with respect to $z\le 0$.*

Proof. First we recall that the expansion

$$\begin{aligned}
w(p,q)&=2((l_1+l_3)|p-p_i|^2+2l_3(p-p_i,q-p_j)+(l_2+l_3)|q-p_j|^2)\\
&\quad+O(|p-p_i|^4)+O(|q-p_j|^4)
\end{aligned}$$

as $|p-p_i|$, $|q-p_j|\to 0$, for $i,j=\overline{1,8}$ implies that there exist $C_1,C_2>0$ and $\delta>0$ such that

$$\begin{aligned}
&C_1(|p-p_i|^2+|q-p_j|^2)\le w(p,q)\le C_2(|p-p_i|^2+|q-p_j|^2),\\
&(p,q)\in U_\delta(p_i)\times U_\delta(p_j)\quad\text{for}\quad i,j=\overline{1,8},\\
&w(p,q)\ge C_1,\quad (p,q)\in\mathbb{T}^2_\delta.
\end{aligned}$$

Applying last estimates and Corollary 3.7 we obtain that there exist $C_1, C_2 > 0$ such that the kernel of the operator $T_{\alpha\beta}(z) - T_{\alpha\beta}(\delta; |z|)$ can be estimated by the square-integrable function $Q(\cdot,\cdot)$ defined on $(\mathbb{T}^3)^2$ as

$$Q(p,q) = C_1 + \frac{|p-p_i|^{-\frac{1}{2}} + |q-p_j|^{-\frac{1}{2}} + 1}{|p-p_i|^2 + (p-p_i, q-p_j) + |q-p_j|^2},$$
$$(p,q) \in U_\delta(p_i) \times U_\delta(p_j), i,j = \overline{1,8},$$
$$Q(p,q) = C_1, \quad (p,q) \notin \bigcup_{i=1}^{8} U_\delta(p_i) \times \bigcup_{j=1}^{8} U_\delta(p_j).$$

Hence, the operator $T_{\alpha\beta}(z) - T_{\alpha\beta}(\delta; |z|)$ belongs to the Hilbert-Schmidt class for all $z \le 0$. In combination with the continuity of the kernel of the operator with respect to $z < 0$, this implies the continuity of $T_{\alpha\beta}(z) - T_{\alpha\beta}(\delta; |z|)$ in the strong operator topology with respect to $z \le 0$. The lemma is proved. □

The following theorem is fundamental for the proof of the asymptotic relation (2.4).

Theorem 5.5. *We have the relation*

$$\lim_{|z|\to 0} n(\gamma, \mathbf{T}(\delta; |z|)) |\log |z||^{-1} = U(\gamma), \quad \gamma > 0. \tag{5.2}$$

Proof. From the definition of the kernel function $T_{\alpha\beta}(\delta, |z|; \cdot, \cdot)$ it follows that the subspace of vector functions $\psi = (\psi_1, \psi_2)$ with components supported by the set $\bigcup_{i=1}^{8} U_\delta(p_i)$ is invariant with respect to the operator $\mathbf{T}(\delta; |z|)$.

Let $\mathbf{T}_0(\delta; |z|)$ be the restriction of the operator $\mathbf{T}(\delta; |z|)$ to the subspace $L_2^{(2)}(\bigcup_{i=1}^{8} U_\delta(p_i))$, that is, 2×2 block operator matrix in $L_2^{(2)}(\bigcup_{i=1}^{8} U_\delta(p_i))$ whose entries $T_{\alpha\beta}^{(0)}(\delta; |z|)$ are integral operators with the kernel $T_{\alpha\beta}^{(0)}(\delta; |z|; \cdot, \cdot)$, where $T_{\alpha\alpha}^{(0)}(\delta; |z|; p, q) = 0$ and the function $T_{\alpha\beta}^{(0)}(\delta; |z|; \cdot, \cdot)$ is defined on $\bigcup_{i=1}^{8} U_\delta(p_i) \times \bigcup_{j=1}^{8} U_\delta(p_j)$ as

$$T_{\alpha\beta}^{(0)}(\delta; |z|; p, q) = \frac{d_0\, (m_\alpha |p-p_i|^2 + |z|/2)^{-\frac{1}{4}} (m_\beta |q-p_j|^2 + |z|/2)^{-\frac{1}{4}}}{(l_\alpha + l_3)|p-p_i|^2 + 2l_3(p-p_i, q-p_j) + (l_\beta + l_3)|q-p_j|^2 + |z|/2},$$
$$(p,q) \in U_\delta(p_i) \times U_\delta(p_j) \quad \text{for} \quad i,j = \overline{1,8}.$$

Since

$$L_2(\bigcup_{i=1}^{8} U_\delta(p_i)) \cong \bigoplus_{i=1}^{8} L_2(U_\delta(p_i)),$$

we can express the integral operator $T_{\alpha\beta}^{(0)}(\delta; |z|)$ as the following block operator matrix $\mathbf{T}_{\alpha\beta}^{(0)}(\delta; |z|)$ acting on $\bigoplus_{i=1}^{8} L_2(U_\delta(p_i))$ as

$$\mathbf{T}_{\alpha\beta}^{(0)}(\delta; |z|) := \begin{pmatrix} T_{\alpha\beta}^{(1,1)}(\delta; |z|) & \ldots & T_{\alpha\beta}^{(1,8)}(\delta; |z|) \\ \vdots & \ddots & \vdots \\ T_{\alpha\beta}^{(8,1)}(\delta; |z|) & \ldots & T_{\alpha\beta}^{(8,8)}(\delta; |z|) \end{pmatrix},$$

where $T^{(i,j)}_{\alpha\beta}(\delta;|z|) : L_2(U_\delta(p_j)) \to L_2(U_\delta(p_i))$ is an integral operator with the kernel $T^{(0)}_{\alpha\beta}(\delta;|z|;p,q)$, $(p,q) \in U_\delta(p_i) \times U_\delta(p_j)$ for $i,j=\overline{1,8}$.

Let us introduce the operator $\mathbf{T}_1(r)$, $r=|z|^{-\frac{1}{2}}$, acting on $L_2^{(8)}(U_r(\mathbf{0}))\oplus L_2^{(8)}(U_r(\mathbf{0}))$ as

$$\mathbf{T}_1(r) := \begin{pmatrix} 0 & \mathbf{T}^{(1)}_{12}(r) \\ \mathbf{T}^{(1)}_{21}(r) & 0 \end{pmatrix}$$

with the entries $\mathbf{T}^{(1)}_{\alpha\beta}(r) : L_2^{(8)}(U_r(\mathbf{0})) \to L_2^{(8)}(U_r(\mathbf{0}))$ (8×8 block operator matrix):

$$\mathbf{T}^{(1)}_{\alpha\beta}(r) := \begin{pmatrix} T^{(1)}_{\alpha\beta}(r) & \dots & T^{(1)}_{\alpha\beta}(r) \\ \vdots & \ddots & \vdots \\ T^{(1)}_{\alpha\beta}(r) & \dots & T^{(1)}_{\alpha\beta}(r) \end{pmatrix},$$

where $T^{(1)}_{\alpha\beta}(r)$ is the integral operator on $L_2(U_r(\mathbf{0}))$ with the kernel

$$\frac{d_0\,(m_\alpha|p|^2+1/(2\delta^2))^{-\frac{1}{4}}(m_\beta|q|^2+1/(2\delta^2))^{-\frac{1}{4}}}{(l_\alpha+l_3)|p|^2+2l_3(p,q)+(l_\beta+l_3)|q|^2+1/(2\delta^2)}.$$

Now we consider the following unitary dilation (16×16 diagonal matrix)

$$\mathbf{B}_r := \mathrm{diag}\{B_r^{(1)},\dots,B_r^{(8)},B_r^{(1)},\dots,B_r^{(8)}\} : \bigoplus_{i=1}^{16} L_2(U_\delta(p_i)) \to L_2^{(16)}(U_r(\mathbf{0})),$$

Here the operator $B_r^{(i)} : L_2(U_\delta(p_i)) \to L_2(U_r(\mathbf{0}))$, $i=\overline{1,8}$ acts as

$$(B_r^{(i)}f)(p) = \Big(\frac{r}{\delta}\Big)^{-\frac{3}{2}} f\Big(\frac{\delta}{r}p+p_i\Big).$$

Then for $i=\overline{1,8}$ we have

$$(B_r^{(i)})^{-1} : L_2(U_r(\mathbf{0})) \to L_2(U_\delta(p_i)), \quad ((B_r^{(i)})^{-1}f)(p) = \Big(\frac{r}{\delta}\Big)^{\frac{3}{2}} f\Big(\frac{r}{\delta}(p-p_i)\Big).$$

Using the definitions of the operators $T^{(1)}_{\alpha\beta}(r)$, $T^{(i,j)}_{\alpha\beta}(\delta;|z|)$ and $B_r^{(i)}$ for $i,j=\overline{1,8}$ we obtain

$$\begin{aligned}
&(B_r^{(i)}T^{(i,j)}_{\alpha\beta}(\delta;|z|)(B_r^{(j)})^{-1}f)(p)\\
&= B_r^{(i)}\left(\int\limits_{U_\delta(p_j)} \frac{d_0\,(m_\alpha|p-p_i|^2+|z|/2)^{-\frac{1}{4}}(m_\beta|q-p_j|^2+|z|/2)^{-\frac{1}{4}} f(\frac{r}{\delta}(q-p_j))dq}{(l_\alpha+l_3)|p-p_i|^2+2l_3(p-p_i,q-p_j)+(l_\beta+l_3)|q-p_j|^2+|z|/2}\right)\\
&= \int\limits_{U_r(\mathbf{0})} \frac{d_0\,(m_\alpha|p|^2+1/(2\delta^2))^{-\frac{1}{4}}(m_\beta|q|^2+1/(2\delta^2))^{-\frac{1}{4}} f(q)dq}{(l_\alpha+l_3)|p|^2+2l_3(p,q)+(l_\beta+l_3)|q|^2+1/(2\delta^2)}\\
&= (T^{(1)}_{\alpha\beta}(r)f)(p),\ \ f\in L_2(U_r(\mathbf{0})).
\end{aligned}$$

Therefore, $\mathbf{T}_1(r) = \mathbf{B}_r \mathbf{T}_0(\delta; |z|) \mathbf{B}_r^{-1}$.

Let us introduce the 2×2 block operator matrices

$$\mathbf{A}_r, \mathbf{E} : L_2^{(16)}(U_r(\mathbf{0})) \to L_2^{(16)}(U_r(\mathbf{0}))$$

of the form

$$\mathbf{A}_r := \begin{pmatrix} 0 & \mathbf{A}_{12}(r) \\ \mathbf{A}_{21}(r) & 0 \end{pmatrix}, \quad \mathbf{E} := \mathrm{diag}\{\mathbf{I}, \mathbf{I}\},$$

where $\mathbf{A}_{\alpha\beta}(r)$ and $\mathbf{I}$ are the 8×1 and 1×8 matrices of the form

$$\mathbf{A}_{\alpha\beta}(r) := \begin{pmatrix} T_{\alpha\beta}^{(1)}(r) \\ \vdots \\ T_{\alpha\beta}^{(1)}(r) \end{pmatrix}, \quad \mathbf{I} := (I, \ldots, I),$$

respectively, here I is the identity operator on $L_2(U_r(\mathbf{0}))$.

It is well known that if B_1, B_2 are bounded operators and $\gamma \neq 0$ is an eigenvalue of $B_1 B_2$, then γ is an eigenvalue for $B_2 B_1$ as well for the same algebraic and geometric multiplicities (see, e.g. [10]). Therefore, $n(\gamma, \mathbf{A}_r \mathbf{E}) = n(\gamma, \mathbf{E}\mathbf{A}_r)$, $\gamma > 0$. Direct calculation shows that $\mathbf{T}_1(r) = \mathbf{A}_r \mathbf{E}$ and

$$\mathbf{E}\mathbf{A}_r : L_2^{(2)}(U_r(\mathbf{0})) \to L_2^{(2)}(U_r(\mathbf{0})), \quad \mathbf{E}\mathbf{A}_r = \begin{pmatrix} 0 & 8T_{12}^{(1)}(r) \\ 8T_{21}^{(1)}(r) & 0 \end{pmatrix}.$$

So, $n(\gamma, \mathbf{T}_1(r)) = n(\gamma, \mathbf{E}\mathbf{A}_r)$, $\gamma > 0$.

Further, we can replace

$$(m_\alpha |p|^2 + 1/(2\delta^2))^{\frac{1}{4}},\ (m_\beta |q|^2 + 1/(2\delta^2))^{\frac{1}{4}},\ (l_\alpha + l_3)|p|^2 + 2l_3(p,q) + (l_\beta + l_3)|q|^2 + 1/(2\delta^2)$$

by the expressions

$$(m_\alpha |p|^2)^{\frac{1}{4}}(1 - \chi_1(p))^{-1},\ (m_\beta |q|^2)^{\frac{1}{4}}(1 - \chi_1(q))^{-1},\ (l_\alpha + l_3)|p|^2 + 2l_3(p,q) + (l_\beta + l_3)|q|^2,$$

respectively, because the corresponding error is a Hilbert-Schmidt operator and continuous in the strong operator topology up to $z = 0$. In this case, we obtain the 2×2 block operator matrix $\mathbf{T}_2(r)$ on $L_2^{(2)}(U_r(\mathbf{0}) \setminus U_1(\mathbf{0}))$ whose entries $T_{\alpha\beta}^{(2)}(r)$ are integral operators with the kernel $T_{\alpha\beta}^{(2)}(r; \cdot, \cdot)$:

$$T_{\alpha\alpha}^{(2)}(r; p, q) = 0, \quad T_{\alpha\beta}^{(2)}(r; p, q) = \frac{8d_0}{(m_1 m_2)^{1/4}} \frac{|p|^{-1/2}|q|^{-1/2}}{(l_\alpha + l_3)|p|^2 + 2l_3(p,q) + (l_\beta + l_3)|q|^2}.$$

Using the dilation

$$\mathbf{M} := \mathrm{diag}\{M, M\} : L_2^{(2)}(U_r(\mathbf{0}) \setminus U_1(\mathbf{0})) \to L_2((0, \mathbf{r}), \sigma^{(2)}),$$
$$(Mf)(x, w) = e^{3x/2} f(e^x w),$$

where $\mathbf{r} = \frac{1}{2}|\log|z||$, $x \in (0, \mathbf{r})$, $w \in \mathbb{S}^2$, one can see that the operator $T_2(r)$ is unitarily equivalent to the integral operator $\mathbf{S_r}$.

Since the difference of the operators $\mathbf{S_r}$ and $\mathbf{T}(\delta;|z|)$ is compact (up to unitary equivalence) and $\mathbf{r} = 1/2|\log|z||$, we obtain the equality

$$\lim_{|z|\to 0} n(\gamma, \mathbf{T}(\delta;|z|))|\log|z||^{-1} = \lim_{\mathbf{r}\to 0} \frac{1}{2}\mathbf{r}^{-1} n(\gamma, \mathbf{S_r}), \quad \gamma > 0.$$

Now Lemma 5.2 and the equality (5.1) completes the proof of Theorem 5.5. □

Proof of Theorem 2.3. Let $\mu = \mu_\alpha^0$, $\alpha = 1, 2$. Using Lemmas 5.2, 5.4 and Theorem 5.5 we have

$$\lim_{|z|\to 0} n(1, \mathbf{T}(z))|\log|z||^{-1} = U(1).$$

Taking into account the last equality and Lemma 5.1, and setting $\mathcal{U}_0 = U(1)$ we complete the proof of Theorem 2.3. □

6. THE LOCATION OF EIGENVALUES OF $\mathbf{H}$

In this section we shall prove Theorem 2.4.

Proof of Theorem 2.4. Let $\mu = \mu_\alpha^0$, $\alpha = 1, 2$. By Theorem 2.3 the operator H_{22} has infinitely many negative eigenvalues $E_1, \ldots, E_n, \ldots$, accumulating at zero. Let $f_2^{(1)}, \ldots, f_2^{(n)}, \ldots$ be the corresponding eigenfunctions.

Denote by $\mathcal{L}_0$ the subspace of all eigenfunctions of H_{22}, corresponding to the negative eigenvalues. We show that $H_{12}|_{\mathcal{L}_0} = 0$. Let f_2 be the eigenfunction of H_{22} corresponding to the eigenvalue $z < 0$, that is, $H_{22} f_2 = z f_2$ or

$$f_2(p, q) = \frac{\mu_1 \varphi_1(p) + \mu_2 \varphi_2(q)}{w(p, q) - z}, \tag{6.1}$$

where

$$\varphi_1(p) := \int_{\mathbb{T}^3} f_2(p, s) ds, \quad \varphi_2(q) := \int_{\mathbb{T}^3} f_2(s, q) ds. \tag{6.2}$$

Substituting the expression (6.1) for f_2 into the equalities (6.2), we obtain

$$\varphi_1(p) = \int_{\mathbb{T}^3} \frac{\mu_1 \varphi_1(p) + \mu_2 \varphi_2(s)}{w(p, s) - z} ds, \quad \varphi_2(q) = \int_{\mathbb{T}^3} \frac{\mu_1 \varphi_1(s) + \mu_2 \varphi_2(q)}{w(s, p) - z} ds,$$

or

$$\varphi_1(p) = \frac{\mu_2}{\Delta_{\mu_1}(p\,; z)} \int_{\mathbb{T}^3} \frac{\varphi_2(s) ds}{w(p, s) - z}, \quad \varphi_2(q) = \frac{\mu_1}{\Delta_{\mu_2}(q\,; z)} \int_{\mathbb{T}^3} \frac{\varphi_1(s) ds}{w(s, q) - z}.$$

This implies that $\varphi_\alpha(\cdot)$, $\alpha = 1, 2$ are periodic functions of each variable with period π. Therefore, the function $f_2(\cdot, \cdot)$, defined by (6.1) is a periodic function of each six variables with period π. By condition (2.2), we obtain $H_{12} f_2 = 0$ for any $f_2 \in \mathcal{L}_0$.

In particular, from here it follows that $H_{12}f_2^{(n)}=0$ for any $n\in\mathbb{N}$. Therefore, the numbers $E_1,\dots,E_n,\dots$ are eigenvalues of $\mathbf{H}$ and the corresponding eigenvectors have the form: $f^{(n)}=(0,f_2^{(n)})$, $n\in\mathbb{N}$.

If $\min_{p\in\mathbb{T}^3}\Delta(p\,;0)\geq 0$, then by Theorem 2.2 we have $\min\sigma_{\rm ess}(\mathbf{H})=0$. In this case the set $\{E_n:n\in\mathbb{N}\}$ is located in below the bottom of the essential spectrum of $\mathbf{H}$ and $\lim_{n\to\infty}E_n=0$. Let $\min_{p\in\mathbb{T}^3}\Delta(p\,;0)<0$ and $\max_{p\in\mathbb{T}^3}\Delta(p\,;0)\geq 0$. Then Theorem 2.2 implies that $\sigma_{\rm ess}(\mathbf{H})\cap(-\infty;M]=[a;M]$ with $a<0$. Hence, the countable (infinite) part of the set $\{E_n:n\in\mathbb{N}\}$ is located in the essential spectrum of $\mathbf{H}$. If $\max_{p\in\mathbb{T}^3}\Delta(p\,;0)<0$, then $\sigma_{\rm ess}(\mathbf{H})\cap(-\infty;M]=[a;b]\cup[0;M]$, $b<0$. It means that the countable (infinite) part of the set $\{E_n:n\in\mathbb{N}\}$ located in $(b;0)$. Theorem 2.4 is proved. □

7. EXAMPLE

We prove one more assertion.

Lemma 7.1. *Let $v_0(\cdot)$ be any continuous function satisfying condition* (2.2) *and*

$$w_1(p):=\varepsilon(p)+1,\quad v(p):=\sqrt{\lambda}v_0(p),\quad \lambda>0.$$

Set

$$\lambda_0:=\left(\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{\varepsilon(s)}\right)^{-1},\quad \lambda_1:=7\left(\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{\varepsilon(p)+\varepsilon(s)}\right)^{-1}.$$

Then the following assertions hold:

(i) *if* $\lambda\in(0;\lambda_0]$, *then* $\min\limits_{p\in\mathbb{T}^3}\Delta(p\,;0)\geq 0$,

(ii) *if* $\lambda\in(\lambda_0;\lambda_1]$, *then* $\min\limits_{p\in\mathbb{T}^3}\Delta(p\,;0)<0$ *and* $\max\limits_{p\in\mathbb{T}^3}\Delta(p\,;0)\geq 0$,

(iii) *if* $\lambda\in(\lambda_1;\infty)$, *then* $\max\limits_{p\in\mathbb{T}^3}\Delta(p\,;0)<0$.

Proof. First we recall that for any $p\in\mathbb{T}^3$ the relations

$$\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{\varepsilon(s)}-\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{\varepsilon(p)+\varepsilon(s)}=\varepsilon(p)\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{(\varepsilon(p)+\varepsilon(s))\varepsilon(s)}\geq 0,$$

$$\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{6+\varepsilon(s)}-\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{\varepsilon(p)+\varepsilon(s)}=(\varepsilon(p)-6)\int\limits_{\mathbb{T}^3}\frac{v_0(s)^2ds}{(\varepsilon(p)+\varepsilon(s))(6+\varepsilon(s))}\leq 0$$

hold. Therefore,

$$\min_{p\in\mathbb{T}^3}\Delta(p\,;0)=1-\lambda\lambda_0^{-1}\quad\text{and}\quad\max_{p\in\mathbb{T}^3}\Delta(p\,;0)=7-7\lambda\lambda_1^{-1}.$$

From here directly follows assertions (i)–(iii) of the Lemma 7.1. □

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