dr n. mat. Paweł IGNACZAK[a], dr n. tech. Andrzej Antoni CZAJKOWSKI[b], dr n. tech. Grzegorz Paweł SKORNY[b], mgr Roman UDAŁA[c]

[a] West Pomeranian University of Technology, Faculty of Economics, Department of Applied Mathematics in Economics
Zachodniopomorski Uniwersytet Technologiczny, Wydział Ekonomiczny, Katedra Zastosowań Matematyki w Ekonomii

[b] Higher School of Technology and Economics in Szczecin, Faculty of Motor Transport
Wyższa Szkoła Techniczno-Ekonomiczna w Szczecinie, Wydział Transportu Samochodowego

[c] Edward Murrow High School, Brooklyn, New York, USA / Szkoła im. Edwarda Murrow, Brooklyn, NowyYork, USA

LEGENDRE POLYNOMIALS APPLICATION FOR EXPANDING FUNCTIONS IN THE SERIES BY THESE POLYNOMIALS

Abstract

Introduction and aim: Selected elementary material about Legendre polynomials have been shown in the paper. The algorithm of expanding functions in the series by Legendre polynomials has been elaborated in the paper.
Material and methods: The selected knowledge about Legendre polynomials have been taken from the right literature. The analytical method has been used in this paper.
Results: Has been shown the theorem describing expanding functions in a series by using Legendre polynomials. It have been shown selected examples of expanding functions in a series by applying Legendre polynomials.
Conclusion: The function f(z) can be expand in the interval $\langle -1,1 \rangle$ in a series according to Legendre polynomials where the unknown coefficients can be determined using the method of undetermined coefficients.

Keywords: Legendre polynomials, function, expanding in a series.
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ZASTOSOWANIE WIELOMIANÓW LEGENDRE'A DO ROZWIJANIA FUNKCJI W SZEREGI WEDŁUG TYCH WIELOMIANÓW

Streszczenie

Wstęp i cel: *W pracy pokazuje się wybrane podstawowe wiadomości o wielomianach Legendre'a. W artykule opracowano algorytm rozwijania funkcji w szereg według wielomianów Legendre'a.*
Materiał i metody: *Wybrane wiadomości o wielomianach Legendre'a zaczerpnięto z literatury przedmiotu. W pracy zastosowano metodę analityczną.*
Wyniki: *W pracy pokazano twierdzenie dotyczące rozwijania funkcji w szereg według wielomianów Legendre'a. Pokazano wybrane przykłady rozwijania funkcji w szereg według wielomianów Legendre'a*
Wniosek: *Funkcja f(z) może być w przedziale $\langle -1,1 \rangle$ rozwinięta w szereg według wielomianów Legendre'a, gdzie nieznane współczynniki można wyznaczyć stosując metodę współczynników nieoznaczonych.*

Słowa kluczowe: *Wielomiany Legendre'a, funkcja, rozwijanie w szereg.*
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1. Introduction to expanding functions in the series by using Legendre polynomials

In many problems it is necessary to expand the function f=f (z) in the interval $\langle -1,1\rangle$ in a series according to Legendre polynomials:

$$f(z) = \sum_{n=0}^{\infty} c_n P_n(z). \quad (1)$$

The coefficients c_n can be determined from the orthogonality of Legenre polynomials. We multiply both sides of equation (1) by $P_m(z)$ [2]-[4]:

$$f(z)P_m(z) = \sum_{n=0}^{\infty} c_n P_n(z)P_m(z). \quad (2)$$

The equation (2) we integrate both sides in the interval $\langle -1,1\rangle$ respect to variable z. Then we have:

$$\int_{-1}^{+1} f(z)P_m(z)\,dz = \int_{-1}^{+1} \sum_{n=0}^{\infty} c_n P_n(z)P_m(z)\,dz. \quad (3)$$

Therefore

$$\int_{-1}^{+1} f(z)P_m(z)\,dz = \sum_{n=0}^{\infty} c_n \int_{-1}^{+1} f(z)P_m(z)\,dz. \quad (4)$$

We use in further consideration of a recursive relationship between Legendre polynomials of the following form [1]:

$$(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0 \quad (5)$$

for n = 1, 2, 3, … .

Now we calculate the value of the integral on the right side of the equation (4) for m=n. So in the formula (4) we replace n by n-1. We therefore:

$$nP_n(z) - (2n-1)zP_{n-1}(z) + (n-1)P_{n-2}(z) = 0 \quad (6)$$

for n = 2, 3, … .

The equation (6) we multiply both sides by the expression $(2n+1)P_n(z)$. Then we have:

$$n(2n+1)P_n^2(z) - (4n^2-1)zP_{n-1}(z)P_n(z) + (n-1)(2n+1)P_{n-2}(z)P_n(z) = 0 \quad (7)$$

for n = 2, 3, … .

The equation (5) we multiply both sides by the expression $(2n-1)P_{n-1}(z)$. Then we have:

$$(n+1)(2n-1)P_{n-1}(z)P_{n+1}(z) - (4n^2-1)zP_{n-1}(z)P_n(z) + n(2n-1)P_{n-1}^2(z) = 0 \quad (8)$$

for n = 1, 2, 3, … .

From equation (7) we subtract the equation (8) by sides. Then we have:

$$\begin{aligned} &n(2n+1)P_n^2(z) - (4n^2-1)zP_{n-1}(z)P_n(z) + (n-1)(2n+1)P_{n-2}(z)P_n(z) + \\ &-(n+1)(2n-1)P_{n-1}(z)P_{n+1}(z) + (4n^2-1)zP_{n-1}(z)P_n(z) - n(2n-1)P_{n-1}^2(z) = 0. \end{aligned} \quad (9)$$

After some simplifications, we have:

$$n(2n+1)P_n^2(z)+(2n^2-n-1)P_{n-2}(z)P_n(z)-(2n^2+n-1)P_{n-1}(z)P_{n+1}(z)+ \\ -n(2n-1)P_{n-1}^2(z)=0 \tag{10}$$

for n = 2, 3, … .

Equation (10) we integrate both sides in the interval $\langle -1,1\rangle$ respect to variable z. So we have the following equality:

$$n(2n+1)\int_{-1}^{+1}P_n^2(z)\,dz+(2n^2-n-1)\int_{-1}^{+1}P_{n-2}(z)P_n(z)\,dz+ \\ -(2n^2+n-1)\int_{-1}^{+1}P_{n-1}(z)P_{n+1}(z)\,dz-n(2n-1)\int_{-1}^{+1}P_{n-1}^2(z)\,dz=\int_{-1}^{+1}0\,dz \tag{11}$$

for n = 2, 3, … .

With orthogonal Legendre polynomials we have that the second and third integral in the equation (11) are equal to zero. Therefore, after appropriate transformations we get:

$$\int_{-1}^{+1}P_n^2(z)\,dz=\frac{2n-1}{2n+1}\int_{-1}^{+1}P_{n-1}^2(z)\,dz \tag{12}$$

for n = 2, 3, … .

By saving formula (12) successively for n = 2, 3, 4, ... we get the following sequence:

$$\int_{-1}^{+1}P_2^2(z)\,dz=\frac{3}{5}\int_{-1}^{+1}P_1^2(z)\,dz\,, \tag{13}$$

$$\int_{-1}^{+1}P_3^2(z)\,dz=\frac{5}{7}\int_{-1}^{+1}P_2^2(z)\,dz=\frac{5}{7}\cdot\frac{3}{5}\int_{-1}^{+1}P_1^2(z)\,dz=\frac{3}{7}\int_{-1}^{+1}P_1^2(z)\,dz=\frac{3}{2\cdot 3+1}\int_{-1}^{+1}P_1^2(z)\,dz\,, \tag{14}$$

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$$\int_{-1}^{+1}P_n^2(z)\,dz=\frac{3}{2n+1}\int_{-1}^{+1}P_1^2(z)\,dz\,. \tag{15}$$

Because

$$P_1(z)=z\,, \tag{16}$$

then

$$\int_{-1}^{+1}P_1^2(z)\,dz=\int_{-1}^{+1}z^2\,dz=\left.\frac{z^3}{3}\right|_{-1}^{+1}=\frac{1}{3}-\left(-\frac{1}{3}\right)=\frac{2}{3}. \tag{17}$$

Therefore:

$$\int_{-1}^{+1}P_n^2(z)\,dz=\frac{3}{2n+1}\cdot\frac{2}{3}=\frac{2}{2n+1}. \tag{18}$$

Thus, the equality (4) has for m = n the following form:

$$\int_{-1}^{+1} f(z) P_n(z)\, dz = \sum_{n=0}^{\infty} c_n \frac{2}{2n+1}. \tag{19}$$

From which it follows that

$$c_n = (n + \tfrac{1}{2}) \int_{-1}^{+1} f(z) P_n(z)\, dz. \tag{20}$$

Theorem 1.

Let f=f (z) be an arbitrary function of the variable z defined in the interval $\langle -1,1 \rangle$ and satisfying the conditions:

- f(z) in the intervals is a smooth function (i.e., has a continuous derivative in the intervals) in the interval $\langle -1,1 \rangle$,
- the integral $\int_{-1}^{+1} f^2(z)\, dz$ has a finite value,

then the series

$$f(z) = \sum_{n=0}^{\infty} c_n P_n(z) \quad \text{for} \quad -1 < z < +1 \tag{21}$$

where

$$c_n = (n + \tfrac{1}{2}) \int_{-1}^{+1} f(z) P_n(z)\, dz \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{22}$$

is convergent and its sum is the function f(z) at each point of the inner interval $\langle -1,1 \rangle$ which is a point of continuity of this function [1].

Therefore by virtue of theorem 1 the series of the form (1) is convergent and its sum is a function f (z).

2. Examples of expanding functions in the series by using Legendre polynomials

<u>Example 1</u>.

Let us consider the following function:

$$f(z) = \begin{cases} 0 & \text{if} \quad -1 \le z < a, \\ 1 & \text{if} \quad a \le z < 1. \end{cases} \tag{23}$$

According to the theorem 1, the function (23) can be expand in following series:

$$f(z) = \sum_{n=0}^{\infty} c_n P_n(z), \tag{24}$$

where the coefficients c_n are as follows:

$$c_n = (n+\tfrac{1}{2})\int_a^1 P_n(z)\,dz\,. \tag{25}$$

For the calculation of the coefficients c_n we use the fact, that $P_n(1) = 1$ and the equation:

$$\frac{dP_{n+1}}{dz} - \frac{dP_{n-1}}{dz} = (2n+1)P_n(z)\,. \tag{26}$$

Therefore

$$P_n(z) = \frac{1}{2n+1}\left(\frac{dP_{n+1}}{dz} - \frac{dP_{n-1}}{dz}\right). \tag{27}$$

Then

$$c_n = \frac{n+\frac{1}{2}}{2n+1}\int_a^1 \left(\frac{dP_{n+1}}{dz} - \frac{dP_{n-1}}{dz}\right)dz\,. \tag{28}$$

Which means further that

$$c_n = \frac{n+\frac{1}{2}}{2(n+\frac{1}{2})}\left(\int_a^1 \frac{dP_{n+1}}{dz}dz - \int_a^1 \frac{dP_{n-1}}{dz}dz\right), \tag{29}$$

and hence

$$c_n = \frac{1}{2}[P_{n+1}(1) - P_{n+1}(a) - P_{n-1}(1) + P_{n-1}(a)]\,. \tag{30}$$

Finally, we get

$$c_n = \frac{1}{2}[P_{n+1}(a) - P_{n-1}(a)]\,. \tag{31}$$

for n = 1, 2, 3, … .

Other hand, when n = 0, we have:

$$c_0 = (0+\tfrac{1}{2})\int_a^1 P_0(z)\,dz, \tag{32}$$

where $P_0(z) = 0$.

Hence finally we get:

$$c_0 = \tfrac{1}{2}(1-a)\,. \tag{33}$$

So the studied function (23) has the following expansion in a series:

$$f(z) = \frac{1}{2}(1-a) - \frac{1}{2}\sum_{n=1}^{\infty}\left[P_{n+1}(a) - P_{n-1}(a)\right]\cdot P_n(z) \tag{34}$$

where $-1 < z < +1$.

Example 2.

Let the function f(z) be a polynomial of n-th degree of the form:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_{m-1} z^{m-1} + a_m z^m = \sum_{n=0}^{m} a_n z^n . \tag{35}$$

Expansion (1) takes the form:

$$f(z) = \sum_{n=0}^{m} c_n P_n(z) . \tag{36}$$

The coefficients c_n can be find from the system of linear equations, which we get by substituting to formula (36) the expressions for Legendre polynomials and equating coefficients of the same powers in the left and right side of the equality.

Let us consider the following function of the 3rd degree:

$$f(z) = z^3 \tag{37}$$

and find its expansion in a series of Legendre polynomials.

Therefore:

$$z^3 = c_0 P_0(z) + c_1 P_1(z) + c_2 P_2(z) + c_3 P_3(z) \equiv c_0 + c_1 z + c_2 \frac{3z^2 - 1}{2} + c_3 \frac{5z^3 - 3z}{2} . \tag{38}$$

After a suitable arrangement of expressions the equality (38) has the form:

$$0 + 0 \cdot z + 0 \cdot z^2 + 1 \cdot z^3 \equiv (c_0 - \tfrac{1}{2} c_2) + (c_1 - \tfrac{3}{2} c_3) z + \tfrac{3}{2} c_2 z^2 + \tfrac{5}{2} c_3 z^3 . \tag{39}$$

After aligning the appropriate coefficients we obtain the following system of equations:

$$\begin{cases} c_0 - \frac{1}{2} c_2 = 0 \\ c_1 - \frac{3}{2} c_3 = 0 \\ \frac{3}{2} c_2 = 0 \\ \frac{5}{2} c_3 = 1. \end{cases} \tag{40}$$

After solving the system of equations (40) relative to c_0, c_1, c_2 and c_3 we obtain:

$$\begin{cases} c_0 = 0 \\ c_1 = \frac{3}{5} \\ c_2 = 0 \\ c_3 = \frac{2}{5}. \end{cases} \tag{41}$$

Thus, equality (38) takes the form:

$$z^3 = \frac{3}{5} z + \frac{2}{5} \frac{5z^3 - 3z}{2} . \tag{42}$$

Considering that

$$P_1(z) = z\,, \tag{43}$$

$$P_3(z) = \frac{5z^3 - 3z}{2} \tag{44}$$

function (37) has the following expansion in a series according to Legendre polynomials:

$$z^3 = \tfrac{3}{5}P_1(z) + \tfrac{2}{5}P_3(z). \tag{45}$$

<u>Example 3</u>. Let us consider the following function of the 4th degree:

$$f(z) = z^4 - 1 \tag{46}$$

and find its expansion in a series of Legendre polynomials. Therefore:

$$z^4 - 1 \equiv c_0 P_0(z) + c_1 P_1(z) + c_2 P_2(z) + c_3 P_3(z) + c_4 P_4(z). \tag{47}$$

Substituting the appropriate expressions for the Legendre polynomials $P_0(z)$, $P_1(z)$, $P_2(z)$ and $P_4(z)$ equality (46) takes the following form:

$$z^4 - 1 \equiv c_0 + c_1 z + c_2\frac{3z^2-1}{2} + c_3\frac{5z^3-3z}{2} + c_4\frac{35z^4-30z^2+3}{8}. \tag{48}$$

After a suitable arrangement of expressions the equality (48) has the form:

$$\begin{aligned} &-1 + 0\cdot z + 0\cdot z^2 + 0\cdot z^3 + 1\cdot z^4 \equiv \\ &\equiv (c_0 - \tfrac{1}{2}c_2 + \tfrac{3}{8}c_4) + (c_1 - \tfrac{3}{2}c_3)z + (\tfrac{3}{2}c_2 - \tfrac{30}{8}c_4)z^2 + \tfrac{5}{2}c_3 z^3 + \tfrac{35}{8}c_4 z^4. \end{aligned} \tag{49}$$

After aligning the appropriate coefficients we obtain the following system of equations:

$$\begin{cases} c_0 - \frac{1}{2}c_2 + \frac{3}{8}c_4 = -1 \\ c_1 - \frac{3}{2}c_3 = 0 \\ \frac{3}{2}c_2 - \frac{30}{8}c_4 = 0 \\ \frac{5}{2}c_3 = 0 \\ \frac{35}{8}c_4 = 1. \end{cases} \tag{50}$$

After solving the system of equations (50) relative to c_0, c_1, c_2, c_3 and c_4 we obtain:

$$\begin{cases} c_0 = -\frac{4}{5} \\ c_1 = 0 \\ c_2 = \frac{4}{7} \\ c_3 = 0 \\ c_4 = \frac{8}{35}. \end{cases} \tag{51}$$

Thus, equality (48) takes the form:

$$z^4 - 1 \equiv -\tfrac{4}{5} + \tfrac{4}{7}\frac{3z^2 - 1}{2} + \tfrac{8}{35}\frac{35z^4 - 30z^2 + 3}{8}. \tag{52}$$

Considering that $P_0(z) = 1$, and

$$P_2(z) = \frac{3z^2 - 1}{2}, \tag{53}$$

$$P_4(z) = \frac{35z^4 - 30z^2 + 3}{8} \tag{54}$$

function (46) has the following expansion in a series according to Legendre polynomials:

$$z^4 - 1 = -\frac{4}{5}P_0(z) + \frac{4}{7}P_2(z) + \frac{8}{35}P_4(z). \tag{55}$$

5. Conclusion

- The function f(z) can be expand in the interval $\langle -1,1 \rangle$ in a series according to Legendre polynomials, i.e. $f(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ where the unknown coefficients c_n can be determined by using the method of undetermined coefficients.

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