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SOME PRACTICAL APPLICATIONS OF GENERATING FUNCTIONS AND LSTS

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Abstract

In many various practical problems we often deal with computing distribution functions of sums of independent non-negative random variables. In applied mathematics (ex. queueing theory) we can find many formulas with Stieltjes convolutions of distribution functions of random variables of the same type. Finding convolutions on the base of definition is not easy and convenient, because there are some technical problems connected with computations. There are some interesting ways to obtain such distribution functions applying other methods. In this paper we present methods connected with applications of generating functions and Laplace-Stieltjes transforms.

1. Introduction

Assume that ξ_1, ξ_2 are two independent non-negative random variables. Distribution functions of these random variables will be denoted by $L_1(x)$ and $L_2(x)$, respectively.

For the random variable $\xi = \xi_1 + \xi_2$ we easily obtain formula for its distribution function $L(x) = P\{\xi < x\}$.

$$L(x) = P\{\xi < x\} = P\{\xi_1 + \xi_2 < x\} =$$
$$= \int_0^x P\{\xi_2 < x - u|\xi_1 \in [u, u + du)\}P\{\xi_1 \in [u, u + du)\} = \tag{1}$$
$$= \int_0^x P\{\xi_2 < x - u\}P\{\xi_1 \in [u, u + du)\} = \int_0^x L_2(x - u)dL_1(u).$$

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In the case of discrete independent random variables taking only integer values with distributions

$$p_k = P\{\xi_1 = k\}, r_k = P\{\xi_2 = k\}, k = 0, 1, \dots, \sum_k p_k = \sum_k r_k = 1$$

we can obtain the distribution of random variable $\xi = \xi_1 + \xi_2$ as follows:

$$q_k = P\{\xi = k\} = P\{\xi_1 + \xi_2 = k\} = \sum_{i=0}^{k} P\{\xi_1 = i, \xi_2 = k - i\} = \tag{2}$$

$$= \sum_{i=0}^{k} P\{\xi_1 = i\} P\{\xi_2 = k - i\}.$$

Then we can finally calculate the distribution function of random variable ξ applying the formula

$$L(x) = \sum_{k<x} q_k.$$

For the arbitrary number of independent random variables we can generalize (1), (2) by induction but final formulas may not be convenient.

For example, if we consider two independent random variables having exponential distribution with the parameter a ($a > 0$) i.e. $L_1(x) = L_2(x) = 1 - e^{-ax}$ for $x > 0$, applying integration by parts, we easily obtain

$$L(x) = \int_0^x \left(1 - e^{-a(x-u)}\right) ae^{-au} du =$$

$$= \int_0^x ae^{-au} du - \int_0^x ae^{-ax} du = 1 - (1 + ax)e^{-ax}.$$

So in this case we obtain 2-Erlang distribution with the parameter a.

In the case of many independent random variables computation becomes very complicated because we have to use parts integration repeatedly.

Consider now two independent random variables ξ_1, ξ_2 which have geometric distribution with the parameter p ($p \in (0, 1)$) i.e. $p_k = r_k = (1 - p)p^k$, $k = 0, 1, \dots$. Then we obtain the following result:

$$q_k = P\{\xi_1 + \xi_2 = k\} = \sum_{i=0}^{k} P\{\xi_1 = i\} P\{\xi_2 = k - i\} =$$

$$= \sum_{i=0}^{k} (1 - p)p^i (1 - p)p^{k-i} = \sum_{i=0}^{k} (1 - p)^2 p^k = (k + 1)(1 - p)^2 p^k.$$

In the case of n independent random variables ($n \geq 3$) computation also becomes more difficult because of complicated sums appearing.

Formulas with Stieltjes convolutions which can be calculated applying (1), (2) appear in applied mathematics very often. For example, they are used in theory of queueing systems with non-homogeneous customers ([3], [4], [5]).

2. Laplace-Stieltjes Transformation (LST) and its Useful Properties

Let ξ denote a non-negative random variable and $L(x)$ its distribution function. For every complex q that has non-negative real part (re $q \geq 0$) we can define the following function ([1], [4]):

$$\alpha(q) = Ee^{-q\xi} = \int_0^\infty e^{-qx} dL(x). \tag{3}$$

The function given by (3) is called Laplace-Stieltjes transformation (LST) of random variable ξ.

Now we present two very interesting properties of LST that can be used in obtaining distribution functions of sums of independent random variables.

Property 1. *Let $\xi_1, \xi_2, \ldots, \xi_n$ denote the sequence of independent non-negative random variables and $\alpha_1(q), \alpha_2(q), \ldots, \alpha_n(q)$ – a sequence of LST of these random variables respectively and $\xi = \xi_1 + \xi_2 + \ldots + \xi_n$ – the sum of random variables $\xi_1, \xi_2, \ldots, \xi_n$. Let $\alpha(q)$ denote the LST of random variable ξ. Then we obtain the following formula:*

$$\alpha(q) = \prod_{i=1}^{n} \alpha_i(q). \tag{4}$$

Proof. Applying the definition of LST, in view of properties of mean value of independent random variables product, we obtain

$$\alpha(q) = Ee^{-q\xi} = Ee^{-q\sum_{i=1}^{n}\xi_i} = E\prod_{i=1}^{n} e^{-q\xi_i} = \prod_{i=1}^{n} Ee^{-q\xi_i} = \prod_{i=1}^{n} \alpha_i(q).$$

□

Property 2. *Assume that ξ is a non-negative random variable and denote as $L(x)$ and $\alpha(q)$ distribution function and LST of random variable ξ consequently. Then we have the following formula:*

$$\alpha(q) = \int_0^\infty e^{-qx} dL(x) = q \int_0^\infty e^{-qx} L(x) dx. \tag{5}$$

Notice that the integral on the right side of (5) is the well known Laplace transformation of the function $L(x)$.

Proof. Calculating the integral from the left side of (5) by parts integration and the basic properties of Stieltjes integral we obtain

$$\alpha(q) = \int_0^\infty e^{-qx} dL(x) = e^{-qx} L(x)\Big|_0^\infty - \int_0^\infty L(x) d(e^{-qx}) =$$

$$= q \int_0^\infty e^{-qx} L(x) dx.$$

Applying the two above properties we can obtain the distribution functions of sums of independent non-negative random variables. First we have to calculate LST $\alpha_i(q)$ for every random variable and in view of (4) we obtain LST

$$\alpha(q) = \prod_{i=1}^n \alpha_i(q)$$

of sum of all variables. Secondly, applying (5), we obtain Laplace transformation of the sum $l(q) = \frac{\alpha(q)}{q}$. Finally, we can use Laplace transform inversion to find distribution function of the sum. In the last step we can use residuum method or Laplace transformation tables or computer algebra systems (*Mathematica* environment). This method is very useful especially in the case of absolutely continuous random variables. □

3. Examples of calculating distribution functions of sums of independent random variables using LST

Let us consider n independent random variables having exponential distribution with parameters a_i $(i = \overline{1, n})$. Applying (4), (5) we obtain

$$l(q) = \frac{\prod_{i=1}^n a_i}{q \prod_{i=1}^n (a_i + q)}. \tag{6}$$

Applying computer algebra systems and inverse Laplace transformation, we obtain distribution function of the sum of above random variables in the form

$$L(x) = 1 + \sum_{i=1}^{n} \frac{\prod_{j\neq i} a_j}{\prod_{i\neq j}(a_i - a_j)} e^{-a_i x}. \tag{7}$$

In the special case of n independent random variables having exponential distribution with the same parameter a we have $\alpha_i(q) = \frac{a}{a+q}$, $i = \overline{1,n}$. In view of (4) and (5) we obtain formula for the Laplace transformation of the sum of these variables

$$l(q) = \frac{a^n}{q(a+q)^n}. \tag{8}$$

Applying residuum method we can obtain the distribution function in the form

$$L(x) = 1 - e^{-ax} \sum_{i=0}^{n-1} \frac{(ax)^i}{i!}. \tag{9}$$

So in this case we obtain n-Erlang distribution with the parameter a.

Assume now that we have n independent random variables having uniform distribution on the interval $[a, b]$ $(0 \le a < b)$ i.e. for every $i = \overline{1,n}$ $L_i(x) = \frac{x-a}{b-a}$ for every $x \in [a, b]$ and $L_i(x) = 0$ if $x < a$ and $L_i(x) = 1$ if $x > b$. Then we have $\alpha_i(q) = \frac{e^{-aq}-e^{-bq}}{q(b-a)}$, $i = \overline{1,n}$. Applying (4) and (5) we obtain

$$l(q) = \frac{(e^{-aq} - e^{-bq})^n}{q^{n+1}(b-a)^n}. \tag{10}$$

Using computer algebra systems we can obtain the distribution function in the form

$$L(x) = \left(\frac{-1}{b-a}\right)^n \sum_{l=0}^{n} \frac{(-1)^l((b-a)l - bn + x)^n H((b-a)l - bn + x)}{l!(n-l)!}, \tag{11}$$

where $H(x)$ is the Heaviside unitstep function.

4. Generating Function (GF) and its Useful Properties

Let us consider a non-negative random variable ξ taking only integer values. Denote as p_k probability that ξ is equal to k i.e. $p_k = P\{\xi = k\}$, $\sum_k p_k = 1$. Then for every complex z that satisfies condition $|z| \le 1$ we can define the following function ([2], [4]):

$$P(z) = Ez^{\xi} = \sum_{k=0}^{\infty} p_k z^k. \tag{12}$$

Because $P(z)$ is analytic we assume that $P(0) = p_0$. The function given by (12) is called the generating function (GF) of random variable ξ.

Now we present two very interesting properties of GF which can be used in obtaining distribution functions of sums of independent random variables taking only integer values.

Property 3. *Let $\xi_1, \xi_2, \ldots, \xi_n$ denote the sequence of independent random variables taking only integer value and $P_1(z), P_2(z), \ldots, P_n(z)$ – a sequence of GF of random variables $\xi_1, \xi_2, \ldots, \xi_n$, respectively.*

If $\xi = \xi_1 + \xi_2 + \ldots + \xi_n$ denotes the sum of these random variables and $P(z)$ is GF of the random variable ξ, then we have the following formula:

$$P(z) = \prod_{i=1}^{n} P_i(z). \tag{13}$$

Proof. Applying the definition of GF, in view of properties of mean value of independent random variables product, we obtain

$$P(z) = Ez^{\sum_{i=1}^{n} \xi_i} = E\prod_{i=1}^{n} z^{\xi_i} = \prod_{i=1}^{n} Ez^{\xi_i} = \prod_{i=1}^{n} P_i(z).$$

□

Property 4. *If we have the GF $P(z)$ of random variable ξ then we can recover distribution $p_k = P\{\xi = k\}$ applying the formula*

$$p_k = \frac{P^{(k)}(0)}{k!}. \tag{14}$$

Proof. Applying the definition of GF we obtain

$$P^{(k)}(z) = \sum_{i=k}^{\infty} i(i-1)(i-2)\ldots(i-k+1)p_i z^{i-k} = k!\sum_{i=k}^{\infty} \binom{i}{k} p_i z^{i-k}. \tag{15}$$

From (15) we easily obtain

$$P^{(k)}(0) = k!p_k,$$

which confirms (14). □

Applying these properties we can obtain the distributions p_k of sums of independent random variables which take only integer values. First we have to calculate GF $P_i(z)$ for every random variable. Then applying (13) and (14) we can obtain distribution of the sum of these variables.

5. Examples of Calculating Distributions of Sums of Independent Random Variables applying GF

Let us consider two independent random variables ξ_1, ξ_2 that are both defined by the following table:

x_i	1	2
p_i	$\frac{1}{3}$	$\frac{2}{3}$

Generating functions of both variables have the form

$$P_1(z) = P_2(z) = \frac{1}{3}z + \frac{2}{3}z^2.$$

Then GF of the sum $\xi_1 + \xi_2$ has the form

$$P(z) = P_1(z)P_2(z) = \left(\frac{1}{3}z + \frac{2}{3}z^2\right)^2 = \frac{1}{9}z^2 + \frac{4}{9}z^3 + \frac{4}{9}z^4. \tag{16}$$

Applying (14) we can find p_k probabilities that are presented in the following table:

x_i	2	3	4
p_i	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{4}{9}$

Similar computations can be leaded for the arbitrary number of independent random variables that are defined in the finite probability tables.

Assume now that we have n independent random variables $\xi_1, \ldots, \xi_n$ having Poisson distribution with parameter μ i.e. $p_k = \frac{\mu^k}{k!}e^{-\mu}$. GF of each random variable has the form: $P_i(z) = e^{-\mu(1-z)}$, $i = \overline{1, n}$. Then the GF of the sum $\xi_1 + \ldots + \xi_n$ has the form

$$P(z) = e^{-n\mu(1-z)}. \tag{17}$$

In view of (17) we obtain

$$P^{(k)}(z) = (n\mu)^k e^{-n\mu(1-z)}.$$

and

$$P^{(k)}(0) = (n\mu)^k e^{-n\mu}.$$

Applying (14) we finally obtain

$$p_k = \frac{(n\mu)^k}{k!} e^{-n\mu}. \tag{18}$$

It follows from (18) that the sum $\xi_1 + \ldots + \xi_n$ has the Poisson distribution with parameter $n\mu$. If $\xi_1, \ldots, \xi_n$ have Poisson distribution with parameter μ_i, $i = \overline{1, n}$, making analogous computation, we can easily obtain that the sum $\xi_1 + \ldots + \xi_n$ have Poisson distribution with the parameter $\mu = \sum_{i=1}^{n} \mu_i$.

Consider now n independent random variables having geometric distribution with parameter p i.e. $p_k = (1-p)p^k$. Then the generating function of each variable has the form: $P_i(z) = \frac{1-p}{1-pz}$, $i = \overline{1, n}$. Then the sum of these random variables has the following GF:

$$P(z) = \frac{(1-p)^n}{(1-pz)^n}. \tag{19}$$

From (19) we easily obtain

$$P^{(k)}(z) = n(n+1)(n+2)\ldots(n+k-1)\frac{p^k(1-p)^n}{(1-pz)^{n+k}}, \tag{20}$$

and in view of (14) we finally obtain

$$p_k = \binom{n+k-1}{k} p^k (1-p)^n. \tag{21}$$

Analogous computations, using GF or LST, can be proceeded also for random variables which have different distribution functions.

For example, let us assume that we have two independent random variables ξ_1, ξ_2. First variable has Poisson distribution with parameter μ and second variable has geometric distribution with parameter p.

Then in view of (13) GF of the sum $\xi_1 + \xi_2$ has the form

$$P(z) = \frac{e^{-\mu(1-z)}(1-p)}{1-pz}. \tag{22}$$

From (22), using *Mathematica* environment, we obtain

$$P^{(k)}(z) = k!(1-p)e^{-\mu(1-z)} \sum_{i=0}^{k} \frac{\mu^i p^{k-i}}{i!(1-pz)^{k+1-i}}, \tag{23}$$

and applying (14) we finally obtain

$$p_k = (1-p)e^{-\mu} \sum_{i=0}^{k} \frac{\mu^i p^{k-i}}{i!} = (1-p)p^k e^{-\mu} \sum_{i=0}^{k} \frac{\left(\frac{\mu}{p}\right)^i}{i!}. \tag{24}$$

6. Case of n Independent Random Vectors

Assume that $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ are two independent non-negative random vectors. Let $L_1(x, y)$ and $L_2(x, y)$ denote the distribution functions of these vectors, respectively. For the random vector

$$\zeta = (\zeta_1, \zeta_2) = \xi + \eta = (\xi_1 + \eta_1, \xi_2 + \eta_2)$$

we obtain the following formula for its distribution function

$$L(x, y) = P\{\zeta_1 < x, \zeta_2 < y\}$$

$$L(x, y) = P\{\zeta_1 < x, \zeta_2 < y\} = P\{\xi_1 + \eta_1 < x, \xi_2 + \eta_2 < y\} =$$
$$= \int_0^x \int_0^y P\{\xi_1 < x - u, \xi_2 < y - v \big| \eta_1 \in [u, u + du), \eta_2 \in [v, v + dv)\} \times$$
$$\times P\{\eta_1 \in [u, u+du), \eta_2 \in [v, v+dv)\} = \int_0^x \int_0^y L_1(x-u, y-v) dL_2(u, v). \tag{25}$$

In the case of two independent random vectors taking only integer values with distributions $p_{ij} = P\{\xi_1 = i, \xi_2 = j\}$, $r_{ij} = P\{\eta_1 = i, \eta_2 = j\}$ we obtain the following formula:

$$q_{ij} = P\{\xi_1 + \eta_1 = i, \xi_2 + \eta_2 = j\} =$$
$$= \sum_{k=0}^{i} \sum_{l=0}^{j} P\{\xi_1 = k, \eta_1 = i - k, \xi_2 = l, \eta_2 = j - l\} =$$
$$= \sum_{k=0}^{i} \sum_{l=0}^{j} P\{\xi_1 = k, \xi_2 = l\} P\{\eta_1 = i - k, \eta_2 = j - l\} =$$
$$= \sum_{k=0}^{i} \sum_{l=0}^{j} p_{kl} r_{i-kj-l}. \tag{26}$$

Then we can finally calculate the distribution function of random vector (ζ_1, ζ_2) applying the formula

$$L(x, y) = \sum_{i<x} \sum_{j<y} q_{ij}.$$

Formulas (25), (26) can be generalized for the arbitrary number of non-negative independent random vectors but computations may be very complicated.

7. LST of Random Vectors and its Properties

Let (ξ, η) denote a non-negative random vector and $L(x, y)$ denote its distribution function. For every complex numbers q, s which satisfy the condition $\operatorname{Re} q \geq 0$, $\operatorname{Re} s \geq 0$ we can define the LST of random vector (ξ, η) as it follows [4]:

$$\alpha(q, s) = Ee^{-q\xi - s\eta} = \int_0^\infty \int_0^\infty e^{-qx-sy} dL(x, y). \tag{27}$$

The function given by (27) is called the double LST of random vector (ξ, η) and has the following properties.

Property 5. *Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)$ be a sequence of independent non-negative random vectors and $\alpha_1(q, s), \alpha_2(q, s), \ldots, \alpha_n(q, s)$ be a sequence of double LST of these random vectors respectively,*

$$(\xi, \eta) = (\xi_1, \eta_1) + (\xi_2, \eta_2) + \ldots + (\xi_n, \eta_n)$$

— the sum of random vectors $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)$. If $\alpha(q, s)$ is the double LST of random vector (ξ, η), then we obtain the following formula:

$$\alpha(q, s) = \prod_{i=1}^{n} \alpha_i(q, s). \tag{28}$$

Property 6. *Let us assume that (ξ, η) is a non-negative random vector and $L(x, y)$ and $\alpha(q, s)$ are distribution function and double LST of random vector (ξ, η) respectively. Then we have the following formula:*

$$\alpha(q, s) = \int_0^\infty \int_0^\infty e^{-qx-sy} dL(x, y) = qs \int_0^\infty \int_0^\infty e^{-qx-sy} L(x, y) dx dy. \tag{29}$$

Let us notice that the integral on the right side of (29) is the well known double Laplace transformation of function $L(x, y)$ ([6]).

Definition of double LST can be generalized for the arbitrary dimensions of random vectors. The properties of such generalized functions stay the same. Applying two above properties we can obtain the distribution functions of independent random vectors in a similar way as we did in the case of independent random variables.

8. Generating Function (GF) of a Random Vector and its Properties

Consider now a non-negative random vector (ξ, η) taking only integer values and introduce the following notation: $p_{ij} = P\{\xi = i, \eta = j\}$, $\sum_{i,j} p_{ij} = 1$. Then for every complex z_1, z_2 that satisfy conditions $|z_1| \leq 1, |z_2| \leq 1$ we can define the following function ([2]):

$$P(z_1, z_2) = Ez_1^{\xi} z_2^{\eta} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} z_1^i z_2^j. \tag{30}$$

The function given by (30) is called the generating function (GF) of random vector (ξ, η).

Now we present two very interesting properties of GF that can be used in obtaining distribution functions of sums of independent random vectors taking only integer values.

Property 7. *Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)$ be a sequence of independent random vectors taking only integer values and*

$$P_1(z_1, z_2), P_2(z_1, z_2), \ldots, P_n(z_1, z_2)$$

denote a sequence of GF of random vectors $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)$ respectively. If $(\xi, \eta) = (\xi_1 + \xi_2 + \ldots + \xi_n, \eta_1 + \eta_2 + \ldots + \eta_n)$ denotes the sum of these random vectors and $P(z_1, z_2)$ – GF of the random vector (ξ, η), then we have the following formula:

$$P(z_1, z_2) = \prod_{i=1}^{n} P_i(z_1, z_2). \tag{31}$$

Property 8. *If we have the GF $P(z_1, z_2)$ of random vector (ξ, η) then we can recover distribution $p_{ij} = P\{\xi = i, \eta = j\}$ applying the formula*

$$p_{ij} = \frac{1}{i!j!} \frac{\partial^{i+j} P(z_1, z_2)}{\partial z_1^i \partial z_2^j} \Big|_{z_1 = z_2 = 0}. \tag{32}$$

In view of those two properties we can compute the distributions of sums of independent random vectors taking only integer values. The definition of GF can be extended for the arbitrary dimensions of random vectors taking only integer values and its properties stay the same.

9. Examples of Calculating Distributions of Sums of Independent Random Vectors applying GF or LST

Assume that $(\xi_1,\eta_1),(\xi_2,\eta_2)$ are two independent random vectors taking only integer values that are defined by the following tables:

(ξ_1,η_1)	1	2
3	$\frac{5}{6}$	$\frac{1}{6}$

(ξ_2,η_2)	3	4
1	$\frac{1}{3}$	$\frac{2}{3}$

The generating functions of the random vectors (ξ_1,η_1) and (ξ_2,η_2) have the form

$$P_1(z_1,z_2)=\frac{5}{6}z_1z_2^3+\frac{1}{6}z_1^2z_2^3,\ P_2(z_1,z_2)=\frac{1}{3}z_1^3z_2+\frac{2}{3}z_1^4z_2.$$

Applying (31) we can obtain the generating function of the sum of these random vectors in the form

$$P(z_1,z_2)=P_1(z_1,z_2)\cdot P_2(z_1,z_2)=\frac{5}{18}z_1^4z_2^4+\frac{11}{18}z_1^5z_2^4+\frac{2}{18}z_1^6z_2^4. \tag{33}$$

Applying (32) and (33) we can recover the distribution of the sum. It is presented in the following table:

$(\xi_1+\xi_2,\eta_1+\eta_2)$	4	5	6
4	$\frac{5}{18}$	$\frac{11}{18}$	$\frac{2}{18}$

Assume now that we have n independent random vectors (ξ,η) having the distribution defined by the formula

$$p_{ij}=\frac{(1-p)p^i}{ej!},\ p\in(0,1). \tag{34}$$

The generating function of the random vector (ξ,η) has the form

$$P(z_1,z_2)=\frac{(1-p)e^{z_2}}{e(1-pz_1)}. \tag{35}$$

GF of the sum has the form

$$Q(z_1,z_2)=\frac{e^{nz_2}(1-p)^n}{(e-epz_1)^n}. \tag{36}$$

Applying (32) we obtain distribution in the form

$$p_{ij}=\frac{1}{i!j!}\left(\frac{1-p}{e}\right)^n p^in^j\prod_{k=0}^{i-1}(n+k). \tag{37}$$

Consider finally two independent non-negative random vectors (ξ, η) having distribution function

$$L(x, y) = P\{\xi < x, \eta < y\} = 1 - e^{-x} - e^{-y} + e^{-x-y}, \ x > 0, \ y > 0. \quad (38)$$

The double LST of each vector has the form:

$$\alpha_1(q, s) = \frac{1}{(q+1)(s+1)}. \quad (39)$$

Then, applying (28), the double LST of the sum of two independent vectors (ξ, η) has the form

$$\alpha(q, s) = \frac{1}{(q+1)^2(s+1)^2}. \quad (39)$$

In view of (29) we obtain formula for the double Laplace transform of this sum

$$l(q, s) = \frac{\alpha(q, s)}{q^2 s^2} = \frac{1}{q^2 s^2 (q+1)^2 (s+1)^2}. \quad (40)$$

If we apply Laplace transform inversion (ex. *Mathematica* environment) we finally obtain formula for the distribution function of the sum

$$L_2(x, y) = e^{-x-y} \left(2 + x + e^x (x-2)\right) \left(2 + y + e^y (y-2)\right). \quad (41)$$

Let us notice that the majority of calculations concern to the situation if the components of the random vectors (ξ, η) are independent. In the discrete case i.e. the distribution of the random vector has the form $p_{ij} = k_i l_j$, where k_i and l_j are the distributions of random variables taking only integer values. In the case of absolutely continuous random vectors the distribution function of each vector has the form $F(x, y) = F_1(x) F_2(y)$, where $F_1(x)$, $F_2(y)$ are the distribution functions of absolutely continuous non-negative random variables.

Calculations in this case are much easier because in the discrete case the GF of the sum has the form $P(z_1, z_2) = (P_1(z_1))^n (P_2(z_2))^n$ and it is not difficult to recover the distribution p_{ij}. In the case of the sum of the absolutely continuous random vectors we have the following formula for its LST:

$$\alpha(q, s) = (\alpha_1(q))^n (\alpha_2(s))^n$$

and in some cases it is easier to inverse the double Laplace transform $\frac{\alpha(q,s)}{qs}$.

If the components of the random vectors are dependent, obtaining general formulas for distribution functions of the sums is much more difficult since the distribution functions of vectors are not the products of distribution functions of their components and during the computations we deal with calculating more complicated sums (i.e. we have to use binomial formula), so finding general formulas is possible only in some special cases.

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