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Heavy Moving Average Distances in Sales Forecasting

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Maricruz Olazabal-Lugo, Luis F. Espinoza-Audelo, Ernesto León-Castro, Luis A. Perez-Arellano, Fabio Blanco-Mesa

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Abstract:
This paper presents a new aggregation operator technique that uses the ordered weighted average (OWA), heavy aggregation operators, Hamming distance, and moving averages. This approach is called heavy ordered weighted moving average distance (HOWMAD). The main advantage of this operator is that it can use the characteristics of the HOWMA operator to under- or overestimate the results according to the expectations and the knowledge of the future scenarios, analyze the historical data of the moving average, and compare the different alternatives with the ideal results of the distance measures. Some of the main families and specific cases using generalized and quasi-arithmetic means are presented, such as the generalized heavy moving average distance and a generalized HOWMAD. This study develops an application of this operator in forecasting the sales growth rate for a commercial company. We find that it is possible to determine whether the company's objectives can be achieved or must be reevaluated in response to the actual situation and future expectations of the enterprise.

Keywords: *Heavy moving average distance, OWA operator, Distance measures, Sales forecasting.*

1. Introduction

In decision-making, various methods can be used to select the best alternative [1, 2]. Of these techniques, distance measures are some of the most common. These include the Hamming Distance [3], the Euclidean distance [4], the Minkowski distance [5], and more generally, all distance methods [6].

The most common problems that can be solved with distance measures are decision-making problems related to comparing alternatives between the actual data and the objective or ideal results [7,8]. The path with the closest distance to zero will be the ideal choice to select. In recent years many authors have developed new extensions such as the logarithmic distance [9], prioritized distances [10], Bonferroni means distances [11, 12], induced Euclidean distances [13], intuitionistic fuzzy induced distances [14], induced heavy aggregation distances [15], probabilistic distances [16] simplified neutrosophic distances [18], attitudinal distances [18], fuzzy linguistic induced distances [19], and many more.

The most basic distance measure in decision-making is the normalized Hamming distance (NHD). Still, it is also possible to combine this traditional technique with other techniques to generate new scenarios for evaluation. For example, one of the most common aggregation techniques is the ordered weighted average (OWA) operator, developed by Yager [20]. The OWA operator has been shown to be a very general and universal aggregation operator that provides tools for human consistent aggregation as shown in Kacprzyk, Yager and Merigo [21], which provides an account of a wide variety of various operators. An example of the power of the ordered weighted averaging operators is Kacprzyk and Zaderozny [22] in which, through the choice of the type of the ordered weighted operator and its weights, various voting procedures can be represented, providing a convenient and powerful representation. Also, it has been used to solve different problems [23, 24] such as the clustering method for classification problems [25] portfolio selection [26], competitiveness [27], econometric forecasting [28], profit of investment [29], business failure [30], financial decisions [31], and so on.

Among the wide variety of OWA operator extensions, this paper focuses on the heavy OWA (HOWA) operator. This operator has the advantage that the weighting vector is not bounded by 1, so in this case, it is possible to under- or overestimate the results and generate more scenarios to consider [32]. Another technique considered in this paper is the moving average (MA), which is useful for representing dynamic information because it can filter out short-term fluctuations [33]. This common technique, used in many different time series problems in different fields, is an important method to forecast future scenarios based on historical data [34].

This paper presents a new aggregation operator that uses the three previously specified techniques. First, we introduce the heavy ordered weighted moving average distance (HOWMAD) operator. It is a distance aggregation operator that considers a parameterized family of distance aggregation operators between the minimum and maximum distances. Some of the most important families and specific cases are also presented, including the heavy weighted moving average distance (HWMAD) operator, the quasi-HOWMAD operator, and the generalized HOWMAD operator.

To analyze the usefulness of the HOWMAD operator, we use it to forecast sales growth rates of a commercial company, identify whether the objectives of the enterprise can be achieved, and analyze whether it is possible to reformulate the objectives based on knowledge about the future and the information provided by the historical data. In the example, we compare the results with other distance measures.

The remainder of the paper is organized as follows. Section 2 reviews aggregation operators and some distance techniques. Section 3 introduces the HOWMAD operator, and Section 4 develops the generalized HOWMAD operator. Section 5 explains the steps for using heavy moving average operators, and Section 6 presents an application of the HOWMAD operator in sales forecasting. Finally, Section 7 summarizes the main conclusions of the paper.

2. Preliminaries

This section briefly reviews some basic concepts to be used throughout the paper, including the aggregation operators, the moving average operators, and the distance techniques.

2.1. Aggregation Operators

One operator that can be used to aggregate information is the OWA operator introduced by Yager [20]. This operator allows the aggregation of information between the maximum and the minimum, and since its introduction, many applications have been described [35]. The definition is as follows.

Definition 1. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ with an associated weight vector W of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, according to the following formula.

$$\mathrm{OWA}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_j is the jth largest element of the collection a_i.

Another extension of the OWA operator can be obtained by increasing or decreasing the values in the weighting vector; in this case, the sum of the weight values is not bounded by 1. This operator is called the Heavy OWA (HOWA) developed by Yager [32]. One of its main characteristics is that due to the inclusion of a weighting vector whose sum can range from $-\infty$ to ∞, the results can be drastically under- or overestimated according to the information and knowledge possessed by the decision-maker. This operator is defined as follows.

Definition 2. A heavy aggregation operator is an extension of the OWA operator for which the sum of weights is bounded by n. Thus, an HOWA operator is a map $R^n \rightarrow R$ that is associated with a weight vector w, with $w_j \in [0,1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$ such that

$$HOWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{2}$$

where b_j is the jth largest element of the collection $a_1, a_2, \dots, a_n$ and the sum of the weights w_j is bounded to n or can be unbounded if the weighting vector $W, -\infty \leq \sum_{j=1}^{n} w_j \leq \infty$. One of the characteristics introduced by Yager [32] is a characterizing parameter, called the beta value of the vector W, which can be defined as $\beta(W) = (|W|-1)/(n-1)$. If $|W| \in [1, n]$, it follows that $\beta \in [0,1]$. Therefore, if $\beta = 1$, we obtain the total operator, and if $\beta = 0$, we obtain the usual OWA operator.

2.2. Moving-average Operators

One common technique that can be used to solve time series smoothing problems is the moving average, which can be extended to the moving-average operators [36]. Some applications of these operators have been described in economics and statistics [34]. The moving average can be defined as follows [37].

Definition 3. Given $\{a_i\}_{i=1}^{N}$, the moving average of dimension n is defined as the sequence $\{s_i\}_{i=1}^{N-n+1}$ obtained by taking the arithmetic mean of the sequence of n terms, such that

$$s_i = \frac{1}{n} \sum_{j=i}^{i+n-1} a_j. \tag{3}$$

Another extension of the usual moving average involves combining it with the HOWMA operator. This operator is called the heavy ordered weighted moving average (HOWMA) operator. The main advantage of this new operator is that it is possible to under- or overestimate the results of the classical moving average according to the expectations of the decision-maker for future scenarios. Therefore, this technique is useful for generating new scenarios, which can help understand different and new alternatives for the future. This operator can be defined as follows [38].

Definition 4. Given a sequence $\{a_i\}_{i=1}^{N}$, the HOWMA operator is defined as the operator that acts on the sequence $\{s_i\}_{i=1}^{N-n+1}$, which is multiplied by a heavy weighting vector, according to

$$HOWMA(s_i) = \sum_{j=1+t}^{m+t} w_j b_j, \tag{4}$$

where b_j is the jth largest element of the collection $a_1, a_2, \dots, a_n$, and W is an associated weighting vector of dimension m that satisfies $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ and $w_i \in [0,1]$. Observe that we can also expand the weighting vector to the range $-\infty$ to ∞. Thus, the weighting vector w becomes unbounded: $-\infty \leq \sum_{j=1}^{n} w_j \leq \infty$.

It is important to note that, as with the OWA operator, it is possible to distinguish between the descending HOWMA (DHOWMA) operator and the ascending HOWMA (AHOWMA) operator, according to the same rules for the weighting vector that were applied to the OWA operator.

2.3. Distance Techniques

A useful technique to calculate the distance between two elements is the Hamming distance [3]. This can be used to calculate the distance between two sets, which can be applied within fuzzy set theory. To define the Hamming distance, it is necessary to define the basic properties of a distance measure:

a) Non-negativity: $D(A_1, A_2) \geq 0$;

b) Commutativity: $D(A_1, A_2) = D(A_2, A_1)$;

c) Reflexivity: $D(A_1, A_2) = 0$;

d) Triangle inequality: $D(A_1, A_2) + D(A_2, A_3) \geq D(A_1, A_3)$.

With these properties, the Hamming distance can be defined as follows (Merigó et al., 2014).

Definition 5. A normalized Hamming distance of dimension n is a mapping $NHD : [0,1]^n x[0,1]^n \rightarrow [0,1]$, such that

$$NHD(A,B) = \left(\frac{1}{N}\sum_{i=1}^{n}|a_i - b_i|\right), \quad (5)$$

where a_i and b_i are the ith arguments of sets A and B, respectively.

Another extension can be obtained by combining the OWA operator and the normalized Hamming distance. This operator is called Hamming ordered weighted average distance (OWAD) operator and can be defined as follows [39].

Definition 6. An OWAD operator of dimension n is a mapping $OWAD$: $[0,1]^n x[0,1]^n \rightarrow [0,1]$ that has an associated weighting vector w, with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ such that

$$OWAD(\langle\mu_1, \mu_1^{(k)}\rangle, \dots, \langle\mu_n, \mu_n^{(k)}\rangle) = \sum_{j=1}^{n} w_j D_j, \quad (6)$$

where D_j is the jth largest of the differences $|x_i - y_i|$, and $|x_i - y_i|$ is the argument variable represented in the form of individual distances.

It is important to note that the distance definition can be combined with the moving average technique; in this situation, we obtain the moving average distance (MAD) operator, which can be defined as follows.

Another extension, the ordered weighted moving average distance (OWMAD) operator, can be obtained using the moving averages, Hamming distances, and OWA operator. The main advantage of this new operator is that we can under- or overestimate the results according to the decision maker's perception. The definition of this operator is as follows [36].

Definition 7. An OWMAD operator of dimension m is a mapping $OWMAD$: $R^m x R^m \rightarrow R$ that has an associated weighting vector W, with $\sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0,1]$, such that

$$OWMAD(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \dots, \langle x_m, y_m\rangle) = \sum_{j=1+t}^{m+t} w_j D_j, \quad (7)$$

where D_j represents the jth largest of the differences $|x_i - y_i|$; x_i and y_i are the ith arguments of the sets X and Y; m is the total number of arguments considered from the whole sample, and t indicates the movement of the average from the initial analysis.

3. Heavy Ordered Weighted Moving Average Distance Operator

3.1. Main Concept

These distance approaches can be extended using the HOWMA operator. In this way, we obtain the heavy ordered weighted moving averaging distance (HOWMAD) operator. This new extension has the advantage that we can under- or overestimate the results according to the decision-maker's perception and the knowledge of the future scenarios of the variables. Like the HOWMA operator, this new extension includes a weighting vector whose values can range from 1 to ∞ or even from $-\infty$ to ∞. It is important to note that it is possible to obtain new scenarios and have a better understanding of different situations because of this property. The definition of the HOWMAD operator is as follows.

Definition 8. A HOWMAD operator of dimension m is a mapping $HOWMAD$: $R^m x R^m \rightarrow R$ that has an associated weighting vector W, with $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ and $w_i \in [0,1]$, such that

$$HOWMAD(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \dots, \langle x_m, y_m\rangle) = \sum_{j=1+t}^{m+t} w_j D_j, \quad (8)$$

where D_j represents the jth largest of the differences $|x_i - y_i|$; x_i and y_i are the ith arguments of the sets X and Y; m is the total number of arguments considered from the whole sample; t indicates the movement in the average from the initial analysis; and the values w_i of the weighting vector can range from 1 to ∞, or even from $-\infty$ to ∞.

It is important to note that the characteristics of the HOWMA operator can also be identified in the HOWMAD operator, so that we can distinguish between the descending HOWMAD (DHOWMAD) and the ascending HOWMAD (AHOWMAD) operators. Additionally, the HOWMAD operator is monotonic and commutative but not bounded if the range of the weighting vector is from $-\infty$ to ∞.

Finally, the characteristics of the beta value of the vector W, which can be defined as $\beta(W) = \frac{|W|-1}{n-1}$, also apply to the HOWMAD operator. If $|W| \in [1, n]$, it follows that $\beta \in [0,1]$. Therefore, if $\beta = 1$, we obtain the total operator, and if $\beta = 0$, we obtain the usual OWA operator.

3.2. Families of the HOWMAD Operator

In this section, different types of HOWMAD operators are presented. Initially, we consider the cases in which the HOWMAD operator becomes the HWMAD operator, the OWMAD operator, the WMAD operator, the MAD operator, among others [40, 41]. To obtain these operators, we must make different changes to the weighting vector:

a) The total distance operator when $\beta = 1$;

b) The minimum distance when $w_n = 1$ and $w_j = 0$, for all $j \neq n$ and $\beta = 0$;

c) The HWMAD operator is obtained when the weighting vector W satisfies $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$, but there is no reordering step according to the jth largest element of $|x_i - y_i|$;
d) The OWMAD operator is obtained when the weighting vector W satisfies $\sum_{j=1+t}^{m+t} w_j = 1$ and $w_j \in [0,1]$, or when $\beta = 0$;
e) The WMAD operator is obtained when the weighting vector W satisfies $\sum_{j=1+t}^{m+t} w_j = 1$, $w_j \in [0,1]$ and there is no reordering step according to jth largest element of $|x_i - y_i|$;
f) The MAD operator is obtained when there is no associated weighting vector;
g) The Olympic-HOWMAD operator is obtained when $w_1 = w_n = 0$ and $w_j = 1/(n-2)$ for all other values of j;
h) The HOWMA operator is obtained if one of the sets is empty;
i) The OWMA operator is obtained if one of the sets is empty and the weighting vector satisfies $\sum_{j=1+t}^{m+t} w_j = 1$;
j) The WMA operator is obtained if one of the sets is empty, the weighting vector satisfies $\sum_{j=1+t}^{m+t} w_j = 1$, and there is no reordering step according to the jth largest element of $|x_i - y_i|$; instead, the reordering is according to the ith initial ordering;
k) The uniform distance allocation is obtained when $w_j = \frac{|W|}{n}$;
l) The push-down allocation is obtained when $w_{n-j+1} = (1 \wedge (|W| - (j-1))) \vee 0$;
m) The push-up allocation is obtained when $w_j = (1 \wedge (|W| - (j-1))) \vee 0$.

4. Generalized Heavy Moving-Average Distances

This section presents an analysis of different generalizations of the HOWMAD operator using quasi-arithmetic means. This is done because, in the quasi-arithmetic means, it is possible to obtain the generalization of a particular case [42]. We can obtain new partial cases through this analysis, thus making it possible to consider the aggregation process more completely. The quasi-HOWMAD (QHOWMAD) operator can be defined as follows.

Definition 9. A Quasi-HOWMAD operator of dimension m is a mapping *Quasi-HOWMAD*: $R^m x R^m \rightarrow R$ that has an associated weighting vector W, with $1 \leq \sum_{i=1+t}^{m+t} w_i \leq n$ and $w_i \in [0,1]$, such that

$$\textit{Quasi-HOWMAD}(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_m, y_m\rangle) = g^{-1} \sum_{j=1+t}^{m+t} w_j g(D_j), \quad (9)$$

where D_j represents the jth largest of the differences $|x_i - y_i|$; x_i and y_i are the ith arguments of the sets X and Y; m is the total number of arguments considered from the whole sample; t indicates the movement in the average from the initial analysis; w_i are the elements of the weighting vector, which can range from 1 to ∞ or even from $-\infty$ to ∞; and $g(D_j)$ is a strictly continuous monotone function.

Also, some families of the Quasi-HOWMAD operator are:

a) The generalized HOWMAD (GHOWMAD) operator, better known as the Minkowski HOWMAD (MHOWMAD) operator, is obtained if $g(D) = D^{\lambda}$.

range from 1 to ∞, or even from $-\infty$ to ∞; and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

b) The HOWMAD operator is obtained if $g(D) = D$.
c) The heavy ordered weighted moving quadratic-average distance (HOWMQAD) operator, or Euclidean HOWMAD (EHOWMAD) operator, is obtained if $g(D) = D^2$.
d) The heavy maximum distance is obtained if $g(D) \rightarrow D^{\lambda}$, for $\lambda \rightarrow \infty$.
e) The heavy minimum distance is obtained if $g(D) \rightarrow D^{\lambda}$, for $\lambda \rightarrow -\infty$.
f) The heavy ordered weighted moving harmonic average distance (HOWMHAD) operator is obtained if $g(D) = D^{-1}$.
g) The heavy ordered weighted moving geometric-average distance operator is obtained if $g(D) \rightarrow D^{\lambda}$, for $\lambda \rightarrow 0$. Note that individual distances equal to zero are not considered in the aggregation because they are neutral elements.

5. Forecasting Sales Using Heavy Moving Average Operators

5.1. Theoretical Approach

Sales forecasting has become one of the most important aspects of planning for companies and usually employs sales control systems, such as rules, policies, and procedures, to reach different corporate objectives [43, 44]. However, if the sales objectives cannot be achieved because they are misaligned with reality, this will negatively affect the sales team. Therefore, a common sales forecasting method considers representative values by using an averaging technique, such as the arithmetic mean or weighted average, to construct different horizons and scenarios.

To obtain a new scenario of the sales in the future, the HOWMAD operator can be used. To use this operator correctly, the following steps can be taken to summarize the application.

Step 1. Analyze and determine the increase or decrease in the sales growth rate based on the historical data that can be considered important and will have a significant impact on the results, over a certain time (e.g., 6 months, 12 months, 3 years).

Step 2. Determine the ideal sales growth rate that the company wants to achieve, based on the objectives set in the manuals or that the administration has set for the future sales growth rate.

Step 3. The difference between the ideal objective and the real results is considered in this step. From this, we obtain the historical distances of the sales, which will help us forecast the future distance between the ideal and the future sales growth rate.

Step 4. Once the difference between the objective and the real data is obtained, we ask the decision-maker to provide two different weighting vectors, one whose sum is equal to 1 and one that is considered a heavy weighting vector, according to their knowledge and expectations for the future of the company.

Step 5. With the historical data and the weighting vectors, we can use different operators to obtain the sales distance forecasts, such as Moving Average Distance (MAD), Weighted Moving Average Distance (WMAD), Ordered Weighted Moving Average Distance (OWMAD) and Heavy Ordered Weighted Moving Average Distance (HOWMAD).

Step 6. With the results obtained with the different operators, it is possible to analyze whether the objective set by the company is reasonable or must be reformulated according to the information obtained using the historical information and the expectations described by the weighting vector.

5.2. Numerical Example

In this section, we investigate a real problem in a Mexican commercial enterprise that wants to know the difference between the historical sales growth rates and their objective because they want to know if they must reevaluate this objective according to future scenarios. To do this, we applied the steps defined in Section 5.1.

Step 1: The Mexican commercial enterprise provides monthly sales growth rates from 2015–2021 (see Table 1).

Step 2: They indicate that the objective sales growth rate is 0.28% per month, for a total of 3.36% per year.

Step 3: We calculate the distance between the objective sales growth rate and the real growth in sales. The results are as follows (see Table 2).

Step 4: To generate different future scenarios, the weighting vector $W = (0.1, 0.15, 0.05, 0.1, 0.15, 0.2, 0.25) = 1$ is provided. Additionally, the heavy weighting vector $W = (0.1, 0.15, 0.1, 0.15, 0.1, 0.15, 0.3) = 1.05$ is used because, according to the expectations and knowledge of the general manager, the company will have better results due to the company's future economic scenarios and competition.

Step 5: The distance aggregation operators are used to forecast the distances for the next five years. The results are as follows (see Tables 3–6).

In Tables 3–6, we use four different aggregation operators to generate different scenarios for the sales distance over the next five years.

To compare the results obtained by the different operators, the following rules are presented by the decision-maker:

1) If the distance is negative, the objective can be achieved easily (A).
2) If the distance is positive but less than 0.05, the objective can be achieved but with a medium degree of difficulty (B).
3) If the distance is positive and more than 0.05, the objective cannot be achieved (C).

The results are as follows (See Tables 7, 8).

As shown in Table 7, different results can be seen, particularly in January, where according to the MAD and WMAD, the objective of sales can be achieved easily in contrast with the OWMAD and HOWMAD results that indicate a medium degree of difficulty. In the other months, the same analysis can be seen; in this sense, when we include more information in the operator, the results can differ one from another.

In the case of Table 8, it is possible to see that in August and December the analysis of the results between the operators is the same. But in the other months, as in the case of Table 7, the same conclusion can be made. When we include more data into the formulation, the results vary, which is the knowledge and expertise of the decision-maker being included in the operator.

Step 6: Analyzing the data reveals that only in December is it clear that it is possible to achieve the desired

Table 1. Sales growth rates, 2015–2021

	2015	2016	2017	2018	2019	2020	2021
January	0.3	0.25	0.31	0.36	0.32	0.24	0.28
February	0.42	0.31	0.29	0.28	0.22	0.32	0.19
March	0.24	0.26	0.31	0.33	0.18	0.22	0.3
April	0.31	0.27	0.29	0.32	0.28	0.27	−0.09
May	0.36	0.29	0.18	0.27	0.16	0.31	0.15
June	0.28	0.33	0.25	0.23	0.14	0.22	0.22
July	0.33	0.25	0.27	0.23	0.21	0.18	0.32
August	0.1	0.15	0.12	0.11	0.17	0.32	0.18
September	0.19	0.2	0.22	0.24	0.23	0.18	0.29
October	0.22	0.23	−0.08	0.21	0.18	0.34	0.35
November	0.24	0.28	0.18	0.23	0.25	0.3	0.28
December	0.35	0.22	0.32	0.29	0.33	0.29	0.4
Total	3.34	3.04	2.66	3.1	2.67	3.19	2.87

Table 2. Distance between objective and real sales growth rates

	2015	2016	2017	2018	2019	2020	2021
January	−0.02	0.03	−0.03	−0.08	−0.04	0.04	0
February	−0.14	−0.03	−0.01	0	0.06	−0.04	0.09
March	0.04	0.02	−0.03	−0.05	0.1	0.06	−0.02
April	−0.03	0.01	−0.01	−0.04	0	0.01	0.37
May	−0.08	−0.01	0.1	0.01	0.12	−0.03	0.13
June	0	−0.05	0.03	0.05	0.14	0.06	0.06
July	−0.05	0.03	0.01	0.05	0.07	0.1	−0.04
August	0.18	0.13	0.16	0.17	0.11	−0.04	0.1
September	0.09	0.08	0.06	0.04	0.05	0.1	−0.01
October	0.06	0.05	0.36	0.07	0.1	−0.06	−0.07
November	0.04	0	0.1	0.05	0.03	−0.02	0
December	−0.07	0.06	−0.04	−0.01	−0.05	−0.01	−0.12

Table 3. Sales growth rate distance forecasting with MAD operator

	2022	2023	2024	2025	2026
January	−0.014	−0.013	−0.020	−0.018	−0.009
February	−0.010	0.009	0.014	0.018	0.020
March	0.017	0.014	0.013	0.019	0.029
April	0.044	0.055	0.061	0.071	0.087
May	0.034	0.051	0.059	0.053	0.060
June	0.041	0.047	0.061	0.066	0.068
July	0.024	0.035	0.036	0.039	0.038
August	0.116	0.107	0.103	0.095	0.084
September	0.059	0.054	0.050	0.049	0.050
October	0.073	0.075	0.078	0.038	0.033
November	0.029	0.027	0.031	0.021	0.017
December	−0.034	−0.029	−0.042	−0.042	−0.047

Table 4. Sales growth rate distance forecasting with WMAD operator

	2022	2023	2024	2025	2026
January	−0.005	−0.005	−0.015	−0.017	−0.007
February	0.005	0.015	0.016	0.024	0.016
March	0.023	0.016	0.006	0.019	0.029
April	0.089	0.095	0.091	0.089	0.083
May	0.041	0.058	0.057	0.062	0.054
June	0.047	0.049	0.054	0.065	0.065
July	0.026	0.027	0.028	0.033	0.040
August	0.096	0.095	0.101	0.100	0.079
September	0.053	0.050	0.046	0.045	0.053
October	0.024	0.056	0.054	0.040	0.029
November	0.015	0.021	0.025	0.021	0.015
December	−0.041	−0.041	−0.045	−0.047	−0.043

Table 5. Sales growth rate distance forecasting with OWMAD operator

	2022	2023	2024	2025	2026
January	0.002	0.006	0.009	0.012	0.013
February	0.019	0.026	0.030	0.035	0.036
March	0.039	0.043	0.047	0.052	0.053
April	0.090	0.097	0.100	0.115	0.115
May	0.065	0.074	0.079	0.086	0.088
June	0.064	0.070	0.073	0.077	0.077
July	0.046	0.051	0.056	0.060	0.060
August	0.142	0.150	0.153	0.156	0.156
September	0.073	0.077	0.079	0.081	0.081
October	0.125	0.137	0.144	0.156	0.157
November	0.045	0.048	0.050	0.054	0.054
December	−0.013	−0.007	−0.004	0.001	0.001

Table 6. Sales growth rate distance forecasting with HOWMAD operator

	2022	2023	2024	2025	2026
January	−0.001	0.001	0.006	0.009	0.010
February	0.014	0.019	0.026	0.031	0.033
March	0.038	0.038	0.045	0.052	0.054
April	0.106	0.086	0.099	0.116	0.117
May	0.062	0.068	0.077	0.085	0.091
June	0.066	0.067	0.074	0.080	0.081
July	0.045	0.046	0.055	0.060	0.062
August	0.140	0.153	0.159	0.165	0.168
September	0.073	0.077	0.082	0.085	0.087
October	0.135	0.125	0.145	0.159	0.161
November	0.046	0.045	0.050	0.056	0.057
December	−0.015	−0.015	−0.009	−0.004	−0.003

Table 7. Comparison between operators from January to June 2022–2026

Month	Operator	2022	2023	2024	2025	2026
January	MAD	A	A	A	A	A
	WMAD	A	A	A	A	A
	OWMAD	B	B	B	B	B
	HOWMAD	A	B	B	B	B
February	MAD	A	B	B	B	B
	WMAD	B	B	B	B	B
	OWMAD	B	B	B	B	B
	HOWMAD	B	B	B	B	B
March	MAD	B	B	B	B	B
	WMAD	B	B	B	B	B
	OWMAD	B	B	B	C	C
	HOWMAD	B	B	B	B	B
April	MAD	B	C	C	C	C
	WMAD	C	C	C	C	C
	OWMAD	C	C	C	C	C
	HOWMAD	C	C	C	C	C
May	MAD	B	C	C	C	C
	WMAD	B	C	C	C	C
	OWMAD	C	C	C	C	C
	HOWMAD	C	C	C	C	C
June	MAD	B	B	C	C	C
	WMAD	B	B	C	C	C
	OWMAD	C	C	C	C	C
	HOWMAD	C	C	C	C	C

objective of the enterprise. This is because the final distance is negative, which indicates that the values in set Y are higher than those in set X. Because X is the objective sales growth rate and Y is the real growth, the company needs to consider a more realistic objective that can be achieved in the future. Additionally, it is possible to construct objectives for each month and the general objective with this information.

With the use of these aggregation operators, we can formulate different expectations for the future based on the information provided by the decision-maker. In this way, the results can include more information and provide more realistic expectations, given the uncertainty that the commercial markets are facing. This idea is seen when the results obtained by the MAD/WMAD/OWMAD/HOWMAD are analyzed and have a significant variation. The applicability of each of the aggregation operators analyzed in the paper are based on the complexity of the problem to analyze, that is because if we use the MAD operator the same result will be obtained always (this because only the historical data is taken into account) but if the decision-maker wants to add some quantitative information such the expectation that the sales increases or decreases in a certain month, a new competitor will appear in the same market, a contraction or expansion of the general economy or any other information that is important and are not taken into account in the historical data, the use of different weighting vectors and ordering process can be used. In this case, the WMAD uses a weighting vector provided by the decision-maker and the association between the attributes and the weights remain the same always, with the OWMAD it is possible to order the attributes and weights association according to an optimist or pessimist expectation of the future and in the HOWMAD the weighting vector can be higher or lower than 1, these will under- or overestimate

Table 8. Comparison between operators from July to December 2022–2026

Month	Operator	2022	2023	2024	2025	2026
July	MAD	B	B	B	B	B
	WMAD	B	B	B	B	B
	OWMAD	B	C	C	C	C
	HOWMAD	B	B	C	C	C
August	MAD	C	C	C	C	C
	WMAD	C	C	C	C	C
	OWMAD	C	C	C	C	C
	HOWMAD	C	C	C	C	C
September	MAD	C	C	C	B	C
	WMAD	C	C	B	B	C
	OWMAD	C	C	C	C	C
	HOWMAD	C	C	C	C	C
October	MAD	B	B	B	B	B
	WMAD	B	B	B	B	B
	OWMAD	C	C	C	C	C
	HOWMAD	C	C	C	C	C
November	MAD	B	B	B	B	B
	WMAD	B	B	B	B	B
	OWMAD	B	B	C	C	C
	HOWMAD	B	B	C	C	C
December	MAD	A	A	A	A	A
	WMAD	A	A	A	A	A
	OWMAD	A	A	A	A	A
	HOWMAD	A	A	A	A	A

the results according to the expectations, aptitude and other quantitative elements that the decision-maker believes will happened.

It is important to note that the results are related to the information and weighting vectors provided by the experts, and this can be an important study limitation. For example, if we make changes to the experts and select different weighting vectors, the results could be significantly different. Additionally, we would like to mention that this limitation is also one of the advantages of using an operator since it is possible to generate a significant number of scenarios when analyzing the same problem from the point of view of different experts.

6. Conclusion

The paper has introduced a new extension of the OWA operator called the heavy ordered weighted moving average distance (HOWMAD) operator. This operator uses the main characteristics of the HOWMA operator and the Hamming distance technique. With this operator, it is possible to combine historical information with the decision-makers' knowledge to forecast the distance between two data sets.

We provided the definition and some of the main properties of the HOWMAD operator. Additionally, we developed a wide range of families of the HOWMAD operator, such as the HWMAD, OWMAD, WMAD, and MAD operators. Additionally, using quasi-arithmetic means, several cases are presented at the end of Section 4, including the GHOWMAD, HOWMQAD, and HOWMCAD operators.

An application of this new approach in sales forecasting has also been developed. We observe that with the HOWMAD operator, we can generate new scenarios to provide the decision-maker with new alternatives and possible plans of action. Under uncertainty, it is very important to analyze many possible alternatives to determine the best decision. In this case, we provide information to the company to reformulate their sales growth rate objectives to something more achievable based on the historical data and the expectations of the general manager.

In future research, we plan to further develop this operator by adding news aggregators, such as the induced ordered weighted average (IOWA) operator, probabilistic ordered weighted average (POWA) operator, weighted averages (WA) operators, fuzzy numbers, interval numbers, or multi-person techniques [45–47]. Also, the use of the operator in complex and dynamic decision problems, such as the consensus reaching process [48, 49].

AUTHORS

Maricruz Olazabal-Lugo – Unidad Regional Culiacán, Universidad Autónoma de Occidente, Culiacán, Sinaloa, México, e-mail: maricruz.olazabal@uadeo.mx.

Luis F. Espinoza-Audelo* – Tecnológico Nacional de México/Instituto Tecnológico de Culiacán, Sinaloa, México, e-mail: luis.ea@culiacan.tecnm.mx.

Ernesto León-Castro – Faculty of Economics and Administrative Sciences, Universidad Católica de la Santísima Concepción, Concepción, Chile, e-mail: eleon@ucsc.cl.

Luis A. Perez-Arellano – Facultad de Psicología, Universidad Autónoma de Sinaloa, Culiacán, Sinaloa, Mexico, e-mail: luyz_@uas.edu.mx.

Fabio Blanco-Mesa – Facultad de Ciencias Económicas y Administrativas, Escuela de Administración de Empresas, Universidad Pedagógica y Tecnológica de Colombia, Av. Central del Norte, 39-115, 150001, Tunja, Colombia, e-mail: fabio.blanco01@uptc.edu.mx.

*Corresponding author

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