UNIWERSYTET TECHNOLOGICZNO-PRZYRODNICZY
IM. JANA I JĘDRZEJA ŚNIADECKICH W BYDGOSZCZY
ZESZYTY NAUKOWE NR 261
TELEKOMUNIKACJA I ELEKTRONIKA 16 (2012), 5-15

COMPARISON OF DESCRIPTIONS OF CONTINUOUS-TIME AND TELETRAFFIC SYSTEMS

Andrzej Borys, Mariusz Aleksiewicz

Faculty of Telecommunications and Electrical Engineering
University of Technology and Life Sciences
Kaliskiego 7, 85-789 Bydgoszcz, Poland
[andrzej.borys, mariusz.aleksiewicz]@utp.edu.pl

Summary: In a fundamental book [5] on the so-called network calculus and research papers using this technique, as for example those cited in this paper, the notion of causal linear time-invariant teletraffic systems (networks) is used. It has been mentioned in [5] that these systems are analogous to the causal linear time-invariant systems (circuits) described by integral convolution (or convolution sum in the case of discrete ones) in classical systems theory. Note that networks considered in the network calculus are described by other type of convolution that uses the infimum operation. Moreover, the algebra used in the above technique is also different. This is the so-called min-plus (or max-plus) algebra. Therefore, it is not obvious that the teletraffic systems (networks) described by the infimum convolution fulfill the following basic properties: linearity, causality, time-invariance, associativity and commutativity of their convolution operator, known from the classical theory of systems. The objective of this paper is to prove or show in detail that the above properties hold.

Keywords: Network calculus, basic properties of teletraffic systems, linearity, causality, time-invariance, associativity, commutativity, infimum convolutions

1. INTRODUCTION

Consider teletraffic system (network) which can be described by means of the so-called Network Calculus [5]. Central to the theory is the use of alternate algebras such as min-plus algebra and max-plus algebra to transform complex network systems into analytically tractable systems [4]. Many detailed examples can be found in the literature, as for example, in [1, 2, 3, 6]. The description of this system is in form of a black box, as depicted in Fig. 1, relating its output traffic $y(t)$ with its input traffic $x(t)$ through the service curve $\beta(t)$ (corresponding to the system impulse response in the classical systems theory dealing with analog or digital signals).

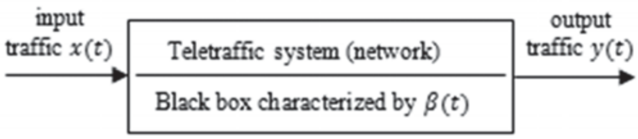

Fig. 1. A teletraffic system (network) input-output model via a black box characterized by a service curve $\boldsymbol{\beta(t)}$

The input and output traffics $x(t)$ and $y(t)$, respectively, are the functions of time variable t. They have the meaning of the cumulative traffic (sums of the bits (packets) arrived in the period from 0 to t). When the relation between $y(t)$ and $x(t)$ has the form

$$y(t) = \inf_{0\le\tau\le t}\{\beta(\tau) + x(t-\tau)\}, \qquad \beta(\tau) \equiv 0 \;\; for \;\; \tau < 0, \tag{1}$$

where inf means the mathematical operation of taking infimum value, then a system described by it is called a causal linear time-invariant one [5]. However, to our best knowledge, there is a lack of evidence in the literature, including [5], that a system described by (1) really fulfills the properties of linearity, causality and time-invariance. This paper fills the above gap. Besides, we show here how the properties mentioned look like in the case of linear systems described by an integral convolution – for reference and comparison. Finally, we compare the forms of associativity and commutativity properties of systems described by convolutions integral with those involving infimum operation.

2. LINEARITY PROPERTY

Let take into account a system described by an integral convolution of the form

$$y(t) = (Gx)(t) = G(x)(t) = \int_0^\infty h(\tau)x(t-\tau)d\tau,\; h(\tau) \equiv 0 \;\; for \;\; \tau < 0 \tag{2}$$

where $y(t)$ and $x(t)$ represent the output and input to the system, respectively, and $h(\tau)$ is the so-called impulse response of this system. The description by (2) (as shown there) can be also expressed in the operator form $y(t) = (Gx)(t) = G(x)(t)$ with G meaning the integral convolution operator. The system represented by (2) is linear when it fulfills the following relation

$$G(ax_1 + bx_2)(t) = aG(x_1)(t) + bG(x_2)(t) = ay_1(t) + by_2(t) \tag{3}$$

where a and b are some real numbers. Moreover, $x_1(t)$ and $x_2(t)$ are two input signals, and $y_1(t)$ and $y_2(t)$ are the corresponding output ones.

Using $x(t) = a \cdot x_1(t) + b \cdot x_2(t)$ (2) gives

$$y(t) = \int_0^\infty h(\tau)[ax_1(t-\tau) + bx_2(t-\tau)]\, d\tau =$$
$$= a\int_0^\infty h(\tau)\, x_1(t-\tau)d\tau + b\int_0^\infty h(\tau)\, x_2(t-\tau)d\tau = ay_1(t) + by_2(t) \tag{4}$$

That is the linearity condition (3) is fulfilled (meaning that the systems described by integral convolutions are linear).

Consider now the systems described by (1), which is also named as infimum convolution. In this case, the algebra used is also defined differently. It is named min-plus algebra. In this algebra, the operation infimum stands for addition in the standard algebra, but the addition, on the other hand, replaces multiplication of standard algebra; for more details, see for example [5]. Therefore, here, the condition of linearity, equivalent of (3), must be written as

$$G\left(\inf_2 \{a + x_1(t), b + x_2(t)\}\right) = \inf_2 \{a + G(x_1)(t), b + G(x_2)(t)\} =$$
$$= \inf_2 \{a + y_1(t), b + y_2(t)\}, \tag{5}$$

where $y_1(t) = G(x_1)(t)$ and $y_2(t) = G(x_2)(t)$ and a and b are real numbers. The subscript 2 under inf operation means calculation of the infimum (in this case minimum) in the set consisting of two elements. In the next step, we check whether (5) is satisfied in the case of G given by (1) or not. To this end, we substitute the signal $x(t) = \; {}^{inf}_{\;2}\{\, a + x_1(t),\; b + x_2(t)\}$ into (1). This gives

$$y(t) = \inf_{0\le\tau\le t} \left\{\beta(\tau) + \inf_2 \{a + x_1(t-\tau), b + x_2(t-\tau)\}\right\} =$$
$$= \inf_2 \left\{\inf_{0\le\tau\le t} \{\beta(\tau) + a + x_1(t-\tau)\}, \inf_{0\le\tau\le t} \{\beta(\tau) + b + x_2(t-\tau)\}\right\} =$$
$$= \inf_2 \left\{a + \inf_{0\le\tau\le t} \{\beta(\tau) + x_1(t-\tau)\}, b + \inf_{0\le\tau\le t} \{\beta(\tau) + x_2(t-\tau)\}\right\} = \tag{6}$$
$$= \inf_2 \;\{a + y_1(t), b + y_2(t)\}$$

So the linearity condition (5) is fulfilled in the case of the G operator given by (1) and min-plus algebra.

Note that in derivation of (6), we have exploited the fact that $\beta(\tau)$ is a constant in the inner expression there (second line) and interchange of inf operations, $\inf\limits_{2}$ with $\inf\limits_{0 \le \tau \le t}$, according to the theorem 3.1.1. "Fubini" formula for infimum in [5].

3. CAUSALITY

The causality property in systems theory is defined as follows: Let be two input signals x_1 and x_2 satisfying $x_1(\tau) = x_2(\tau)$ for all $\tau \leq t$. Then the operator G describing a system is called causal if it satisfies

$$y_1(t) = G(x_1)(t) = G(x_2)(t) = y_2(t) \tag{7}$$

In the case of systems described by the convolution integral (2), we have

$$\begin{aligned} y_1(t) - y_2(t) &= \int_0^{\infty} h(\tau)[x_1(t-\tau) - x_2(t-\tau)]d\tau = \\ &= -\int_t^{-\infty} h(t-\tau')[x_1(\tau') - x_2(\tau')]d\tau' = \\ &= \int_{-\infty}^{t} h(t-\tau')[x_1(\tau') - x_2(\tau')]d\tau' = \\ &= \int_0^{t} h(t-\tau')[x_1(\tau') - x_2(\tau')]d\tau' = \\ &= \int_0^{t} h(t-\tau')\,[\,0\,]\,d\tau' = 0 \end{aligned} \tag{8}$$

So, really, the condition (7) is fulfilled, meaning that the systems described by (2) are causal.

Note that in derivation of (8) we have used the substitution $\tau' = t - \tau$ and the fact that $h(\tau) \equiv 0$ for $\tau < 0$.

From the equality $x_1(\tau) = x_2(\tau)$ for all $\tau \leq t$, where now x_1 and x_2 mean the input traffics, and relation (1), it follows immediately that

$$\begin{aligned} & y_1(t) - y_2(t) = \\ & = \inf_{0\leq\tau\leq t} \{\beta(\tau) + x_1(t-\tau)\} - \inf_{0\leq\tau\leq t} \{\beta(\tau) + x_2(t-\tau)\} = 0 \end{aligned} \tag{9}$$

That is the teletraffic systems described by (1) are causal.

4. TIME-INVARIANCE

To define the time-invariance property, let us first introduce a delay operator U_T given by

$$U_T(x)(t) = x(t-T) \tag{10}$$

for signals starting at $t = 0$ that is for which $x(t) \equiv 0$ for $t < 0$. T in (10) means delaying the signal $x(t)$ by T seconds.

Generally, we say that a system described by a G operator is time-invariant if the following:

$$y_T(t) = G(x(t-T)) = G(U_T x)(t) = G(x)(t-T) = y(t-T) \quad (11)$$

holds.

Now, introducing the delayed signal $U_T(x)(t) = x(t-T)$ in (2), gives

$$y_T(t) = \int_0^{\infty} h(\tau)x(t-T-\tau)d\tau = $$
$$= \int_0^{t-T} h(\tau)x(t-T-\tau)d\tau = y(t-T) \quad (12)$$

because assumptions $x(t) \equiv 0$ and $h(t) \equiv 0$ for $t < 0$ follow that only the times $t-T-\tau > 0$ and $\tau > 0$ make sense. This leads to the upper limit in the integral $t-T$ and $t > T$. Otherwise, $y_T(t) \equiv 0$. Finally, we conclude that relation (12) proves the time-invariance of systems described by the integral convolution (2).

Consider now the teletraffic systems described by (1) and introduce there the delayed input traffic $x(t-T)$ (given by similar expression as (10)). We get then

$$y_T(t) = \inf_{0 \le \tau \le t} \{\beta(\tau) + x(t-T-\tau)\} \quad (13)$$

Furthermore, note that the restrictions regarding (13) are similar to those occurring in the previous case, that is $t-T-\tau > 0$ and $\tau > 0$. They lead to the upper limit $t-T$ instead of t under the symbol inf in (13) and the inequality $t > T$. Otherwise, $y_T(t) \equiv 0$. So, from (13) we get

$$y_T(t) = \inf_{0 \le \tau \le t-T} \{\beta(\tau) + x(t-T-\tau)\} = y(t-T) \quad (14)$$

which proves the time-invariance property of teletraffic systems described by (1).

5. ASSOCIATIVITY AND COMMUTATIVITY OF INTEGRAL AND INFIMUM CONVOLUTION

Associativity and commutativity of the integral convolution are very useful properties exploited in calculations. Let us recall their derivation here, especially, that the derivation of the first one is not obvious.

We begin with the commutativity property, which proves immediately, when we apply the assumption $x(t) \equiv 0$ for $t < 0$, introduce a new variable

$\tau' = t - \tau$, rename τ' as τ, and finally take into account the fact that $h(t) \equiv 0$ for $t < 0$. In effect, we get from (2)

$$\begin{aligned} y_T(t) &= \int_0^{\infty} h(\tau)x(t-\tau)d\tau = \int_0^{t} h(\tau)x(t-\tau)d\tau = \\ &= -\int_t^{0} h(t-\tau')x(\tau')d\tau' = \int_0^{t} x(\tau')h(t-\tau')d\tau' = \\ &= \int_0^{t} x(\tau)h(t-\tau)d\tau = \int_0^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned} \tag{15}$$

Consider now a cascade of two systems characterized by their impulse responses $h_1(t)$ and $h_2(t)$ as depicted in Fig. 2, with $x(t)$ being the input signal to the cascade and the corresponding output signals $y_1(t)$ and $y_2(t)$.

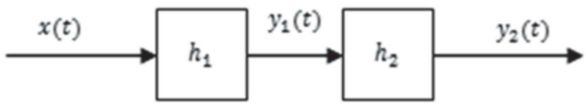

Fig. 2. A cascade of two systems for illustration of associativity property

To proceed further, observe first that according to (15) the descriptions

$$\int_0^{\infty} h(\tau)x(t-\tau)d\tau \quad \text{and} \quad \int_0^{t} h(\tau)x(t-\tau)d\tau$$

in our setting, are equivalent to each other. In what follows, we shall use the latter. So, for the cascade in Fig, 2, using additionally the commutativity property (15), we can write

$$\begin{aligned} y_2(t) &= \int_0^{t} y_1(\tau_2)\, h_2(t-\tau_2)d\tau_2 = \\ &= \int_0^{t}\int_0^{\tau_2} x(\tau_1)h_1(\tau_2-\tau_1)h_2(t-\tau_2)d\tau_1 d\tau_2 \end{aligned} \tag{16a}$$

Associating h_1 with h_2 would give

$$h(t-\tau_1) = \int_0^{t-\tau_1} h_2(\tau_3)\, h_1(t-\tau_1-\tau_3)d\tau_3 \tag{16b}$$

and then it should be possible to express $y_2(t)$ as

$$y_2(t) = \int_0^t x(\tau_1)\, h(t - \tau_1) d\tau_1 \tag{16c}$$

With $h(\cdot)$ given by (16b). In what follows, we will answer whether the above is true. For this purpose, consider an area of iteration of the double iterated integral occurring in (16a). It is illustrated in Fig. 3, and follows from the fact in our setting $h_1(t) \equiv 0$, $h_2(t) \equiv 0$ and $x(t) \equiv 0$ for all $t < 0$. This gives $t \geq \tau_2 \geq \tau_1 \geq 0$.

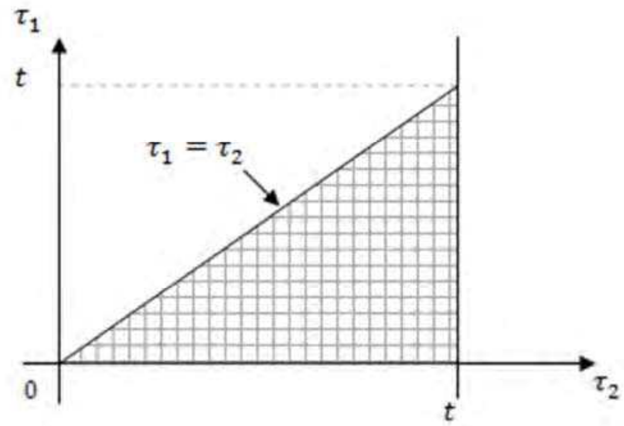

Fig. 3. The area of integration of an iterated integral in (16a)

Introduction of a new variable $\tau_3 = t - \tau_2$ rearranges the set of inequalities $t \geq \tau_2 \geq \tau_1 \geq 0$ to $t \geq \tau_1 \geq 0$ and $t - \tau_1 \geq \tau_3 \geq 0$, which describes a new area of integration (with τ_3 instead of τ_2). This area is shown in Fig. 4.

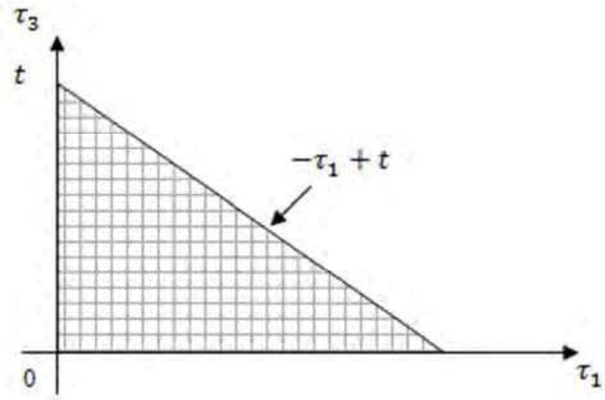

Fig. 4. The area of integration of an iterated integral in (16a) after introducing a new variable $\boldsymbol{\tau_3}$ instead of $\boldsymbol{\tau_2}$

Taking into account in (16a) the area of integration shown in Fig. 4, we arrive at

$$y_2(t) = \int_0^t \int_0^{t-\tau_1} x(\tau_1)h_1(t-\tau_1-\tau_3)h_2(\tau_3)d\tau_3 d\tau_1 = \\ = \int_0^t x(\tau_1) \int_0^{t-\tau_1} h_1(t-\tau_1-\tau_3)\, h_2(\tau_3)d\tau_3 d\tau_1 \tag{17}$$

Further, we deduce from (17) that, really, the output signal $y_2(t)$ can be expressed as in (16c) with $h(t)$ given by (16b). And this finally proves the associativity property of the integrated convolution given by (2).

Consider now the question of commutativity of the infimum convolution defined in (1). Introducing in (1) a new variable $\tau' = t - \tau$ gives

$$y(t) = \inf_{0\le\tau\le t}\{\beta(\tau) + x(t-\tau)\} = \inf_{0\le\tau'\le t}\{\beta(t-\tau') + x(\tau')\} = \\ = \inf_{0\le\tau\le t}\{x(\tau) + \beta(t-\tau)\} \tag{18}$$

with τ' renamed as τ in the last equality in (18). Looking at (18), we see that, really, the infimum convolution is commutative.

To show that the associativity property holds in the case of infimum convolution we need to prove first the following theorem.

Theorem 1. Let $S = S(\tau_1, \tau_2)$ be a set consisting of pairs of $\tau_1, \tau_2 \in R$ and let $S_{\tau_2}(\tau_1)$ and $S_{\tau_1}(\tau_2)$ be subsets of $S(\tau_1, \tau_2)$ chosen such that τ_2 or τ_1, respectively, is constant. Then, the following

$$\inf_{\tau_2\in S}\left\{\inf_{\tau_1\in S}\left\{f\left(S_{\tau_2}(\tau_1)\right)\right\}\right\} = \inf_{\tau_1\in S}\left\{\inf_{\tau_2\in S}\{f(S_{\tau_1}(\tau_2))\}\right\} \tag{19}$$

holds, where $f(\cdot)$ is a function of τ_1 and τ_2 belonging to the set S.

Proof. Using the notation introduce above ,we can write the range of the function f as range of

$$f = f(S) \tag{20}$$

Furthermore, note that

$$S = S(\tau_1, \tau_2) = \bigcup_{\tau_2\in S} S_{\tau_2}(\tau_1) = \bigcup_{\tau_1\in S} S_{\tau_1}(\tau_2) \tag{21}$$

Applying now the infimum operation to the set given by (20) with (21), we obtain

$$\inf\{f(S)\} = \inf_{\tau_1,\tau_2 \in S}\{f(S(\tau_1,\tau_2))\} = \inf_{\tau_1,\tau_2 \in S}\left\{\bigcup_{\tau_2 \in S} S_{\tau_2}(\tau_1)\right\} = \\ = \inf_{\tau_2 \in S}\left\{\inf_{\tau_1 \in S}\{f(S_{\tau_2}(\tau_1))\}\right\} \tag{22a}$$

or, similarly,

$$\inf\{f(S)\} = \inf_{\tau_1 \in S}\left\{\inf_{\tau_2 \in S}\{f(S_{\tau_1}(\tau_2))\}\right\} \tag{22b}$$

Therefore, we can conclude that the expressions on the most right-hand sides of (22a) and (22b) are equal each to other, and this constitutes the equality (19).

Concluding, we say that one can interchange the subscripts τ_1 and τ_2 in the way as shown in (19), when performing infimum operation iteratively. Moreover, note that our theorem 1 is generalization of the theorem 3.1.1 “Fubini” formula for infimum presented in [5] for any set consisting of two variables τ_1 and $\tau_2 \in R$.

Consider now a cascade of two teletraffic systems as shown in Fig. 5.

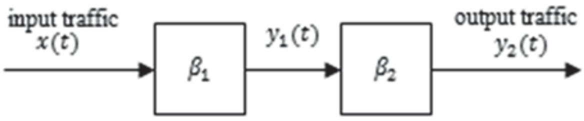

Fig. 5. A cascade of two teletraffic systems for illustration of associativity property

$$\begin{aligned} y_2(t) &= \inf_{0\le\tau_2\le t}\{\beta_2(\tau_2) + y_1(t-\tau_2)\} = \\ &= \inf_{0\le\tau_2\le t}\left\{\beta_2(\tau_2) + \inf_{0\le\tau_1\le t-\tau_2}\{\beta_1(\tau_1) + x(t-\tau_2-\tau_1)\}\right\} = \\ &= \inf_{0\le\tau_2\le t}\left\{\inf_{0\le\tau_1\le t-\tau_2}\{\beta_2(\tau_2) + \beta_1(\tau_1) + x(t-\tau_2-\tau_1)\}\right\} = \\ &= \inf_{0\le\tau_2\le t}\left\{\inf_{0\le\tau_1\le t-\tau_2}\{\beta_2(\tau_2) + \beta_1(t-\tau_2-\tau_1) + x(\tau_1)\}\right\} \end{aligned} \tag{23}$$

Note that the last expression in (23) holds because $\beta_2(\tau_2)$ does not depend upon τ_1. Moreover, the commutativity property of the infimum convolution has been used in it.

In the next step, we introduce a new variable $\tau'_1 = t - \tau_1$ in (23). This leads to

$$y_2(t) = \inf_{0 \le \tau_2 \le t} \left\{ \inf_{\tau_2 \le \tau'_1 \le t} \{\beta_2(\tau_2) + \beta_1(\tau'_1 - \tau_2) + x(t - \tau'_1)\} \right\} = \tag{24}$$

$$= \inf_{0 \le \tau'_1 \le t} \left\{ x(t - \tau'_1) + \inf_{0 \le \tau_2 \le \tau'_1} \{\beta_2(\tau_2) + \beta_1(\tau'_1 - \tau_2)\} \right\}$$

Note that the interchange of the subscript τ_2 and τ'_1 under the infimum operations in (24) have been used, according to theorem 1. And finally, we conclude that (24) proves fulfillment of the associativity property by the infimum convolution.

6. DIGITAL SIGNAL-BASED SYSTEMS

The digital linear time-invariant causal systems are described by a sum convolution as

$$y(k) = \sum_{i=0}^{\infty} h(i)x(k-i), \quad \text{with } h(i) \equiv 0, \;\; x(i) \equiv 0 \text{ for } i < 0 \tag{25}$$

in place of the integral convolution for time-continuous systems. In (25), $h(\cdot)$, $y(\cdot)$ and $x(\cdot)$ are the samples of the system impulse response, its output signal, and its input signal, respectively. Moreover, k means a discrete time.

Also the systems characterized by (25) possess the properties we dealt with in the previous section. This can be easily shown using the approach applied before for analogous ones.

7. CONCLUSION

In this paper, we have analyzed systematically the basic properties of teletraffic systems that can be described by an infimum convolution. We have shown that these systems are linear, causal, and time-invariant. Moreover, their convolution operator fulfills the properties of associativity and commutativity. To our best knowledge, the derivations presented here have not been presented in the literature, at least in such a systematic form. For reference and comparison, we have also presented analogous derivations for systems of which input-output descriptions are in form of an integral convolution.

BIBLIOGRAPHY

[1] Agrawal R., Cruz R. L., Okino C., Rajan R., 1999, Performance Bounds for Flow Control Protocols. IEEE/ACM TRANSACTIONS ON NETWORKING, Vol. 7, No. 3.

[2] Fidler M., 2010, A Survey of Deterministic and Stochastic Service Curve Models in the Network Calculus. IEEE COMMUNICATIONS SURVEYS & TUTORIALS, Vol. 12, No. 1.
[3] Ghiassi Y., Liebeherr J., 2009, Output Characterization of Constant Bit Rate Traffic in FIFO Networks. IEEE COMMUNICATIONS LETTERS, Vol. 13, No. 8.
[4] Jiang Y., 2009, Network Calculus and Queueing Theory: Two Sides of One Coin. VALUETOOLS, Pisa, Italy.
[5] Le Boudec J.-Y., Thiran P., 2004, Network Calculus. A Theory of Deterministic Queuing Systems for the Internet, Springer Verlag, Berlin, Germany.
[6] Liebeherr J., Valaee S., 2010, A system-Theoretic Approach to Bandwidth Estimation. IEEE/ACM TRANSACTIONS ON NETWORKING, Vol. 18, No. 4.

PORÓWNANIE OPISÓW SYSTEMÓW TELEINFORMATYCZNYCH ORAZ CIĄGŁYCH W CZASIE

Streszczenie

W znanej monografii nt. rachunku sieciowego (network calculus), napisanej przez J.-Y. Le Boudeca i P. Thirana, zostało wprowadzone pojęcie liniowych systemów teleinformatycznych niezależnych od czasu. Wskazano w niej na podobieństwa istniejące pomiędzy powyższą klasą systemów a liniowymi systemami analogowymi niezależnymi od czasu, jednakże zrobiono to w sposób dosyć pobieżny. W tym artykule podobieństwa te są przeanalizowane w sposób systematyczny, a także bez uciekania się do bardzo abstrakcyjnej teorii systemów opisywanych za pomocą algebry min-plus – jedynie przy wykorzystaniu elementarnych pojęć matematyki wyższej. Wiele przedstawionych tutaj wyprowadzeń nie było dotychczas nigdzie publikowanych, jak na przykład twierdzenie 1.

Słowa kluczowe: rachunek sieciowy (network calculus), liniowe i niezależne od czasu systemy teleinformatyczne, własności splotu w algebrze min-plus