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ON b-VERTEX AND b-EDGE CRITICAL GRAPHS

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Abstract. A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes, and the b-chromatic number $b(G)$ of a graph G is the largest integer k such that G admits a b-coloring with k colors. A simple graph G is called b^+-vertex (edge) critical if the removal of any vertex (edge) of G increases its b-chromatic number. In this note, we explain some properties in b^+-vertex (edge) critical graphs, and we conclude with two open problems.

Keywords: b-coloring, b-chromatic number, critical graphs.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

A proper coloring of a simple graph G is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. The *chromatic number* of G is the minimum integer $\chi(G)$ such that G has a proper coloring with $\chi(G)$ colors.

A *b-coloring* of a graph G by k colors is a proper coloring of the vertices of G such that in each color class there exists a vertex having neighbors in all the other $k-1$ colors classes. We call any such vertex a *b-vertex*. The b-chromatic number $b(G)$ of a graph G is the largest integer such that G admits a b-coloring with k colors. The concept of b-coloring has been introduced by R.W. Irving and D.F. Manlove ([14, 21]). They proved that determining $b(G)$ is NP-hard for general graphs, even when it is restricted to the class of bipartite graphs ([20]), but it is polynomial for trees ([14, 21]). The NP-completeness results have incited researchers to establish bounds on the b-chromatic number in general or to find its exact values for subclasses of graphs (see [2, 3, 6–8, 10, 12, 15, 18–20, 22, 23]).

The b-chromatic number of a graph G may increase, decrease or remain unchanged when G is modified by removing a vertex or an edge. In this context, Ikhlef Eschouf ([13]) and Blidia *et al.* ([5]) have characterized the class of P_4-sparse graphs, quasi-line graphs, P_5-free graphs and d-regular graphs for which $b(G-e) < b(G)$ holds for every edge e in G. They also proved that deciding if a graph is in this class is NP-hard for general graphs ([5]), even when it is restricted to the subclass of P_5-free graphs formed by the graphs that are the union of two split graphs. The same authors [4] have recently characterized trees for which $b(G-v) < b(G)$ holds for each vertex v in G ([4]). The focus of this paper involves studying the graphs in which removing of any vertex (edge) of G increases its b-chromatic number.

In the remainder of this section, we introduce some definitions and notation. Consider a graph $G = (V, E)$. For any $A \subset V$, let $G[A]$ denote the subgraph of G induced by A. For any vertex v of G, the *neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E\}$ (or $N(v)$ if there is no confusion), and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. Let $\Delta(G)$ (respectively, $\delta(G)$ be the maximum (respectively, minimum) degree in G. Let $\omega(G)$ denote the size of a maximum clique of G. If G and H are two vertex-disjoint graphs, the *union* of G and H is the graph $G+H$ whose vertex-set is $V(G) \cup V(H)$ and edge-set is $E(G) \cup E(H)$. For an integer $p \geq 2$, the union of p copies of a graph G is denoted pG. The *join* of graphs G and H is the graph denoted $G \vee H$ obtained from $G+H$ by adding all edges between G and H. The *join* of $k \geq 2$ copies of H is the graph $G = H \vee H \vee \ldots . \vee H$ obtained by taking k copies of H, and adding all edges between any two different copies. In case $k = 1, G = H$. The *cartesian product* of two graphs G and H denoted by $G \square H$, is a simple graph with $V(G) \times V(H)$ as its vertex set and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ if and only if either $u_1 = u_2$ and v_1, v_2 are adjacent in H, or u_1, u_2 are adjacent in G and $v_1 = v_2$. The girth $g(G)$ of G is the length of a shortest cycle in G. For further terminology on graphs we refer to the book by Berge [1].

Definition 1.1. A graph is said to be b^+-vertex critical if $b(G-v) > b(G)$ holds for every vertex v in G, and is said to be b^+-edge critical if $b(G-e) > b(G)$ holds for every edge e in G

In this paper, we describe some particular graphs that are b^+-vertex (edge) critical, and we mention some other graphs that are not in such classes. We conclude the paper by posing two open problems.

We now present some known results which will be used in the rest of the paper.

2. SOME KNOWN RESULTS

It is known that every graph G satisfies

$$\omega(G) \leq b(G) \leq \Delta(G) + 1. \tag{2.1}$$

The following theorem is proven by Jakovac and Klavzar in [15]. They showed that, except for four simple graphs, the b-chromatic number of connected cubic graph is 4. Let P, F_1, F_2 and $K_{3,3}$ be the graphs depicted in the Figure 1.

Theorem 2.1 ([15]). *Let G be a connected cubic graph. Then $b(G) = 4$ unless G is P, F_1, F_2 or $K_{3,3}$. In these cases, $b(P) = b(F_1) = b(F_2) = 3$ and $b(K_{3,3}) = 2$.*

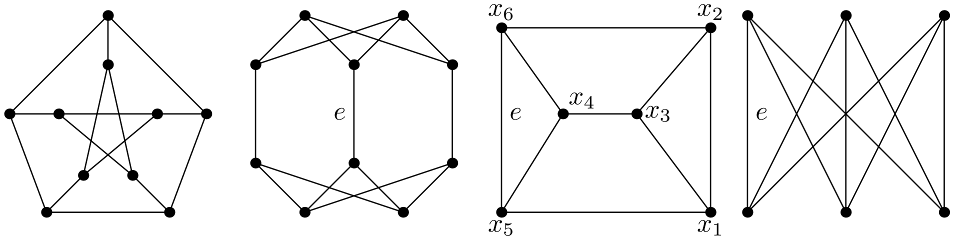

Fig. 1. Cubic graphs whose b-chromatic number is less than 4

The next result was established in [12].

Lemma 2.2 ([12]). *Let G_1, G_2 be two vertex-disjoint graphs. Then the join $G_1 \vee G_2$ has $b(G_1 \vee G_2) = b(G_1) + b(G_2)$.*

The following result on graphs of girth greater than 5 was proved in [17].

Proposition 2.3 ([17]). *Let G be a graph with girth at least 6. Then $b(G) \geq \delta(G)$. Moreover, if G is d-regular, then $b(G) = d + 1$.*

The b-chromatic number of the cartesian product of some graphs was studied in [16, 18]. In particular, R. Javadi and B. Omoomi [16] showed that the b-chromatic number of $K_3 \Box K_3$ is equal to 3.

Proposition 2.4 ([16]). $b(K_3 \Box K_3) = 3$.

3. REMOVING VERTEX

In this section we look at the effect of vertex removal on the b-chromatic number of a graph. More precisely, we are interested in graphs for which removing of any vertex increase the b-chromatic number. We first give some properties of b^+-vertex critical graphs, and as consequence, we conclude that graphs with girth at least 6, chordal graphs and connected cubic graphs are not b^+-vertex critical. However, we prove that the join of two b^+-vertex critical graphs is b^+-vertex critical; in particular, we show that $K_3 \Box K_3$ and the join of $k \geq 1$ copies of $K_3 \Box K_3$ are b^+-vertex critical graphs.

Recall that a graph G is chordal ([11, 24]) if every cycle of length at least four in G has a chord (an edge between non-consecutive vertices of the cycle). As usual, we say that a vertex is simplicial if its neighborhood induces a clique. It is well known that any chordal graph contains at least one simplicial vertex.

Proposition 3.1. *If G is b^+-vertex critical graph, then:*

(i) $b(G) \leq \delta(G) - 1$,
(ii) *G does not contain simplicial vertices,*
(iii) $g(G) \leq 5$.

Proof. (i) Let G be a b^+-vertex critical graph and $v \in V(G)$ be a vertex of minimum degree in G. Set $b(G-v) = k$ and consider a b-coloring c of $G-v$ with k colors. Suppose on the contrary that $b(G) \geq \delta(G)$. So $k > \delta(G)$. We define a coloring π of G with k colors obtained from c as follows. All vertices of G keep their colors except vertex v which is colored with a missing color in its neighborhood, this is possible because $d_G(v) \leq k-1$. We obtain a b-coloring with k colors such that each b-vertex of c is also a b-vertex of π, which implies that $b(G) \geq k$, a contradiction.

(ii) Suppose that G contains a simplicial vertex x and set $b(G-x) = k$. Then by (2.1), we have $k > b(G) \geq \omega(G) > d_G(x)$. So with an argument similar to that used in (i), one can show that G admits a b-coloring with k colors, implying that $b(G) \geq k$, a contradiction.

(iii) This follows immediately from Proposition 2.3 and item (i) of Proposition 3.1. □

Items (ii) and (iii) of Proposition 3.1 imply the two next results.

Corollary 3.2. *Chordal graphs are not b^+-vertex critical.*

Corollary 3.3. *Graphs with girth at least* 6 *are not b^+-vertex critical.*

Further, it is not difficult to see that $K_{3,3}$ is not b^+-vertex critical because $b(K_{3,3}) = b(K_{2,3}) = 2$. Also, in view of Theorem 2.1 and item (i) of Proposition 3.1, we conclude that P, F_1 and F_2 are not b^+-vertex critical, and furthermore, we have the following result.

Corollary 3.4. *Connected cubic graphs are not b^+-vertex critical.*

Proposition 3.5. *Let G_1 and G_2 be two vertex-disjoint graphs. Then $G_1 \vee G_2$ is b^+-vertex critical graph if and only if G_1, G_2 are b^+-vertex critical graphs.*

Proof. Let $G = G_1 \vee G_2$ and v be any vertex of G. Suppose without loss of generality that $v \in V(G_1)$. By virtue of Lemma 2.2, we have $b(G-v) = b((G_1 - v) \vee G_2) = b(G_1 - v) + b(G_2) > b(G_1) + b(G_2) = b(G)$. Thus G is b^+-vertex critical graph. Let us now prove the converse. Let v be any vertex of G_i ($i = 1$ or 2), and for $j = 1, 2$, set $G_j = G \setminus G_i$ ($j \neq i$). As G is b^+-vertex critical graph, $b(G-v) > b(G)$. This yields $b((G_i - v) \vee G_j) > b(G_i \vee G_j)$, and by Lemma 2.2, we get $b(G_i - v) + b(G_j) > b(G_i) + b(G_j)$. This immediately implies that $b(G_i - v) > b(G_i)$ and so for $i = 1, 2$, G_i is b^+-vertex critical graph. □

Proposition 3.6. *$K_3 \square K_3$ is b^+-vertex critical graph.*

Proof. Let $G = K_3 \square K_3$ with vertices x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} such that for each $i \in \{1, 2, 3\}$, x_{i1}, x_{i2}, x_{i3} induce a 3-cycle in this order, and x_{1i}, x_{2i}, x_{3i} induce a 3-cycle in this order. By assigning color 1 to x_{12}, x_{31}, color 2 to x_{21}, x_{33}, color 3 to x_{23}, x_{32} and color 4 to x_{11}, x_{22}, we obtain a b-coloring of $G - x_{13}$ with 4 colors in which $x_{21}, x_{22}, x_{31}, x_{32}$ are b-vertices. So up to symmetry $b(G-v) \geq 4$ holds for each vertex v of G. Since, by Proposition 2.4, $b(G) = 3$, it follows that G is b^+-vertex critical graph. □

The next result is a direct consequence of Propositions 3.5 and 3.6.

Corollary 3.7. *The join of $k \geq 2$ copies of $K_3 \square K_3$ is b^+-vertex critical graph.*

4. REMOVING EDGE

We now turn our attention to the effect of edge removal on the b-chromatic number. Unlike vertex removal, we are interested in graphs for which removing any edge increase the b-chromatic number. As for b^+-vertex critical graphs, we show that chordal graphs are not b^+-edge critical graphs. We also prove that Petersen graph, cartesian product of two cliques K_3 and its joins are b^+-edge critical graphs. Moreover, some other partial results were obtained. We start this section by the following proposition in which we give some properties of b^+-edge critical graphs.

Proposition 4.1. *Let G be a b^+-edge critical graphs. Then the following three properties hold:*

(i) $b(G) \leq \Delta(G)$,
(ii) *G does not contain simplicial vertices,*
(iii) *if G is a d-regular graph, then $g(G) \leq 5$.*

Proof. (i) Suppose that the first part is not true; so by (2.1), $b(G) = \Delta(G) + 1$. As $\Delta(G - e) \leq \Delta(G)$, it follows that $b(G - e) \leq \Delta(G) + 1 = b(G)$, a contradiction.

(ii) Assume that G contains a simplicial vertex x. Let $e = xy$ be the removed edge from G such that y is any neighbor of x. Consider a b-coloring c of $G - e$ with k colors and set $k = b(G - e)$. Vertices x, y have the same color, otherwise c remains a b-coloring of G with k colors and thus $b(G) \geq k$, a contradiction. Consequently, any b-vertex of c has a neighbor (different from x) of color $c(x)$. If $\omega(G) \geq k$, then (2.1) implies that $b(G) \geq \omega(G) \geq k$, a contradiction. If $\omega(G) < k$, then $d_G(x) \leq \omega(G) - 1 < k - 1$. Therefore, one can recolor x by a missing color in its neighborhood. But then c remains a b-coloring of G with k colors, a contradiction.

(iii) This follows immediately from Proposition 2.3 and item (i) of Proposition 4.1. $\square$

The following corollary is immediate.

Corollary 4.2. *Chordal graphs are not b^+-edge critical graphs.*

It was shown in [6] that d-regular graphs for $d \leq 6$ with girth at least 5, different from the Petersen graph, have b-chromatic number $d + 1$; so Proposition 4.1 item (i) implies the next corollary.

Corollary 4.3. *If G is a d-regular graph with girth $g(G) \geq 5$, different from the Petersen graph, and with $d \leq 6$, then G is not b^+-edge critical.*

Proposition 4.4. *Petersen graph and $K_3 \square K_3$ are b^+-edge critical graphs.*

Proof. Let P be the Petersen graph with vertices x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 such that x_1, x_2, x_3, x_4, x_5 induce a 5-cycle in this order, y_1, y_3, y_5, y_2, y_4 induce a

5-cycle in this order, and x_iy_i is an edge for each $i \in \{1, \dots 5\}$. Let $K_3 \Box K_3$ be the cartesian product of two complete graphs of order 3 with vertices as in the proof of Proposition 3.6. By Theorem 2.1 and Proposition 2.4, we have $b(P) = b(K_3 \Box K_3) = 3$. By assigning color 1 to x_2, y_4, y_5, color 2 to x_5, y_2, y_3, color 3 to x_4, and color 4 to x_1, x_3, y_1, we obtain a b-coloring of $P - x_1y_1$ with 4 colors in which y_4, x_5, x_4, x_3 are b-vertices. Also, by assigning color 1 to x_{31}, x_{23}, color 2 to x_{21}, x_{32}, color 3 to x_{12}, x_{33} and color 4 to x_{11}, x_{13}, x_{22}, we obtain a b-coloring of $K_3 \Box K_3 - x_{11}x_{13}$ with 4 colors in which $x_{31}, x_{32}, x_{33}, x_{11}$ are b-vertices. Since all edges of P (respectively, $K_3 \Box K_3$) play the same role, it follows that $b(P - e) \geq 4$ (respectively, $b(K_3 \Box K_3 - e) \geq 4$) for any edge of P (respectively, $K_3 \Box K_3$). Thus Petersen graph and $K_3 \Box K_3$ are b^+-edge critical graphs. $\Box$

Now, in contrast, we show that graphs depicted in Figure 1, except Petersen graph, are not b^+-edge critical graphs. Before presenting this result, we recall some additional definitions and known results.

Remark first that if a graph G admits a b-coloring with k colors, then G has at least k vertices of degree at least $k-1$. Irving and Manlove [14,21] define the m-degree $m(G)$ of G to be the largest integer t such that G has at least t vertices of degree at least $t-1$. Thus every graph G satisfies the following.

Proposition 4.5 ([14,21])**.** *$b(G) \leq m(G)$.*

In [9], the author gave the following definition.

Definition 4.6 ([9])**.** Let $G = (V, E)$ be a graph with n vertices, and $(x_1, x_2, \dots, x_n)$ an ordering of V giving a nonincreasing sequence of degrees (i.e., if d_i is the degree of x_i, we have $d_1 \geq d_2 \geq \dots \geq d_n$). If we delete in this ordering every vertex x_i such that there exists $j < i$ with $N(x_i) \subset N(x_j)$, the nonincreasing sequence of remaining degrees is called the modified degree sequence of G.

It was noted in [9] that the set $(x_{i_1}, \dots, x_{i_k})$ obtained after deletion of subordinate vertices may depend on the initial ordering, but the sequence of degrees $d'_1 \geq \dots \geq d'_k$ in which d'_j is the degree of x_{i_j}, is the same for any choice of the initial ordering.

Using the notion of modified degree sequence, T. Faik ([9]) has introduced a new parameter, denoted $m'(G)$, giving a bound for $b(G)$ improving the bound of Proposition 4.5.

Definition 4.7 ([9])**.** Let $d'_1 \geq \dots \geq d'_k$ be the modified degree sequence of a graph G. Then $m'(G) = max\{i : d'_i \geq i - 1\}$.

Proposition 4.8 ([9])**.** *$b(G) \leq m'(G)$.*

Proposition 4.9. *F_1, F_2 and $K_{3,3}$ are not b^+-edge critical graphs.*

Proof. Let $G = F_1$ or F_2 or $K_{3,3}$ and e be an edge of G as shown in the Figure 1. If $G = F_1$ or $K_{3,3}$, then one can verify easily that $m'(F_1 - e) = 3$ and $m'(K_{3,3} - e) = 2$, and so by Proposition 4.8, we have $b(F_1 - e) \leq 3$ and $b(K_{3,3} - e) \leq 2$. Suppose now that $G = F_2$ and let $x_1, x_2, x_3, x_4, x_5, x_6$ be the vertices of G as shown in Figure 1. Then $m(G - e) = 4$ because $G - e$ has four vertices of degree three and two vertices of degree two. Hence, according to Proposition 4.5, we have $b(G - e) \leq 4$. We claim that $b(G - e) \neq 4$. Suppose not; hence clearly, for any b-coloring c of $G - e$ with 4 colors, x_1, x_2, x_3, x_4 are the unique b-vertices of c. Therefore, suppose without loss of generality that $c(x_i) = i$ for each i in $\{1, 2, 3, 4\}$. Since x_1 needs all colors on its neighbors, x_5 must be colored with the color 4, which is not possible as x_5 is adjacent to x_4 and $c(x_4) = c(x_5)$; so $b(G - e) \leq 3$. Thus, in view of Theorem 2.1, F_1, F_2 and $K_{3,3}$ are not b^+-edge critical graphs. □

Using Theorem 2.1, Proposition 4.1 item (i) and Propositions 4.4 and 4.9, we conclude the following.

Corollary 4.10. *The Petersen graph is the only connected cubic graph that is b^+-edge critical.*

Proposition 4.11. *Let G_1, G_2 be two graphs such that for $i = 1, 2$, G_i is in $\{P, K_3 \square K_3\}$. Then $G_1 \vee G_2$ is b^+-edge critical graph.*

Proof. Let $G = G_1 \vee G_2$. Setting $E_1 = E(G_1) \cup E(G_2)$ and $E_2 = E(G_1 \vee G_2) \setminus (E(G_1) \cup E(G_2))$. So $E(G) = E_1 \cup E_2$. Since all edges of E_1 (respectively, E_2) play the same role, there are two types of edges to consider. Let e be any edge of G.

Case 1. $e \in E_1$. Then for $i, j \in \{1, 2\}, (j \neq i)$, $b(G - e) = b((G_i - e) \vee G_j)$. By virtue of Lemma 2.2, we have $b(G - e) = b(G_i - e) + b(G_j)$, and by Proposition 4.4, we get $b(G - e) > b(G_i) + b(G_j) = b(G)$.

Case 2. $e \in E_2$. Suppose first that G_1 and G_2 are two copies of Petersen graphs. For $i = 1, 2$, let x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, y_1^i, y_2^i, y_3^i, y_4^i, y_5^i be the vertices of G_i defined as in the proof of Proposition 4.4. For $i = 1, 2$, let c_i be a proper coloring of G_i with 4 colors defined as follows. Assign color 1 to x_3^1, y_4^1, y_5^1, color 2 to x_1^1, x_4^1, y_3^1, color 3 to x_5^1, y_1^1, y_2^1, and color 4 to x_2^1. For the second copy, set $c_2(x_2^2) = c_1(x_2^1)$ and $c_2(x_j^2) = 4 + c_1(x_j^1)$, $c_2(y_j^2) = 4 + c_1(y_j^1)$ for each $j \in \{1, 3, 4, 5\}$. By combining c_1 and c_2 we obtain a b-coloring of $G - x_2^1x_2^2$ with 7 colors in which $x_2^1, y_3^1, y_4^1, y_1^1, y_1^2, y_3^2, y_4^2$ are b-vertices. Thus up to symmetry $b(G - e) \geq 7$ for any edge $e \in E_2$. According to Theorem 2.1 and Lemma 2.2, $b(G) = b(G_1) + b(G_2) = 6$. Therefore $b(G) < b(G - e)$.

Suppose now that G_1 is a Petersen graph and G_2 is the cartesian product of two cliques K_3 where G_1 and G_2 are defined as in the proof of Propositions 4.4 and 3.6, respectively. For $i = 1, 2$, let c_i be a proper coloring of G_i with 4 colors defined as follows. Assign color 4 to x_{12}, color 5 to x_{11}, x_{22}, x_{33}, color 6 to x_{13}, x_{21}, x_{32} and color 7 to x_{23}, x_{31}. Combination of c_1 and c_3 give a b-coloring of $G - x_2^1x_{12}$ with 7 colors such that $x_2^1, y_3^1, y_4^1, y_1^1, x_{23}, x_{32}, x_{33}$ are b-vertices. Up to symmetry, $b(G - e) \geq 7$ for any edge $e \in E_2$. According to Theorem 2.1, Lemma 2.2 and Proposition 2.4, we have $b(G) = b(G_1) + b(G_2) = 6$. Thus $b(G) < b(G - e)$ for any edge $e \in E_2$.

Finally, suppose that G_1 and G_2 are two copies of the cartesian product of two cliques K_3. For $i = 1, 2$, let x_{11}^i, x_{12}^i, x_{13}^i, x_{21}^i, x_{22}^i, x_{23}^i, x_{31}^i, x_{32}^i, x_{33}^i be the vertices of G_i with 5 colors defined as in the proof of Proposition 3.6. For $i = 1, 2$, let c_i be a proper coloring of G_i defined as follows. Assign color 1 to x_{13}^1, x_{22}^1, color 2 to x_{23}^1, x_{31}^1, color 3 to x_{12}^1, x_{33}^1, color 4 to x_{21}^1, x_{32}^1 and color 9 to x_{11}^1. For the second copy, color the vertex x_{11}^2 by 9, and for the remaining vertices, set $c_2(x_{ij}^2) = c_1(x_{ij}^1) + 4$

where i, j are in $\{1,2,3\}$ and $(i,j) \neq (1,1)$. By combining c_1 and c_2 we obtain a b-coloring of $G - x^1_{11}x^2_{11}$ with 9 colors in which $x^1_{11}, x^1_{22}, x^1_{23}, x^1_{32}, x^1_{33}, x^2_{22}, x^2_{23}, x^2_{32}, x^2_{33}$ are b-vertices. Thus up to symmetry $b(G-e) \geq 9$ for any edge $e \in E_2$. According to Theorem 2.1, Lemma 2.2 and Proposition 2.4, we have $b(G) = b(G_1) + b(G_2) = 6$. Therefore $b(G) < b(G-e)$ for any edge $e \in E_2$.

Thus in either case, we have $b(G) < b(G-e)$ for any edge $e \in E$ implying that G is b^+-edge critical. □

Using Propositions 4.4 and 4.11, we obtain the next result.

Corollary 4.12. *Let $G_1, G_2, \ldots, G_k$ $(k \geq 3)$ be k graphs such that each of them is a Petersen graph or the cartesian product of two cliques K_3. Then $G_1 \vee G_2 \vee \ldots \vee G_k$ is b^+-edge critical graph.*

Proof. Setting $G = G_1 \vee G_2 \vee \ldots \vee G_k$ and let $e = uv$ be any edge of G. If $u, v \in V(G_i)$, $(1 \leq i \leq k)$, then

$$b(G-e) = b((G_i - e) \vee (G \setminus G_i)).$$

Lemma 2.2 implies that

$$b(G-e) = b(G_i - e) + b(G \setminus G_i),$$

and by Proposition 4.4, we get

$$b(G-e) > b(G_i) + b(G \setminus G_i) = b(G).$$

If $u \in V(G_i)$ and $v \in V(G_j)$, $(1 \leq i \neq j \leq k)$, then

$$b(G-e) = b(G_i \vee G_j - e) \vee (G \setminus G_i \vee G_j)).$$

By virtue of Lemma 2.2,

$$b(G-e) = b(G_i \vee G_j - e) + b(G \setminus G_i \vee G_j),$$

and by Proposition 4.11, we have

$$b(G-e) > b(G_i \vee G_j) + b(G \setminus G_i \vee G_j) = b(G).$$

Thus G is b^+-edge critical graph. □

The previous results motivate the following problems.

Problem 4.13. Is it true that a graph G is b^+-vertex critical graph if and only if $G = K_3 \square K_3$ or it is the join of $k \geq 2$ copies of $K_3 \square K_3$?

Problem 4.14. Is it true that a graph G is b^+-edge critical graph if and only if $G = P$, $K_3 \square K_3$, or the join of several graphs such that each of them is P or $K_3 \square K_3$.

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