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LAGUERRE POLYNOMIALS APPLICATION FOR EXPANDING FUNCTIONS IN THE SERIES BY THESE POLYNOMIALS

Abstract

Introduction and aim: Selected elementary material about Laguerre polynomials have been shown in the paper. The algorithm of expanding functions in the series by Laguerre polynomials has been elaborated in the paper.
Material and methods: The selected knowledge about Laguerre polynomials have been taken from the right literature. The analytical method has been used in this paper.
Results: Has been shown the theorem describing expanding functions in a series by using Laguerre polynomials. It have been shown selected examples of expanding functions in a series by applying Laguerre polynomials, e.g. functions z^k and exp(-az).
Conclusion: The function f(z) can be expand in the interval $\langle 0,+\infty)$ in a series according to Laguerre polynomials where the unknown coefficients can be determined from the orthogonality of Laguerre polynomials

Keywords: Laguerre polynomials, function of complex variable, expanding functions in a series by using Laguerre polynomials.

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ZASTOSOWANIE WIELOMIANÓW LAGUERRE'A DO ROZWIJANIA FUNKCJI W SZEREGI WEDŁUG TYCH WIELOMIANÓW

Streszczenie

Wstęp i cel: *W pracy pokazuje się wybrane podstawowe wiadomości o wielomianach Laguerre'a. W artykule opracowano algorytm rozwijania funkcji w szereg według wielomianów Laguerre'a.*
Materiał i metody: *Wybrane wiadomości o wielomianach Laguerre'a zaczerpnięto z literatury przedmiotu. W pracy zastosowano metodę analityczną.*
Wyniki: *W pracy pokazano twierdzenie dotyczące rozwijania funkcji w szereg według wielomianów Laguerre'a. Pokazano wybrane przykłady rozwijania funkcji w szereg według wielomianów Laguerre'a m.in. funkcji* z^k *i exp(-az).*
Wniosek: *Funkcja f(z) może być w przedziale* $\langle 0,+\infty)$ *rozwinięta w szereg według wielomianów Laguerre'a, gdzie nieznane współczynniki można wyznaczyć korzystając z ortogonalności wielomianów Laguerre'a.*

Słowa kluczowe: *Wielomiany Laguerre'a, funkcja zmiennej zespolonej, rozwijanie funkcji w szereg według wielomianów Laguerre'a.*

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1. Introduction

Definition 1.

Lauguerre polynomials $L_n^a(z)$ for complex variable z and for the real values $a > -1$ have the following form [1]-[5]:

$$L_n^a(z) = \exp(z)\frac{1}{n!z^a}\frac{d^n}{dz^n}[\exp(-z)z^{n+a}] \quad \text{for} \quad n = 0,1,2,\ldots. \tag{1}$$

Laguerre polynomials calculated directly from definition (5) provide the following system of functions [1]-[5]:

$$L_0^a(z) = 1, \tag{2}$$

$$L_1^a(z) = 1 + a - z, \tag{3}$$

$$L_2^a(z) = \tfrac{1}{2}[(a+1)(a+2) - 2(a+2)z + z^2], \tag{4}$$

....

$$L_{n-1}^a(z) = \exp(z)\frac{1}{(n-1)!z^a}\frac{d^{n-1}}{dz^{n-1}}[\exp(-z)z^{n-1+a}], \tag{5}$$

$$L_n^a(z) = \exp(z)\frac{1}{n!z^a}\frac{d^n}{dz^n}[\exp(-z)z^{n+a}], \tag{6}$$

....

and so on.

Theorem 1. *(Generating function)*

Function

$$w(z,t) = \frac{1}{(1-t)^{a+1}}\cdot\exp\left(\frac{-zt}{1-t}\right), \quad a > -1 \tag{7}$$

is the generating function for Laguerre polynomials, i.e. for the values $|t| < 1$, there is an expanding in the series [1]:

$$w(z,t) = \frac{1}{(1-t)^{a+t}}\cdot\exp\left(\frac{-zt}{1-t}\right) \equiv \sum_{n=0}^{\infty} L_n^a(z)\, t^n dz \quad \text{for} \quad a > -1. \tag{8}$$

Theorem 2. *(The recurrence equation for Laguerre polynomials)*

If $L_{n-1}^a(z), L_n^a(z)$ and $L_{n+1}^a(z)$ are Laguerre polynomials, then [1]:

$$(n+1)L_{n+1}^a(z) + (z - a - 2n - 1)L_n^a(z) + (n+a)L_{n-1}^a(z) = 0 \tag{9}$$

for $a > -1$ and n = 1, 2, … .

Definition 2. *(gamma-Euler function).*

Gamma-Euler function for any complex values z define in the following form:

$$\Gamma(z) \equiv \int_0^\infty \exp(-t) \cdot t^{z-1} dt, \quad \mathrm{Re}(z) > 0 \tag{10}$$

where R(z) means the real part of complex number z [1].

Function gamma-Euler can be written in the other following form [2]-[5]:

$$\Gamma(z) \equiv \int_0^1 \exp(-t) \cdot t^{z-1} dt + \int_1^\infty \exp(-t) \cdot t^{z-1} dt. \tag{11}$$

Theorem 3. *(Properties of the gamma-Euler function)*

Function gamma-Euler has the following properties [2]-[5]:

$$\Gamma(z+1) = z\Gamma(z), \tag{12}$$

$$\Gamma(z)\Gamma(z-1) = \frac{\pi}{\sin(\pi z)}, \tag{13}$$

$$2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \sqrt{\pi}\Gamma(2z). \tag{14}$$

2. Epanding functions in the series by using Laguerre polynomials

One of the most important properties of Laguerre polynomials is the ability to show any function f (z) defined in the interval $\langle 0,+\infty)$ in a series [1]-[5]:

$$f(z) = \sum_{n=0}^{\infty} c_n L_n^a(z) \tag{15}$$

where $0 < z < +\infty$.

The coefficients c_n can be determined from orthogonality of Laguerre polynomials. Let us multiply both sides of the equality (15) by the expression $L_m^a(z)$:

$$f(z)L_m^a(z) = \sum_{n=0}^{\infty} c_n L_n^a(z) L_m^a(z) \tag{16}$$

where $0 < z < +\infty$. Both sides of the equality (16) we multiply by the expression $\exp(-z)z^a$:

$$f(z)\exp(-z)z^a L_m^a(z) = \sum_{n=0}^{\infty} c_n \exp(-z)z^a L_n^a(z) L_m^a(z) \tag{17}$$

where $0 < z < +\infty$.

The equation (17) we integrate in the interval $\langle 0,+\infty)$ with respect to variable z:

$$\int_0^{+\infty} f(z)\exp(-z)z^a L_m^a(z)\, dz = \sum_{n=0}^{\infty} c_n \int_0^{+\infty} \exp(-z)z^a L_n^a(z) L_m^a(z)\, dz\,. \tag{18}$$

When m = n the formula (18) has the following form:

$$\int_0^{+\infty} f(z)\exp(-z)z^a L_m^a(z)\,dz = \sum_{n=0}^{\infty} c_n \int_0^{+\infty} \exp(-z)z^a [L_n^a(z)]^2\,dz\,. \tag{19}$$

Now we calculate the integral located under the sign of the sum on the right side of the equation (13). For this purpose, in recurrence equation (9) we replace n with n–1 and we get:

$$nL_n^a(z) + (z-a-2n+1)L_{n-1}^a(z) + (n+a-1)L_{n-2}^a(z) = 0 \tag{20}$$

for n = 2, 3, … . We multiply the obtained equation (20) by $L_n^a(z)$, then

$$n[L_n^a(z)]^2 + (z-a-2n+1)L_{n-1}^a(z)L_n^a(z) + (n+a-1)L_{n-2}^a(z)L_n^a(z) = 0 \tag{21}$$

dla n = 2, 3, … . Now the both sides of the equation (9) we multiply by the expression $L_{n-1}^a(z)$. Then we have:

$$(n+1)L_{n+1}^a(z)L_{n-1}^a(z) + (z-a-2n-1)L_{n-1}^a(z)L_n^a(z) + (n+a)[L_{n-1}^a(z)]^2 = 0 \tag{22}$$

for n = 2, 3, … . From the equation (21) we subtract by sides the equation (22) and we get:

$$\begin{aligned} &n[L_n^a(z)]^2 + (z-a-2n+1)L_{n-1}^a(z)L_n^a(z) + (n+a-1)L_{n-2}^a(z)L_n^a(z) + \\ &-(n+1)L_{n+1}^a(z)L_{n-1}^a(z) - (z-a-2n-1)L_{n-1}^a(z)L_n^a(z) - (n+a)[L_{n-1}^a(z)]^2 = 0 \end{aligned} \tag{23}$$

for n = 2, 3, … . We multiply the above equation (22) by expression $z^a\exp(-z)$, then

$$\begin{aligned} &nz^a\exp(-z)[L_n^a(z)]^2 + (z-a-2n+1)z^a\exp(-z)L_{n-1}^a(z)L_n^a(z) + \\ &+(n+a-1)z^a\exp(-z)L_{n-2}^a(z)L_n^a(z) - (n+1)z^a\exp(-z)L_{n+1}^a(z)L_{n-1}^a(z) + \\ &-(z-a-2n-1)z^a\exp(-z)L_{n-1}^a(z)L_n^a(z) - (n+a)z^a\exp(-z)[L_{n-1}^a(z)]^2 = 0 \end{aligned} \tag{24}$$

After reduction expressions we have:

$$\begin{aligned} &nz^a\exp(-z)[L_n^a(z)]^2 + 2z^a\exp(-z)L_{n-1}^a(z)L_n^a(z) + (n+a-1)z^a\exp(-z)L_{n-2}^a(z)L_n^a(z) + \\ &-(n+1)z^a\exp(-z)L_{n+1}^a(z)L_{n-1}^a(z) - (n+a)z^a\exp(-z)[L_{n-1}^a(z)]^2 = 0 \end{aligned} \tag{25}$$

Next we integrate the equation (25) in the interval $\langle 0,+\infty)$ with respect to variable z.

$$\begin{aligned} &n\int_0^{+\infty} z^a\exp(-z)[L_n^a(z)]^2dz + 2\int_0^{+\infty} z^a\exp(-z)L_{n-1}^a(z)L_n^a(z)\,dz + \\ &+(n+a-1)\int_0^{+\infty} z^a\exp(-z)L_{n-2}^a(z)L_n^a(z)\,dz + \\ &-(n+1)\int_0^{+\infty} z^a\exp(-z)L_{n+1}^a(z)L_{n-1}^a(z)\,dz - (n+a)\int_0^{+\infty} z^a\exp(-z)[L_{n-1}^a(z)]^2dz = 0 \end{aligned} \tag{26}$$

Due to the orthogonality of Laugerre polynomials, integrals containing the product of polynomials with different indexes are equal zero. Thus the equation (26) has the following form:

$$n\int_0^{+\infty} z^a \exp(-z)[L_n^a(z)]^2\,dz-(n+a)\int_0^{+\infty} z^a \exp(-z)[L_{n-1}^a(z)]^2\,dz=0 \tag{27}$$

for n = 2, 3, …, n .

By using formula (27) successively for n = 2, 3, ..., n we get the following sequence of equalities:

$$\int_0^{+\infty} z^a \exp(-z)[L_2^a(z)]^2\,dz=\frac{2+a}{2}\int_0^{+\infty} z^a \exp(-z)[L_1^a(z)]^2\,dz, \tag{28}$$

$$\int_0^{+\infty} z^a \exp(-z)[L_3^a(z)]^2\,dz=\frac{(3+a)(2+a)}{3\cdot 2}\int_0^{+\infty} z^a \exp(-z)[L_1^a(z)]^2\,dz, \tag{29}$$

. . . .

$$\int_0^{+\infty} z^a \exp(-z)[L_n^a(z)]^2\,dz=\frac{(n+a)(n+a-1)\cdot\ldots\cdot(a+3)(a+2)}{n(n-1)(n-2)\cdot\ldots\cdot 3\cdot 2}\int_0^{+\infty} z^a \exp(-z)[L_1^a(z)]^2\,dz. \tag{30}$$

We use the following formula:

$$\int_0^{+\infty} z^a \exp(-z)[L_n^a(z)]^2\,dz=\frac{\Gamma(n+a+1)}{n!} \tag{31}$$

for a>–1 and n = 0, 1, … . Function gamma-Euler is defined by the formula (10).

Returning to equality (19) we have that

$$\int_0^{+\infty} f(z)\exp(-z)z^a L_n^a(z)\,dz=c_n\,\frac{\Gamma(n+a+1)}{n!}. \tag{32}$$

Therefore:

$$c_n=\frac{n!}{\Gamma(n+a+1)}\int_0^{+\infty} f(z)\exp(-z)z^a L_n^a(z)\,dz. \tag{33}$$

for a>–1 and n = 0, 1, … .

<u>Theorem 4</u>.

Let f (z) be an arbitrary function of the variable z defined in the interval ⟨0,+∞) and satisfying the conditions [1]:

- f(z) in the intervals is a smooth function (i.e., has a continuous derivative in the intervals) in the finite interval (0,+a),
- the integral $\int_0^{+\infty} \exp\left(-\frac{z}{2}\right) z^{\frac{a}{2}-\frac{1}{4}}\,|f(z)|\,dz$ has a finite value,

then the series

$$f(z)=\sum_{n=0}^{\infty} c_n L_n^a(z) \quad \text{for} \quad 0<z<+\infty \tag{34}$$

where

$$c_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{+\infty} f(z)\exp(-z)z^a L_n^a(z)\,dz \tag{35}$$

is convergent and its sum is the function f(z) at each point z which given function is continuity [1]. Proof of that theorem is possible to see in [1].

3. Examples of expanding functions in the series by using Laguerre polynomials

Example 1.

Let us consider the following function [1]:

$$f(z) = z^k. \tag{36}$$

The assumptions of theorem (4) will be met when the exponent $k > -\frac{1}{2}(1+a)$. We show that the function (36) expands in the following series:

$$z^k = \sum_{n=0}^{\infty} c_n L_n^a(z) \tag{37}$$

where

$$c_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{+\infty} \exp(-z)\, z^{a+k} L_n^a(z)\,dz. \tag{38}$$

Using the definition (1) now we have that:

$$c_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{+\infty} \left\{ \exp(-z)\, z^{a+k} \exp(z) \frac{1}{n! z^a} \frac{d^n}{dz^n}\left[\exp(-z)\, z^{n+a}\right] \right\} dz, \tag{39}$$

$$c_n = \frac{1}{\Gamma(n+a+1)} \int_0^{+\infty} z^k \frac{d^n}{dz^n}\left[\exp(-z)\, z^{n+a}\right] dz. \tag{40}$$

Using the formula for integration by parts n times we get that:

$$c_n = \frac{(-1)^n k(k-1)(k-2)\cdot \ldots \cdot (k-n+1)}{\Gamma(n+a+1)} \int_0^{+\infty} \exp(-z)\, z^{a+k}\,dz. \tag{41}$$

Given the definition (2) and formula Γ(n +1) = n! we have:

$$c_n = \frac{(-1)^n \Gamma(k+a+1)\cdot\Gamma(k+1)}{\Gamma(n+a+1)\cdot\Gamma(k-n+1)}. \tag{42}$$

Returning to expansion of the function (36) in a series according to Laguerre polynomials, we substitute the calculated coefficients c_n in (37) finally we have:

$$z^k = \Gamma(k+a+1)\cdot\Gamma(k+1) \sum_{n=0}^{\infty} \frac{(-1)^n L_n^a(z)}{\Gamma(n+a+1)\cdot\Gamma(k-n+1)} \tag{43}$$

for z>0 and a>–1.

Example 2.

We have to expand in series according to Laguerre polynomials the function [1]:

$$f(z) = \exp(-az) \tag{44}$$

where a is any real or complex number.

The function (44) we expand in the series:

$$f(z) = \sum_{n=1}^{+\infty} c_n L_n^a(z). \tag{45}$$

where c_n coefficients are calculated from the formula:

$$c_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{+\infty} \exp[-(a+1)z] z^a L_n^a(z)\, dz. \tag{46}$$

Using the definition (1) we obtain:

$$c_n = \frac{n!}{\Gamma(n+a+1)} \int_0^{+\infty} \left\{ \exp[-(a+1)z] z^a \exp(z) \frac{1}{n! z^a} \frac{d^n}{dz^n} \left[\exp(-z) z^{n+a} \right] \right\} dz, \tag{47}$$

$$c_n = \frac{1}{\Gamma(n+a+1)} \int_0^{+\infty} \left\{ \exp(-az) \frac{d^n}{dz^n} \left[\exp(-z) z^{n+a} \right] \right\} dz, \tag{48}$$

for n = 0, 1, … .

Using the formula for integration by parts n times we get that:

$$c_n = \frac{a^n}{\Gamma(n+a+1)} \int_0^{+\infty} \exp[-(a+1)z] z^{n+a} dz \tag{49}$$

for n = 0, 1, … .

Given the definition (2) and formula Γ(n +1) = n! we have:

$$c_n = \frac{(-1)^n \Gamma(k+a+1) \cdot \Gamma(k+1)}{\Gamma(n+a+1) \cdot \Gamma(k-n+1)}. \tag{42}$$

We continue to consider the definition of gamma-Euler functions and we have:

$$c_n = \frac{a^n \Gamma(n+a+1)}{\Gamma(n+a+1)(a+1)^{n+a+1}} = \frac{a^n}{(a+1)^{n+a+1}}. \tag{43}$$

Returning to expansion of the function (44) in a series according to Laguerre polynomials, we substitute the calculated coefficients c_n in (43) finally we have:

$$\exp(-az) = \sum_{n=0}^{\infty} \frac{a^n}{(a+1)^{n+a+1}} \cdot L_n^a(z) \tag{44}$$

$$\exp(-az) = \frac{1}{(a+1)^{a+1}} \sum_{n=0}^{\infty} \left(\frac{a}{a+1} \right)^n \cdot L_n^a(z) \tag{45}$$

for 0< z < ∞ and a>−1.

Example 3.

In this example we show how the function [1]:

$$f(z) = \exp(-az), \quad a > -1 \tag{46}$$

expand in series according to Laguerre polynomials.

We only use the generating function for these polynomials given in the form (7).

We choose substitution:

$$t = \frac{a}{a+1} \tag{47}$$

in the formula (8):

$$-\frac{1}{\left(\frac{a}{a+1}-1\right)^{a+t}} \cdot \exp\left(\frac{-\frac{a}{a+1}z}{1-\frac{a}{a+1}}\right) \equiv \sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^{n} L_n^a(z)dz, \quad |t|<1, \tag{48}$$

$$(a+1)^{a+1} \cdot \exp(-az) \equiv \sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^{n} L_n^a(z)dz, \quad |t|<1. \tag{49}$$

Finally the expansion of the function (46) has the following form:

$$\exp(-az) = \frac{1}{(a+1)^{a+1}}\sum_{n=0}^{\infty}\left(\frac{a}{a+1}\right)^{n} \cdot L_n^a(z) \tag{50}$$

for $0< z < \infty$ and $a>-1$.

4. Conclusion

The function f(z) can be expand in the interval $\langle 0,+\infty)$ in a series according to Laguerre polynomials, i.e. $f(z) = \sum_{n=0}^{\infty} c_n L_n^a(z)$ where the unknown coefficients c_n can be determined from the orthogonality of Laguerre polynomials.

Literature

[1] Лебедев Н.Н.: *Специальные функци и их приложения*, Государственное Издательсто Физико-Математческой Литературы, Москва-Ленинград 1963, издание 2.

[2] Leja F.: *Complex functions*, Mathematical Library Volume 29. Warsaw: PWN, 1973.

[3] Meyberg K., Vachenauer P.: *Höhere Mathematik 1. Differential- und Integralrechnung. Vektor- und Matrizenrechnung.* Berlin Heidelberg New York Barcelona Hongkong London Mailand Paris Tokio: Springer, 2001, Sechste, korrigierte Auflage.

[4] Smirnow W.I.: *Higher mathematics, Volume 3, Part 2.* Warsaw: PWN, 1965.

[5] Whittaker E.T., Watson G.N.: *A course of modern analysis, Part 2.* Warsaw: PWN, 1968.