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The use of fractional order operators in modelling of RC electrical systems

by

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Abstract: The present study concerns the analysis of using the integro-differential fractional operators in the process of modelling of electrical systems. As the object of study, the RC circuit with the ultracapacitor was used. A mathematical model of the super capacitor has been introduced, based on the integro-differential fractional order operator. Throughout the modelling it was assumed that the ultracapacitor was ideal. The main goal of the work was to carry out the experimental tests. Static and dynamic characteristics of the RC circuit with the ultracapacitor were determined with a prepared laboratory test rig. The obtained experimental results were compared with simulation tests of the ultracapacitor dynamics. The analysis, modelling and the obtained results allowed for the assessment of applicability of the fractional order operators in modelling of electrical systems. The proposed fractional order model yielded more precise description of dynamic properties of the system.

Keywords: fractional order operators, ultracapacitor, identification

1. Introduction

The first and most important step, during control system design process, is the development of the appropriate and suitable mathematical model of the analyzed object. Such a model can be derived either from physical laws or from experimental data and the identification process. Compared with the classical integer models, fractional models provide often good tools for modeling and description of properties of various materials and processes,. A typical example of the object, which shows fractional behavior and which can be modeled in a better way by the non-integer operators is the voltage-current relation of a semi-infinite lossy transmission line (see Wang, 1987), or the diffusion of the heat into a semi-infinite solid, where heat flow is equal to the semi-derivative of temperature (see Podlubny et al., 1995). We can easily find more examples of this kind of fractional objects in various technical fields, like control engineering (see Das, 2008; Kaczorek, 2011; Ortiqueira, 2011), signal processing (see Matasu, 2012), system modeling (see Podlubny, 1999), electronics and electrical engineering (see Dzieliński et al., 2011; Sierociuk et al., 2013a, b) etc.

Considering the increase in practical use of the fractional integro-derivatives or fractional differences in systems modelling of real behaviors, there has recently been a growing interest in further development of this topic, from both theoretical and practical points of view. In modelling of real phenomena, the respective authors emphatically use generalizations of n-th order differences to their fractional forms, see, for example, Kaczorek (2011), Dzieliński et al. (2011), Buslowicz and Ruszewski (2013), Mozyrska and Pawluszewicz (2013). On the other hand, one can find different definitions of fractional derivatives, among which the most popular and used are Caputo-, Riemann-Liouville- and Grünwald-Letnikov-type operators, see, for example, Das (2008), Matasu (2012, Podlubny (1999), Chen et al. (2009), Ostalczyk (2008). Ferreira and Torres (2011) and Mozyrska et al. (2013) adopted the more general Riemann-Liouville, Caputo-, and Grünwald-Letnikov-type h difference operators. The presence of h in these operators is important from both engineering and numerical points of view. On the one hand, h represents a sample step, on the other – when h tends to zero, the solutions of the fractional difference equation may be seen as approximations to the solutions of corresponding fractional equations, see for example Mozyrska and Wyrwas (2015) or Podlubny (2002). A question that quite often arises when considering the practical use of the fractional differential or difference operators is what the operator should applied to. Usually the answer depends on the problem and the observed data. But, nevertheless, the right choice is not easy, see for example Dzieliński et al. (2011), or Sierociuk et al. (2013a).

The main goal of this study was to assess the usability of the fractional order operators in modeling of the electrical systems. The assessment was based on the experimental tests, their results being compared with those of the simulation studies of the RC bridges. The inspiration to this work was constituted by the results, presented in Dzieliński et al. (2011) and Sierociuk et al. (2013a).

2. Differential operators of fractional order

In the general case, a fractional order operator of order α, can be defined as (see Chen et al., 2009):

$$ {}_aD_t^{\alpha} := \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}} & Re\,\alpha > 0 \\ 1 & Re\,\alpha = 0 \\ \int_a^t (d\tau)^{-\alpha} & Re\,\alpha < 0 \end{cases} \tag{1} $$

It is usually assumed that α is a real number, but it could also be a complex one. For the purposes of this section, as the first, let us recall that Euler's

Gamma function $\Gamma : (0, +\infty) \to R$, the generalization of the classical factorial, is a function defined by the following expression:

$$\Gamma(t) := \int_0^\infty \tau^{t-1} e^{-\tau} d\tau \tag{2}$$

which convergers in the right half of the complex plane $Re(t) > 0$. Then, the Riemann-Liouville fractional order α, $\alpha \in [n-1, n]$, $n \in N$, differential operator is defined as (see, for example, Podlubny et al., 1995; Matasu, 2012, or Sierociuk et al., 2013a):

$$ {}_a^{RL}D_t^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & \alpha > 0 \\ \frac{1}{\Gamma(-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1}} d\tau, & \alpha < 0 \end{cases}. \tag{3}$$

The Laplace transform of this operator is given as (see, for example, Podlubny et al., 1995; Sierociuk et al., 2013a, or Ostalczyk, 2008):

$$L\left[{}_0^{RL}D_t^\alpha \tau f(t)\right] =$$

$$= \begin{cases} s^n F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0^{RL}D_t^{\alpha-k-1} \tau f(t)\right]_{t=0} & \text{for } \alpha > 0 \\ s^n F(s) & \text{for } \alpha < 0 \end{cases}. \tag{4}$$

The fractional order Grünwald–Letnikov operator of order $\alpha \in (n-1, n)$, $n \in N$, is defined as (see for example Das, 2008; Kaczorek; 2011 or Podlubny, 1999):

$$ {}_a^{GL}D_t^\alpha f(t) := \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh) \tag{5}$$

where $h > 0$ and $\begin{pmatrix} \alpha \\ j \end{pmatrix} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$, $j = 1, 2, 3, \ldots$, denotes the binomial coefficient. Note that ${}_0^{tRL}D_t^\alpha f(t) = {}_0^{tGL}D_t^\alpha f(t)$ for any continosly differentiable function f. The Laplace transform of Grünwald–Letnikov operator exists only for $\alpha \in (0, 1]$ and is given by (see Kaczorek, 2011; Podlubny; 1999 or Ostalczyk, 2008):

$$L\left[{}_0^{GL}D_t^\alpha \mathrm{t} f(t)\right] = s^n F(s). \tag{6}$$

As the third type of the fractional order differential operator let us consider the Caputo operator of order $\alpha \in (n-1, n]$, $n \in N$:

$$ {}_a^{C\alpha}D_t^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)^{(n)}}{(t-\tau)^{\alpha-n+1}} d\tau. \tag{7}$$

It can be shown (see Podlubny, 1999), that for $\alpha \to n$ the Caputo differential operator becomes a convential n-th order derivative of function f. The Laplace transform of the fractional Caputo operator of order $\alpha \in (n-1, n]$, $n \in N$, is given as (see, for example, Podlubny et al., 1995; Sierociuk et al., 2013a, or Ostalczyk, 2008):

$$L\left[{}_0^C D_t^\alpha f(t)\right] = s^n F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). \tag{8}$$

Geometrical and physical interpretations of this fact are problematic, but some relevant approach can be found in Podlubny (2002).

As this was mentioned before, in practice, the proper choice of fractional operator is not easy. It depends on the problem and on the observed data. In this context, note also that in the Laplace transform of the Caputo fractional order operator appears in initial values of classical integer-order derivatives only, while the Laplace transform of the Riemann–Liouville order operator leads to initial conditions containing the limit values of this operator, i.e. containing

$$\lim_{t\to 0} {}_0^{RL} D_t^{\alpha-1} \mathrm{t} f(t) = a_1, \ldots, {}_0^{RL} D_t^{\alpha-n} f(t) = a_n$$

where $a_1, \ldots, a_n$ are given constants.

Taking into account the ultracapacitor scheme with included resistance of the electrodes, see Fig. 2a, and the simplified circuit (first mesh), see Fig. 2b, that are the object of our studies, we are going to concentrate on the Riemann–Louville fractional operator, but we assume that $[{}_0^{RL} D_t^{\alpha-k-1} f(t)]_{t=0} = 0$, i.e. we assume zero initial conditions. The motivation of this choice originates from the fact that behavior and properties of the ith mesh are influenced by the behaviors of earlies meshes. Taking into account the experiment process, setting of zero initial conditions comes in a natural way.

3. The mathematical model of the ultracapacitor

The capacitor is a passive electrical device, which is generally used to store electrostatic potential energy. The fundamental parameter, which describes the capacitor is the capacity c, defined basically as:

$$c = \frac{q}{u}. \tag{9}$$

Capacitors are usually made use of in electronic systems as upper or down pass filters or in backup power systems. The new types of IT electronic devices need capacitors with a much higher capacitance. The solution proved to be constituted by the so-called ultracapacitors. They are some kind of electrolytic capacitors, which feature very high capacitance due to the use of a new material in their construction – graphene. Owing to low resistance, graphene turned out to be a perfect material for electrodes (see Wang, 1987). In addition, electrodes in ultracapacitors have porous surface, what causes that they exhibit good relation of the surface area to mass (about 1 000 m^2/g).

Based on the above information, we can create two mathematical models allowing for the description of the ultracapacitor. The first model assumes that

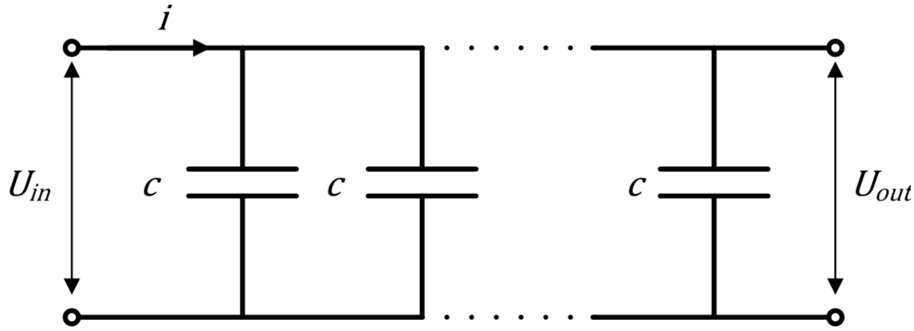

Figure 1. Ultracapacitor scheme with neglected resistance of the electrodes

the graphene electrodes have a negligible electrical resistance, so ultracapacitor is a battery of capacitors connected parallelly, as shown in Fig. 1.

Using the definition of the electric current, which can be written as:

$$i\left(t\right)=\frac{dq\left(t\right)}{dt}, \tag{10}$$

equation (9) can be written down in the following manner:

$$u\left(t\right)=\frac{1}{nc}\int i(t)dt. \tag{11}$$

Using the Laplace transform of both sides in eq. (11) we get:

$$U\left(s\right)=\frac{1}{ncs}I(s). \tag{12}$$

Assuming that the input signal is the current, charging the capacitor, and the output signal is the voltage, stored on the capacitor's covers, the transfer function model of this system can be written as follows:

$$G_C\left(s\right)=\frac{U_{in}(s)}{I(s)}=\frac{1}{ncs}=\frac{1}{Cs}, \tag{13}$$

where n is the number of capacitors connected parallelly, c is the capacitance of the ultracapacitor component, and C is the total capacitance of ultracapacitor.

The second model of the ultracapacitor assumes that the resistance of the electrodes is not negligible, and so it is built of n RC bridges, which are connected together (see Sierociuk, 2013a). The scheme of this model is shown in Fig. 2a.

Basing on the laws of Ohm and Kirchoff for the first mesh (denoted as I), presented in Fig. 2b, and taking the Laplace transform, we get:

$$\begin{cases} U_1\left(s\right)=rI_1\left(s\right)+U_2\left(s\right) \\ I_1\left(s\right)=I_2\left(s\right)+csU_2\left(s\right) \end{cases}. \tag{14}$$

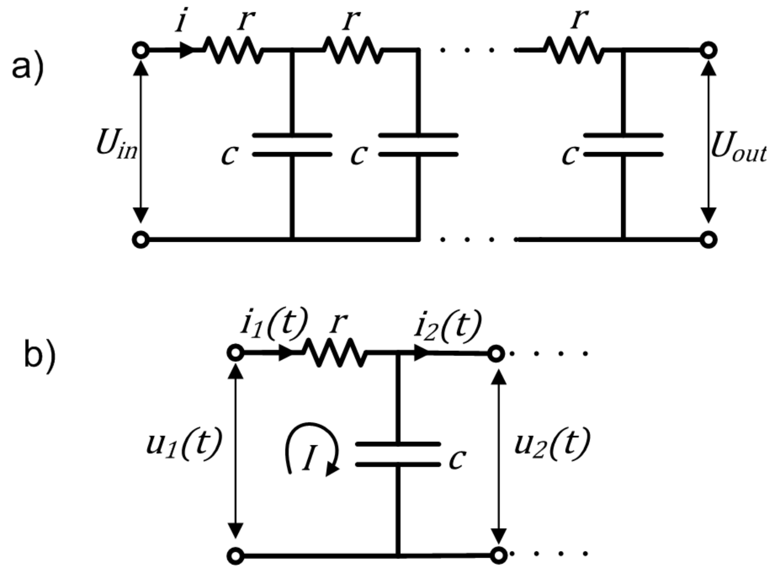

Figure 2. a) Ultracapacitor scheme with included resistance of the electrodes, b) the simplified circuit (first mesh)

Using eq. (14) and assuming that the input signal to the system is the current $i_1(t)$ and the output signal is $u_1(t)$, we can derive the transfer function model:

$$G(s)=\frac{U_1(s)}{I_1(s)}=r+\frac{U_2(s)}{I_2(s)+csU_2(s)}=r+\frac{1}{cs+\frac{1}{\frac{U_2(s)}{I_2(s)}}}. \tag{15}$$

Assuming that the number of the RC bridges is very large (tends to infinity), and taking into account eq. (15), the transfer function of the ultracapacitor in the general form can be described as follows:

$$G_C(s)=\frac{U_{in}(s)}{I(s)}=r+\frac{1}{cs+\frac{1}{r+\frac{1}{cs}+\frac{1}{r+\frac{1}{cs+\dots}}}}, \tag{16}$$

where r is the resistance of the electrode component and c is the capacitance of the ultracapacitor component.

Upon solving that fractal and assuming that resistance of electrodes is low, we get the transfer function with fractional order in the following form:

$$G_C(s)=\frac{U_{in}(s)}{I(s)}=\frac{R}{n}+\frac{1}{\sqrt{\frac{C}{R}}s^{0,5}}, \tag{17}$$

where R is the resistance of the electrodes, C is the capacitance of the ultracapacitor, and n is number of the RC bridges.

In view of the fact that many assumptions and simplifications were made, we have to introduce an amendment part into the transfer function (17), and so, finally, the transfer function of the ultracapacitor takes the form:

$$G_c(s) = \frac{U_{in}(s)}{I(s)} = \frac{R}{n} + \frac{1}{\sqrt{\frac{C}{R}} s^{\alpha}}, \tag{18}$$

where α is the parameter, satisfying the condition $0 < \alpha < 1$.

4. The analyzed object

As the research object, the RC bridge with ultracapacitor was used. The scheme of the bridge is as shown in Fig. 3.

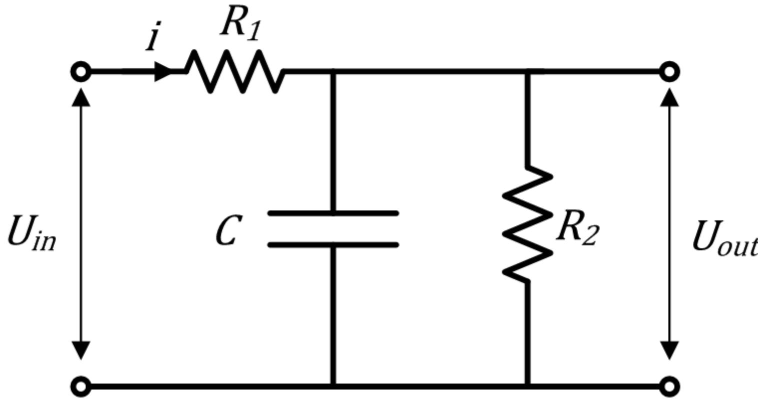

Figure 3. RC bridge scheme with ultracapacitor

Using Kirchoff's laws we got the transfer function of this study object for both models of the ultracapacitor. For the model without the resistance of electrodes (i.e. the one that uses eq. (13)) we get following transfer function:

$$G(s) = \frac{U_{out}(s)}{U_{in}(s)} = \frac{\frac{1}{1+\frac{R1}{R2}}}{\frac{R1C}{1+\frac{R1}{R2}} s + 1} = \frac{k}{Ts+1} \quad . \tag{19}$$

The model with resistance of the electrodes in the ultracapacitor, based on eq. (18), is presented below:

$$G(s) = \frac{U_{out}(s)}{U_{in}(s)} = \frac{\frac{R2R}{n(R1+R2)}\sqrt{\frac{C}{R}} s^{\alpha} + \frac{R2}{R1+R2}}{\left(\frac{R1R2+\frac{R}{n}(R1+R2)}{R1+R2}\right)\sqrt{\frac{C}{R}} s^{\alpha} + 1} = \frac{T_1 s^{\alpha} + k}{T_2 s^{\alpha} + 1} \quad . \tag{20}$$

5. Measurement system setup

In order to determine the properties of the analyzed RC bridge with ultracapacitor, an adequate measurement system was designed and prepared. The constructed measurement setup allows for generation of standard excitation signals, like step or sinusoidal signals, for purposes of determination of the experimental static and dynamic characteristics. In order to check the correctness of the models described by the transfer functions (19) and (20), we had to analyze Bode plots of the RC bridge, obtained experimentally. The measurement setup scheme used is presented in Fig. 4.

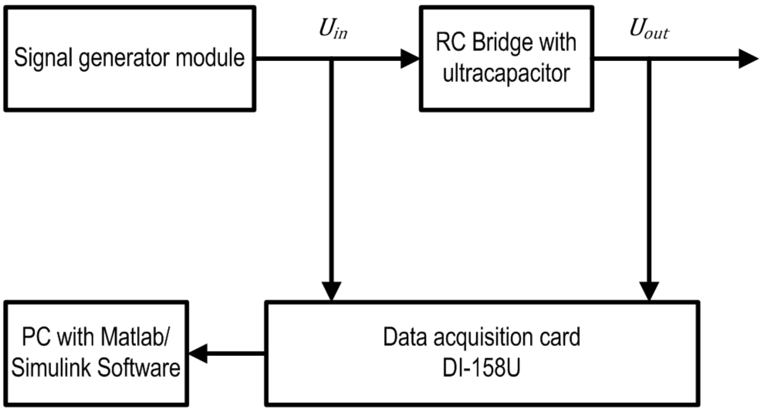

Figure 4. Measurement system scheme

The signal generator module was based on EVB 5.1 board with Atmega 32A microcontroller featuring 32kB of flash memory, 16MHz clock and 3 PWM outputs. This microcontroller enables generating sine wave signals modulated in frequency and amplitude by PWM. Signals were also amplified by the Darlington circuit. As the data acquisition block, the low cost, four channel DI-158U card was used with WinDaq and Matlab/Simulink as the recording and playback software. This equipment allowed for data monitoring and recording with 12 bits of measurement accuracy at the rates up to 14,400 samples/second.

In our studies we have analyzed capacitors produced by Panasonic company, having capacities, respectively, as follows: BUC 0.1F, EECSOHD334V 0.33F and BUC 1F. Finally, 100 Ω resistors were used for compensation of resistance of the connections.

6. Results

The measurement setup, described in the previous section, makes it possible to analyse the static characteristics of the examined plant and to consider the dynamic properties, regarding both frequency and time domain dynamic characteristics. The experimental characteristics, obtained for the RC bridge with various capacities are presented in Figs. 5 - 13. The real point characteristics were compared with responses from the mathematical models (transfer functions defined by eqs. (19) and (20)) for better visibility of the quality of integer and fractional order models. Figures 5 - 7 contain Bode plots of RC bridge with various capacities.

When analyzing Figs. 5 - 7 we can see that the model with included electrode resistance describes the real object better. In fact, the classical capacitor with less capacitance is described like the model corresponding to eq. (13), meaning that ultracapacitors have different properties than normal (standard) capacitors.

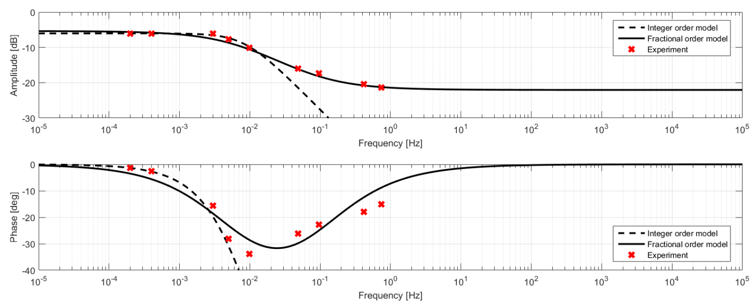

Figure 5. Bode plot of the RC bridge with 0.1 F ultracapacitor

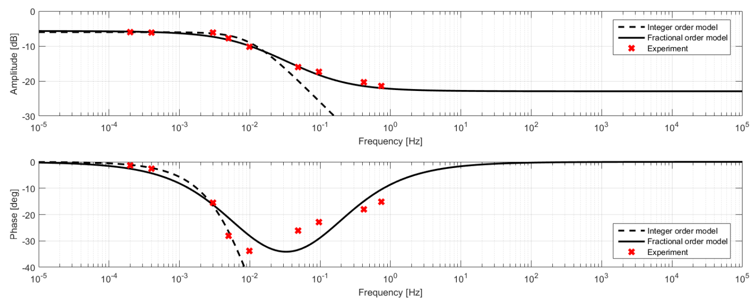

Figure 6. Bode plot of the RC bridge with 0.33 F ultracapacitor

In Figs. 8–10, the step responses of the RC bridge are shown. Also in this case, the fractional order model, when compared with experimental data, appears to reflect the real object dynamics better.

Additionally, the static characteristics of the RC bridge were determined and are presented in Figs. 11-13. This analysis allows for assessing the use of fractional order operators for expressing the behavior of the object in the steady states.

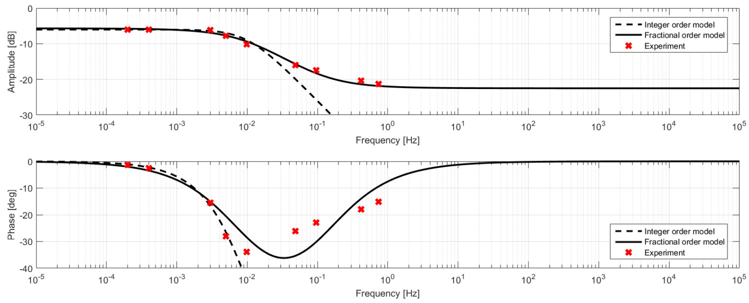

Figure 7. Bode plot of the RC bridge with 1 F ultracapacitor

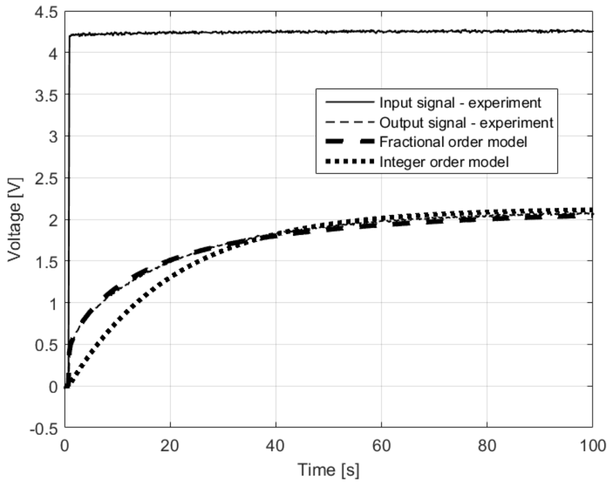

Figure 8. Step response of the RC bridge with 0.1 F capacitor

7. Conclusions

In this paper, we considered the use of the integro-differential fractional operator in the process of electrical systems modelling. The results, obtained from model building, were compared with those from experimental studies. Taking into account the nature of the here analysed model, we proposed to use the Riemann-Louville fractional operator. We assumed zero initial conditions. As the study object, the RC bridge with supercapacitor was used. This study object was described using the classical approach and the integer order operator, as well as using the relatively new mathematical tool - fractional order operators. The experiments performed show that the fractional order models express system dynamics better. As the identification method, the frequency response

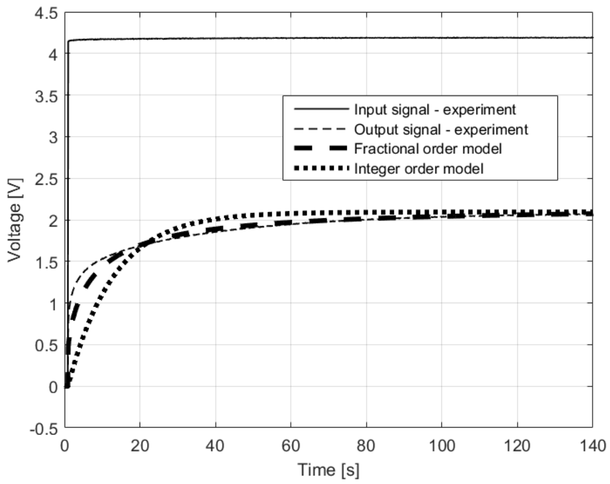

Figure 9. Step response of the RC bridge with 0.33 F capacitor

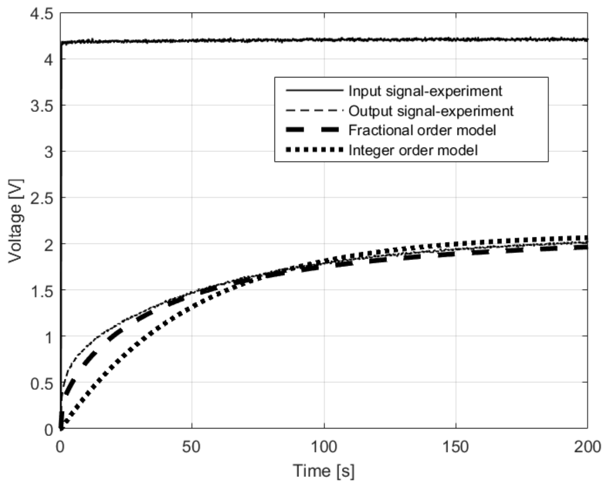

Figure 10. Step response of the RC bridge with 1 F capacitor

technique was used. Experimental test were carried out for the frequencies between 0.0001 Hz and 1 Hz. Thus, the obtained characteristics can be compared primarily within this frequency band. Nevertheless, taking into account only this part of the frequency range, it can be observed clearly that the fractional order model gives better results. Moreover, when analyzing the Bode plots, an interesting property of RC bridge with ultracapacitor can be observed, namely

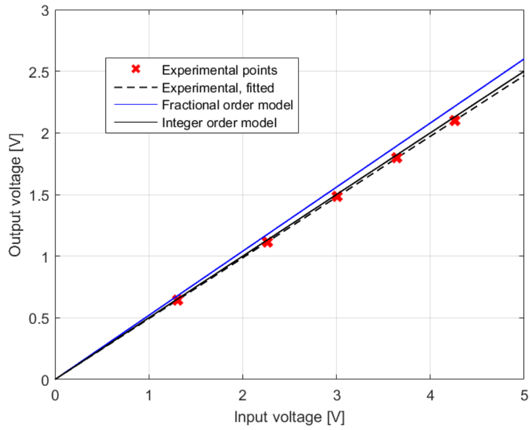

Figure 11. Static characteristic of the RC bridge with 0.1 F capacitor

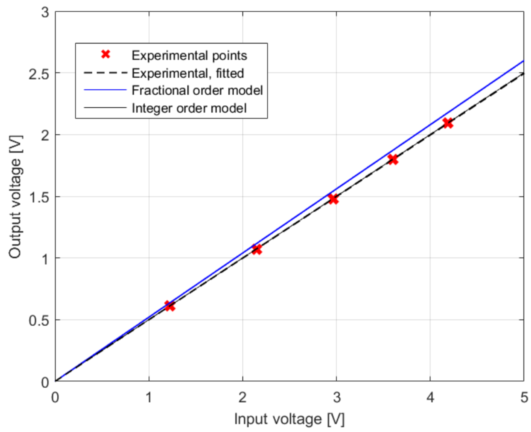

Figure 12. Static characteristic of the RC bridge with 0.33 F capacitor

that significant phase shifts occur only in a narrow frequency band. This also means that the tested object does not delay the input signals even in high frequencies. The step responses of the RC bridge also confirm that the fractional order model fits the experimental responses better. The static characteristics of the RC bridge were taken, as well. These results show that steady states were modeled better by the integer system model, but the differences are insignificant.

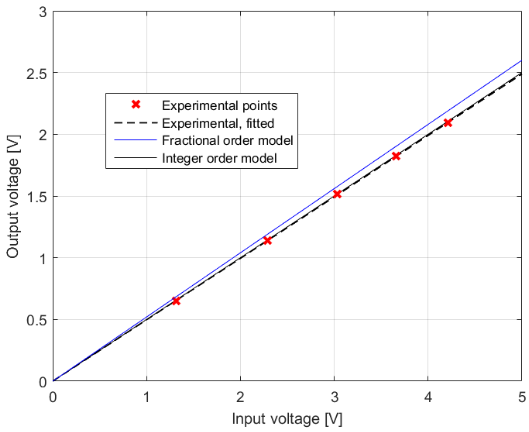

Figure 13. Static characteristic of the RC bridge with 1 F capacitor

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