Application of Aitken's extrapolation in numerical analysis

Jan Purczyński
West Pomeranian University of Technology
71-126 Szczecin, ul. 26 Kwietnia 10, e-mail: janpurczynski@ps.pl

1. Introduction

The paper is a continuation and elaboration of the subject of [5], which investigated the application of algorithm ε and Aitken's extrapolation to solve large sets of linear equations. In this paper the application of algorithm ε [1], [2] was abandoned since in the following examples under study it yielded worse results than Aitken's extrapolation.

The acceleration of sequence (x_n) convergence through Aitken's extrapolation consists in creating sequence (x'_n) given by dependence:

$$x'_n = x_n - \frac{(x_n - x_{n-1})^2}{x_n - 2x_{n-1} + x_{n-2}} \tag{1}$$

which is convergent faster than the sequence (x_n) [1], [2], [4].

To the sequence (x'_n) formula (1) can be once more applied, which is referred to as multiple Aitken's extrapolation and can be written as:

$$a_{0,n} = x_n$$

$$a_{i+1,n} = a_{i,n} - \frac{(a_{i,n} - a_{i,n-1})^2}{a_{i,n} - 2a_{i,n-1} + a_{i,n-2}} \tag{2}$$

2. Approximate solving of large sets of linear equations

The following set of equations was investigated:

$$AX = B \tag{3}$$

where: $A_{J\times J} = [a_{j,k}]$; $B = [b_j]$.

The method of accelerating convergence was applied to the sequence of solutions obtained through Jacobi's method [4], [6]:

$$X^{\langle 0\rangle} = d \qquad X^{\langle n\rangle} = C \cdot X^{\langle n-1\rangle} + d \tag{4}$$

where: $d_j = \frac{b_j}{a_{j.j}}$; $c_{j,k} = \frac{a_{j,k}}{a_{j.j}}[\delta(j,k) - 1]$; $\delta(j,k)$ - Kronecker's delta.

Matrix B entries were obtained using a generator of random numbers - with uniform distribution within the range (0, 1000) - installed in MathCad program. Matrix A entries lying outside the main diagonal were determined using the same method. However, the diagonal entries satisfy the following condition:

$$a_{j,j} = p \sum_{j \neq k} \left| a_{j,k} \right| \tag{5}$$

Parameter p determines the spectral radius of matrix C, hence determines the convergence of results obtained in Jacobi's method. For a sufficiently large J (matrix degree) the following approximately applies

$$\rho(C) = \frac{1}{p} \tag{6}$$

In this paper the matrices under study were of a degree $J = 200 \div 1200$. The maximum relative error resulting from the application of (6) did not exceed the value $2 \cdot 10^{-5}$.

The largest characteristic value as to its absolute value (spectral radius) can be determined by the power method [4]. Having limited oneself to one iteration, an approximate formula is obtained:

$$\rho(C) = \left| \frac{\sum_{k=1}^{J} c_{1,k} S_k}{S_1} \right| \tag{7}$$

where: $S_j = \sum_{k=1}^{J} c_{j,k}$.

For the matrices under study, the relative error resulting from the application of (7) fell within the range $(10^{-6} \div 10^{-15})$.

While doing the acceleration of sequence convergence of solutions obtained through Jacobi's method, it was concluded that the best results were given by the single Aitken's extrapolation. Formula (1) was applied vectorially, i.e. in relation to subsequent vectors $X^{\langle n \rangle}$ (formula (4)). Treating the solution obtained through Gaussian elimination method as an exact one, the relative error for all solutions of the set (3) was determined and the maximum error BM was calculated. Assuming $BM = 10^{-4}$, an adequate number of iterations n was determined as shown in Table 1.

In accordance with (1), Aitken's extrapolation is applied to the last three terms of a sequence. In Table 1 the values of spectral radius within the range $(1 \div 5)$ are provided which means that it is a divergent sequence of solutions obtained through Jacobi's method (the convergence condition is $\rho < 1$). For $\rho > 5$ error $BM > 10^{-4}$ was obtained. The results in Tab. 1 show that the application of Aitken's extrapolation enables the abandonment of a dominant main diagonal condition with reference to matrix A (formula (3)). Figure 1 presents maximum values of relative error BM depending on a number n of iteration being subject to Aitken's extrapolation. The figure was done for J = 1000 and $\rho = 4$; for other values J and ρ a similar dependence of BM on n is obtained.

Table 1. Minimum number of iterations in Jacobi's method

Spectral radius ρ	Number of equations in a set J					
	400	600	800	1000	1200	J > 1000
1	6	6	6	6	6	6
2	8	8	8	7	7	7
3	9	8	8	8	8	8
4	10	10	8	8	9	9
5	12	9	9	10	9	10

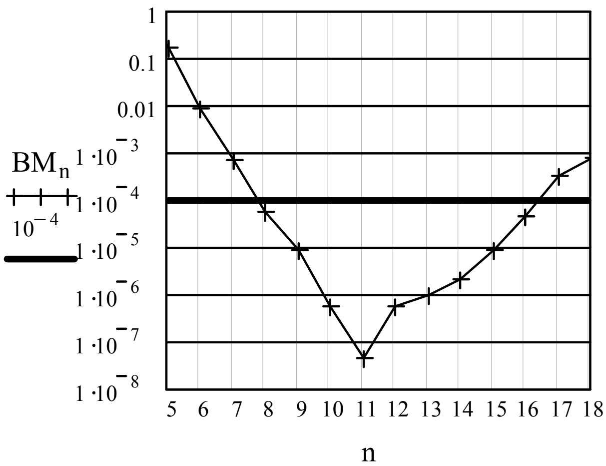

Fig. 1. Maximum values of relative error BM depending on number n of iteration being subject to Aitken's extrapolation

3. Determining the principal value of singular integral

As an example the following singular integral was considered

$$I = \int_{-1}^{1} \frac{x}{x-a} dx \tag{8}$$

where $a \in (-1,1)$.

By calculating the integral

$$I(h) = \int_{-1}^{a-h} \frac{x}{x-a}\,dx + \int_{a+h}^{1} \frac{x}{x-a}\,dx \tag{9}$$

we obtain

$$I(h) = 2 - 2h + a\ln\left(\frac{1-a}{1+a}\right) \tag{10}$$

By reaching in (10) limit $h \to 0$, the principal value of the singular integral (8) is obtained, which is given by

$$i = 2 + a\ln\left(\frac{1-a}{1+a}\right) \tag{11}$$

In order to apply Aitken's extrapolation, and using the procedure of numerical integration from MathCad program, the value of the integral was determined

$$I_m = \int_{-1}^{0,2-h_m} \frac{x}{x-0,2}\,dx + \int_{0,2+h_m}^{1} \frac{x}{x-0.2}\,dx \tag{12}$$

for m = 1, 2, 3; $h_m = 2^{-m}$, and subsequently formula (1) was applied.

The value of relative error obtained for the result (in relation to (11)) equaled $1{,}5 \cdot 10^{-15}$. In the case of using the parabola method to calculate the integrals present in formula (12) the obtained value of relative error equaled $1{,}2 \cdot 10^{-10}$.

Figure 2 shows the integrand function curve $f(x) = \frac{x}{x-a}$ (for a=0,2) present in the integral (8).

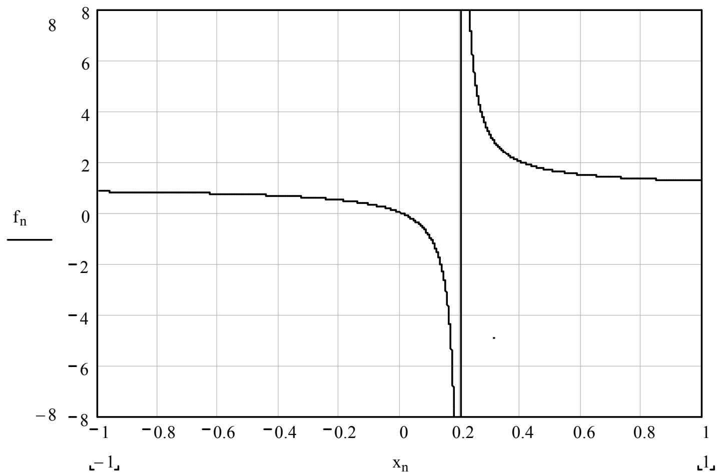

Fig. 2. The curve of function $f(x) = \frac{x}{x-a}$; $a = 0{,}2$

As another example the following singular integral was considered

$$I = \int_{-1}^{1} \frac{x^2}{x-a} dx \tag{13}$$

where $a \in (-1,1)$.

The principal value of the singular integral (13) was given by

$$i = 2 \cdot a + a^2 \ln\left(\frac{1-a}{1+a}\right) \tag{14}$$

Using the procedure of numerical integration from MathCad program, the value of the integral was determined:

$$I_m = \int_{-1}^{a-h_m} \frac{x^2}{x-a} dx + \int_{a+h_m}^{1} \frac{x^2}{x-a} dx \tag{15}$$

for a = -0,8; m = 1, 2, 3; $h_m = 2^{-m}$, and subsequently formula (1) was applied.

The value of relative error obtained for the result equaled $7{,}33 \cdot 10^{-15}$.

4. Determining the value of improper integral

As an example the following improper integral was considered

$$I = \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \tag{16}$$

An integration interval $x \in \langle 0, N\pi \rangle$ was divided into N sub-intervals of a size $h = \pi$. The rectangle method was used to calculate the value of function $f(x) = \frac{\sin(x)}{x}$ in points $x_i = \frac{2i+1}{2}\pi$ where: i=0,1,...,N.

In accordance with the rectangle method, the value of the integral was determined for varying n; n = 0,1,...,N

$$S_n = h \sum_{i=0}^{n} f(x_i) \quad \text{for n=0,1,...,N} \tag{17}$$

To the sequence of partial sums (17) multiple Aitken's extrapolation (formula (2)) was applied and the obtained result was burdened with an error equal to $5 \cdot 10^{-15}$.

5. Accelerating the convergence of solution in one-point iteration method

Several methods of approximate solving of nonlinear equations boil down to the one-point iteration method, which is given by

$$x_{n+1} = \phi(x_n) \tag{18}$$

Also in this case, Aitken's extrapolation helps to accelerate the convergence of formula (18) solution.

As an example, the determination of the root of the following equation was investigated:

$$f(x) = 2x + \ln(x) = 0 \tag{19}$$

Equation (19) can be transformed into an equivalent form

$$x_{n+1} = \exp(-2x_n) \tag{20}$$

For the obtained sequence (x_n), where: $x_0 = 0{,}5$; $n = 0,1,\ldots,N$; $N = 10$ multiple Aitken's extrapolation is used – formula (2). The result obtained in this way is burdened with a relative error $4{,}6 \cdot 10^{-9}$. In order to obtain such accuracy while using formula (20) N=121 terms of the sequence would have to be taken into account. When considering only N=10 in (20), the obtained result is burdened with a relative error 3,38%.

Interesting qualities characterize the modified Aitken's extrapolation method, the so called active Aitken's extrapolation [4]. In this method, starting from value x_0, x_1 and x_2 are consecutively derived in accordance with formula (18). Then formula (1) is used to obtain value x'_2. Beginning with value x'_2 the next two terms of the sequence are calculated based on formula (18).

The described algorithm can be written as

$$x_{n+3} = \begin{cases} x_{n+2} - \dfrac{(x_{n+2} - x_{n+1})^2}{x_{n+2} - 2x_{n+1} + x_n} & \mathit{for}\ \mathrm{mod}(n,3) = 0 \\ \phi(x_{n+2}) & \mathit{for}\ \mathrm{mod}(n,3) \neq 0 \end{cases} \tag{21}$$

Applying formula (21) to equation (20), for $x_0 = 0{,}5$; $n = 0,1,\ldots,N$; $N = 10$, the solution burdened with a relative error $-1{,}83 \cdot 10^{-12}$ is obtained. The error is more than 1000 smaller than the error obtained in the multiple extrapolation method ($4{,}6 \cdot 10^{-9}$). The modified Aitken's extrapolation method has yet another advantage, namely, it can be applied to a divergent sequence. To illustrate that, formula (19) can be used, which can be transformed into the following form

$$x_{n+1} = -\frac{\ln(x_n)}{2} \tag{22}$$

The application of algorithm (21) to formula (22) provides the solution (N = 10) with a relative error $2{,}28 \cdot 10^{-10}$. It should be noticed that the sequence (x_n) derived from formula (22) is divergent since the convergence condition $|\phi'(x_n)| < 1$ [4] is not met. A similar quality belongs to the multiple Aitken's extrapolation method which provides solution for the divergent sequence $\phi(x_n)$, yet it presents slower convergence than the active Aitken's extrapolation method – in the example under study the relative error of the solution is $2{,}85 \cdot 10^{-4}$.

The best effects of accelerating the convergence of the solution of formula (18) were obtained using Wegstein's method which is given by [7]:

$$x_1 = \phi(x_0) \; ; \; x_{n+2} = \phi(x_{n+1}) - \frac{\phi(x_{n+1}) - x_{n+1}}{\dfrac{x_{n+1} - x_n}{\phi(x_{n+1}) - \phi(x_n)}} \tag{23}$$

Applying formula (23), to both (20) and (22), the relative error of solution for N = 8 was smaller than $1 \cdot 10^{-15}$.

6. Conclusions

The paper investigated the application of Aitken's extrapolation to solve selected problems of numerical analysis.

In the case of approximate solving of large sets of linear equations, Aitken's extrapolation proved useful for the sequence of solutions obtained through Jacobi's method. Thanks to the application of the single Aitken's extrapolation to the last three terms of the sequence, the relative error of solutions was smaller than $1 \cdot 10^{-4}$. Taking into account the fact that the amount of the considered terms of the sequence did not exceed 12, the proposed method is competitive with other methods of solving the sets of linear equations with respect to calculation complexity. An additional advantage of this method is the possibility of abandoning the dominant main diagonal condition in relation to matrix A (formula (3)), which limits applicability of Jacobi's method. Aitken's extrapolation method can be applied to spectral radius $\rho \in (0{,}5)$.

The following two examples concerned determining the principal value of singular and improper intervals. In the case of the singular integral, the integrand function's discontinuity within the integration interval was present. The substantial accuracy of results was reached with little calculation complexity – the relative error did not exceed the value of $1{,}2 \cdot 10^{-10}$. The fact that the integral in section IV was improper resulted from the infinity of the integration interval. The application of the multiple Aitken's extrapolation to the sequence obtained through the rectangle method provided the result's relative error equal to $5 \cdot 10^{-15}$.

In section V the acceleration of the sequence convergence of equation (18) solutions was discussed. Furthermore, the usefulness of the multiple Aitken's extrapolation and active Aitken's extrapolation was ascertained. Both of these methods can be used in the case when the sequence of solutions of equation (18) is a divergent sequence.

References

[1] Brezinski C., *Acceleration de la convergence en anlyse numerique*. Berlin 1977.
[2] Brezinski C., *Algorithmes d`acceleration de la convergence. Etude numerique*. Paris 1978.
[3] Chua L.O. Pen- Minlin, *Komputerowa analiza układów elektronicznych*. *Algorytmy i metody obliczeniowe*. WNT Warszawa 1981.
[4] Dahlqiust G., Bjorck A., *Metody numeryczne*. Warszawa: PWN 1983.
[5] Purczyński J., *Zastosowanie algorytmu* ε *i ekstrapolacji Aitkena do rozwiązywania dużych układów równań liniowych*. Mat. Konf. ZkwE`97, Poznań- Kiekrz 1997, s.89-92.
[6] Stoer J., Bulirsch R., *Wstęp do metod numerycznych*. PWN 1975.
[7] Wierzbicki W.M., *Czisliennyje mietody*. Wysszaja szkoła, Moskwa, 2000.