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FREQUENCY RESPONSE OF THE PRESSURE TRANSDUCER MODEL DESCRIBED BY THE FRACTIONAL ORDER DIFFERENTIAL EQUATION

Abstract

The recent dynamic development of research into the application of the fractional calculus to analyse dynamic systems made the authors of this paper attempt its application to the analysis and modelling of transducers used in pneumatic systems. The paper compares logarithmic frequency responses of the pressure transducer described by the ordinary differential equation with the one described by the fractional order differential equation. Simulation tests were made in MATLAB programme.

INTRODUCTION

A mathematical model and frequency analysis of the pressure transducer described by the second order differential equation is well-known today [5], [7], [8], [10] and [12]. The physical phenomenon of pressure dynamics in a transducer is of continuous character. Thus we can pose a question: why is a pressure transmitter of continuous character described by the second order differential equation, that is the order from a discrete set and not a continuous set?

This paper is an attempt at a mathematical description and frequency analysis of a transmitter of continuous quantities, like for example pressure, with the use of the fractional order differential equations. The basic feature of the fractional order derivatives is the fact that they are determined on a section – not at a point as it is the case with classical derivatives. Such a property allows us to analyse phenomena in more detail, especially phenomena of continuous character. The hypothesis which was put forward inspired the authors of this paper to take advantage of a mathematical tool which is the differential-integral calculus of fractional order to model measuring transducers used in pneumatic systems [3] and [6].

1. MEMBRANE PRESSURE TRANSDUCER

To examine dynamic properties of the pressure transducer, a model of a pressure chamber with an inlet pipe was made (Fig. 1).

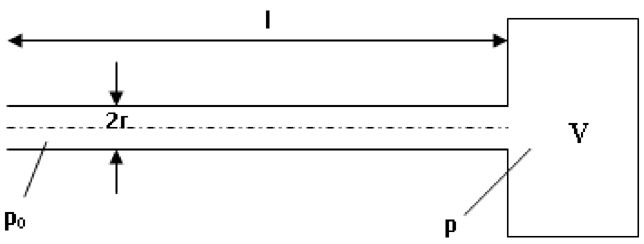

Fig. 1. *Pressure chamber with an inlet pipe: r, l – pipe dimensions, p_0 – inlet pressure, p – pressure in the transmitter's chamber [6]*

The differential equation constituting the mathematical model of the analysed pneumatic system, in which the fractional order differential equation is applied, looks as follows [6]:

$$\frac{d^{(\nu_2)}}{dt^{(\nu_2)}}p(t)+2\xi\omega_0\frac{d^{(\nu_1)}}{dt^{(\nu_1)}}p(t)+\omega_0^{\ 2}\frac{d^{(\nu_0)}}{dt^{(\nu_0)}}p(t)=$$
$$=\omega_0^{\ 2}p_0(t) \qquad (1)$$

where:
p(t) – pressure in the transducer's chamber,
$p_0(t)$ – inlet pressure,
ξ – damping coefficient,
ω_0 – pulsation,
v>0 – order of derivative.

To determine the derivative of a continuous function, i.e. pressure in the transducer's chamber, we used the Riemann-Liouville definition of fractional derivative [5] and [7]:

$${}_0^{RL}D_t^{\alpha}f(t)=\frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_a^t(t-\tau)^{k-\alpha-1}f(\tau)d\tau \qquad (2)$$

where:
α – the order of integration within the limits (a,t) of the function $f(t)$, $k-1\le\alpha\le k$ and: $\alpha\in R^+$, $\Gamma(x)=\int_0^{\infty}e^{-t}t^{x-1}dt$ – the Euler gamma function.

The Laplace transform for the Riemann-Liouville fractional derivative is [5] and [7]:

$$L\left[{}_0^{R-L}D_t^{\alpha}f(t)\right]=s^{\alpha}F(s)-\sum_{k=0}^{j-1}s^k\,{}_0^{R-L}D_t^{\alpha-k-1}f(0) \qquad (3)$$

where: $j-1\le\alpha\le j\in N$

Applying the Laplace transform to equation (1), for zero initial conditions, we obtain:

$$s^{2\nu}p(s)+2\xi\omega_0s^{\nu}p(s)+\omega_0^{\ 2}p(s)=\omega_0^{\ 2}p_0(s)\,. \qquad (4)$$

Hence:

$$p(s)=\frac{\omega_0^{\ 2}}{\left(s^{2\nu}+2\xi\omega_0s^{\nu}+\omega_0^{\ 2}\right)}p_0(s) \qquad (5)$$

From equation (5) we obtain the transfer function of the analysed pressure transmitter:

$$G^{(v)}(s)=\frac{p(s)}{p_0(s)}=\frac{\omega_0{}^2}{s^{2v}+2\xi\omega_0 s^v+\omega_0{}^2} \tag{6}$$

Substituting:

$$s=j\omega=\omega e^{j\frac{\pi}{2}}=\omega\left[\cos\left(\frac{\pi}{2}\right)+j\sin\left(\frac{\pi}{2}\right)\right] \tag{7}$$

in the formula (6), we obtain the spectral transfer of the transducer:

$$G^{(v)}(j\omega)=\frac{\omega_0{}^2}{(j\omega)^{2v}+2\xi\omega_0(j\omega)^v+\omega_0{}^2} \tag{8}$$

Owing to elementary transformations we can calculate the real and imaginary parts of the spectral transform function:

$$G^{(v)}(j\omega)=P^{(v)}(\omega)+jQ^{(v)}(\omega) \tag{9}$$

Knowing the real and imaginary part of the spectral transform of the transducer, we can determine the equation describing the logarithmic amplitude function:

$$M^{(v)}(\omega)=20\log\sqrt{\left[P^{(v)}(\omega)\right]^2+\left[Q^{(v)}(\omega)\right]^2} \tag{10}$$

as well as the equation describing the logarithmic phase characteristic:

$$\varphi^{(v)}(\omega)=arctg\left[\frac{Q^{(v)}(\omega)}{P^{(v)}(\omega)}\right]. \tag{11}$$

2. NUMERICAL TESTS AND SIMULATIONS

To present frequency responses of logarithmic functions of amplitude and phase of the pneumatic pressure transducer, a programme was written in MATLAB environment [1] and [11] which calculates the function values for the given parameters of the system and selected orders and plots their characteristics.

In order to verify the dependencies describing logarithmic functions of amplitude (10) and phase (11) of the tested transducer, a pneumatic pressure transducer was modelled in the MATLAB environment described by means of an ordinary differential equation and a fractional order differential equation. While describing the transducer with the use of the fractional order differential equation we adopted parameter v=1 and compared the obtained logarithmic functions of the amplitude and phase with the logarithmic functions of amplitude and phase obtained from a description of the pressure transducer made with an ordinary differential equation.

In simulations we adopted:

– pulsation ω_0 = 500 [rad/s];
– damping coefficient ξ = 0.7.

3. FREQUENCY RESPONSES OF THE TRANSDUCER MODELLED WITH THE USE OF ORDINARY AND FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

The transfer function of the pneumatic pressure transducer described with the use of the ordinary differential equation looks as follows:

$$G(s)=\frac{p(s)}{p_0(s)}=\frac{\omega_0{}^2}{s^2+2\xi\omega_0 s+\omega_0{}^2} \tag{12}$$

While conducting simulation of equation (12), which represents dynamics of phenomena occurring in the analysed pneumatic system, in MATLAB environment, we obtained the frequency responses outlined in Figure 2:

When simulating in MATLAB environment equations (10) and (11), describing the pneumatic pressure transducer with the use of the fractional order differential equation and adopting the parameter v=1, we obtained the functions outlined in Figure 3.

Logarithmic frequency responses of amplitude and phase represented by means of the ordinary differential equation simulation (Figure 2), overlap with the frequency responses obtained by simulation of equations describing the amplitude function (10) and the phase function (11), obtained from the equation for the tested transducer described by the fractional order differential equation (Figure 3).

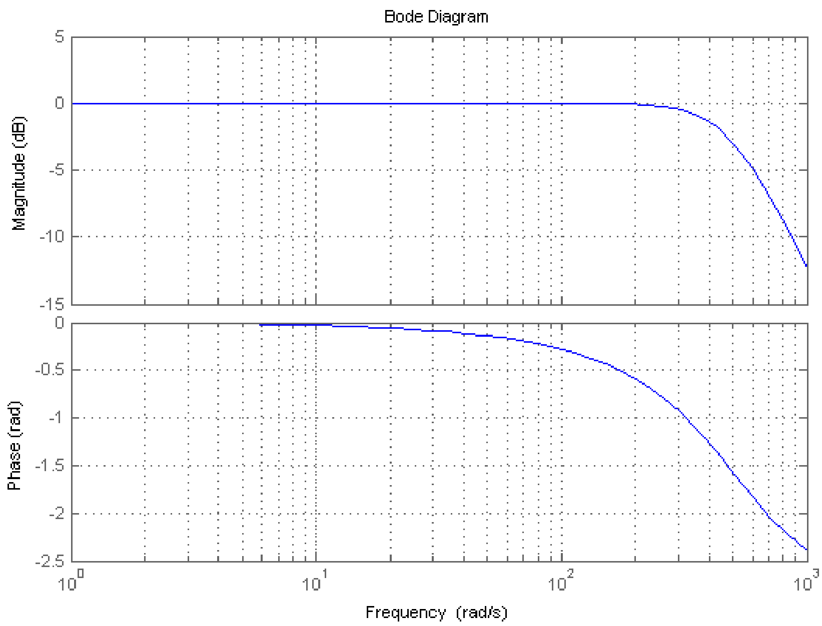

Fig. 2. *Logarithmic frequency responses of the pneumatic transducer described by the ordinary differential equation*

4. FREQUENCY RESPONSES OF THE PRESSURE TRANSDUCER DESCRIBED BY THE FRACTIONAL ORDER DIFFERENTIAL EQUATION

To obtain Bode plots, equations (10) and (11) underwent simulations by writing a suitable programme in MATLAB environment.

The program written in MATLAB environment enables analysis of the transducer for differentials of different orders, with an optional time step, as the order was given as parameter *v*. Simulation results for selected values of parameter *v*, are outlined in Figures 4 and 5 [6].

CONCLUSIONS

By frequency simulation of equation (12), representing the transfer function of the pressure transducer described by the ordinary differential equation and equation (6), representing the transfer function of the transducer described by the fractional order differential equation, it was shown that logarithmic frequency responses of amplitude and phase coincide for parameter $v=1$. This proves correctness of the mathematical model and algorithm creating these functions.

Analyses of the functions indicate that for $v<1$ logarithmic amplitude functions are monotonically decreasing functions; for $v>1$ logarithmic amplitude functions have the maximum depending on the order of the differential. The maximum is achieved at the resonant pulsation frequency.

When the order of the differential increases, the frequency responses acquire the character of the inertial second-order member; when the order of the differential decreases, the frequency responses acquire the character of the inertial first-order member.

On the basis of the frequency response analysis we can state that increasing the *v* parameter above 1 causes that the system behaves like the inertial second-order system with damping $\xi<0.707$. The higher the order is, the lower damping is.

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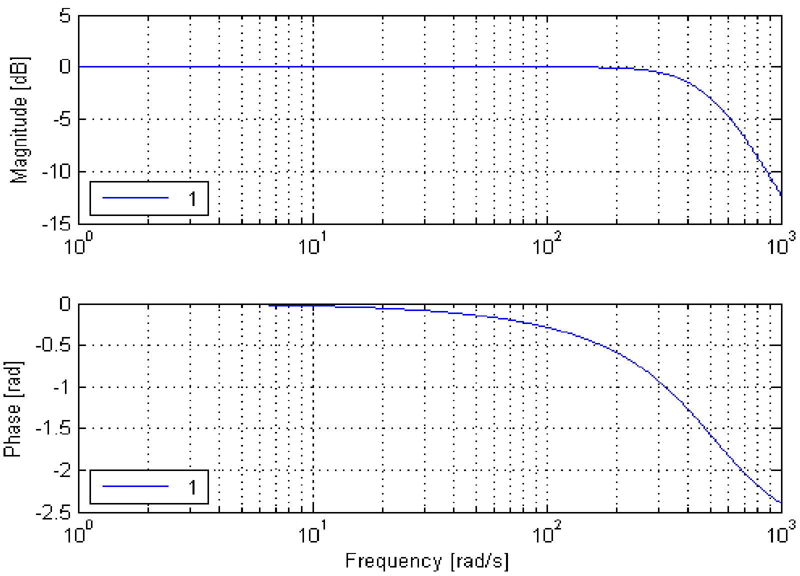

Fig. 3. *Logarithmic frequency responses of the pneumatic transducer described by the fractional order differential equations*

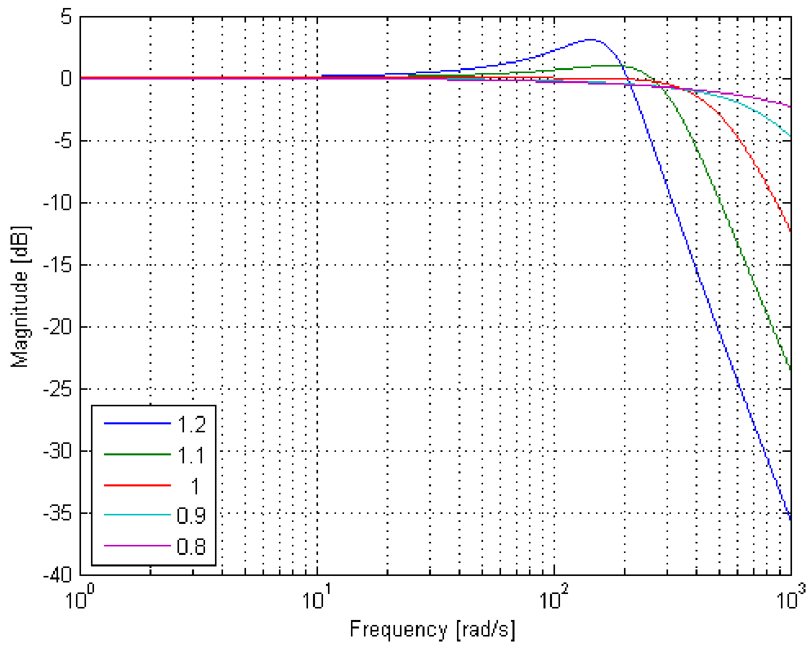

Fig. 4. Logarithmic amplitude functions of the pneumatic transducer described by the fractional order differential equation for parameter v from the (0.8–1.2) range

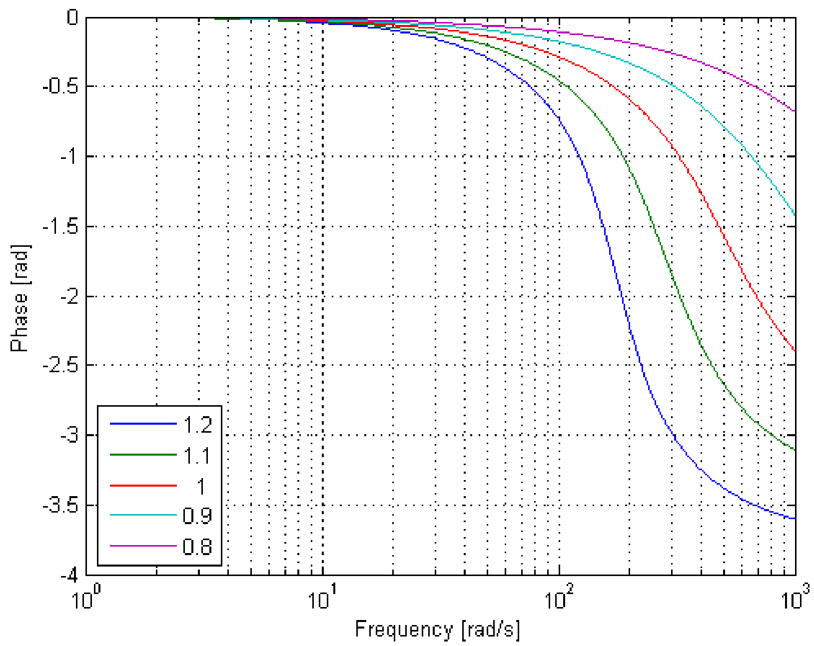

Fig. 5. Logarithmic phase functions of the pneumatic transducer described by the fractional order differential equation for the parameter v from the (0.8–1.2) range

CHARAKTERYSTYKI CZĘSTOTLIWOŚCIOWE MODELU PRZETWORNIKA CIŚNIENIA OPISANEGO RÓWNANIEM RÓŻNICZKOWYM NIECAŁKOWITYCH RZĘDÓW

Streszczenie

Dynamiczny rozwój badań w ostatnich latach nad zastosowaniem rachunku różniczkowo-całkowego niecałkowitych rzędów do analizy układów dynamicznych, skłonił autorów artykułu do podjęcia próby jego zastosowania w analizie i modelowaniu przetworników stosowanych w układach pneumatycznych. Przetworniki pneumatyczne znajdują szerokie zastosowanie w transporcie, m.in. pojazdach szynowych, w autobusach, w układach hamulcowych.

W artykule przedstawiono model matematyczny *przetwornika pneumatycznego, pobudzonego stałym wymuszeniem, opisany rachunkiem różniczkowo- całkowym niecałkowitego rzędu oraz jego charakterystyki częstotliwościowe. Porównano modele przetwornika ciśnienia całkowitego i niecałkowitego rzędu. Badania wykonano za pomocą oprogramowania MATLAB.*

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