Scientific Issues
Jan Długosz University
in Częstochowa
Mathematics XVII (2012)
87–96

THE STABILITY OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

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Abstract

In the present paper we deal with the Dhombres-type trigonometric difference

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[f(y)+g(y)\big],$$

assuming that its absolute value is majorized by some constant. Our aim is to find functions $\widetilde{f}$ and $\widetilde{g}$ which satisfy the Dhombres-type trigonometric functional equation and for which the differences $\widetilde{f}-f$ and $\widetilde{g}-g$ are uniformly bounded.

1. Introduction

Stability problems concerning classical functional equations have been treated by several authors (see, e.g., [5]–[7]). The cosine functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{1}$$

and the sine functional equation

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)f(y) \tag{2}$$

are both stable (in fact, they are even superstable) in the Hyers-Ulam sense. In [3], Baker studied the stability of the cosine functional equation (1), while Cholewa established the stability of the sine functional equation (2) in [4]. The

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results about the superstability can be obtained as corollaries from theorems by Badora and Ger. Namely, the following theorems hold.

Theorem 1 (Badora and Ger, see [2]). *Let $(G,+)$ be an Abelian group and let $f\colon G\to\mathbb{C}$ and $\varphi\colon G\to\mathbb{R}$ satisfy the inequality*

$$\big|f(x+y)+f(x-y)-2f(x)f(y)\big|\le\varphi(x)\quad\textit{for all}\quad x,y\in G.$$

Then either f is bounded or

$$f(x+y)+f(x-y)=2f(x)f(y)\quad\textit{for all}\quad x,y\in G.$$

Theorem 2 (Badora and Ger, see [2]). *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $f\colon G\to\mathbb{C}$ and $\varphi\colon G\to\mathbb{R}$ satisfy the inequality*

$$\left|f(x)f(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\le\varphi(x)\quad\textit{for all}\quad x,y\in G.$$

Then either f is bounded or

$$f(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2\quad\textit{for all}\quad x,y\in G.$$

From now on, we denote the odd and the even parts of a function f by f_o and f_e, respectively. The next lemma (due to Wilson, see [11]) provides general solutions of an equation that generalizes the equation (1).

Lemma 1 (see also [1] and [8]). *Let $(G,+)$ be an Abelian group. Then functions $f,g\colon G\to\mathbb{C}$ satisfy the functional equation*

$$f(x+y)+f(x-y)=2f(x)g(y)\tag{3}$$

if and only if one of the following conditions holds:

(i) *the function g is arbitrary and $f=0$;*

(ii) *there exist an additive function $a\colon G\to\mathbb{C}$ and a constant $\alpha\in\mathbb{C}$ such that*

$$f(x)=a(x)+\alpha\quad\textit{and}\quad g(x)=1\quad\textit{for all}\quad x\in G;$$

(iii) *there exist an exponential function $m\colon G\to\mathbb{C}$ and constants $\beta,\gamma\in\mathbb{C}$ such that*

$$f(x)=\beta m_o(x)+\gamma m_e(x)\quad\textit{and}\quad g(x)=m_e(x)\quad\textit{for all}\quad x\in G.$$

In [8], Székelyhidi studied the Hyers-Ulam stability of the equation (3), obtaining the following result.

Theorem 3. *Let $(G,+)$ be an Abelian group and let $\varepsilon \geq 0$. If functions $f,g\colon G \to \mathbb{C}$ satisfy the inequality*

$$\big|f(x+y)+f(x-y)-2f(x)g(y)\big| \leq \varepsilon \quad \text{for all} \quad x,y \in G,$$

then one of the following conditions holds:

(i) *if $f=0$, then g is arbitrary;*

(ii) *if $f \neq 0$ is bounded, then g is bounded, as well;*

(iii) *if g is bounded and f is unbounded, then $g=1$ and there exist an additive function $A\colon G \to \mathbb{C}$ and a constant $\delta \in \mathbb{C}$ such that*

$$\big|f(x)-A(x)\big| \leq \delta \quad \text{for all} \quad x \in G;$$

(iv) *if $f \neq 0$ and g is unbounded, then f is unbounded, as well. Moreover, functions f and g satisfy the equation* (3).

The above theorems allow us to formulate the following result concerning superstability.

Corollary 1. *Let unbounded functions $f,g\colon G \to \mathbb{C}$ satisfy the inequality*

$$\big|f(x+y)+f(x-y)-2f(x)g(y)\big| \leq \varepsilon$$

for all $x,y \in G$ and for some $\varepsilon \geq 0$. Then f and g satisfy the equation (3).

The aim of this paper is to study stability properties of the Dhombres-type trigonometric functional equation, i.e.,

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y)+f(x-y) = f(x)\big[f(y)+g(y)\big]. \tag{4}$$

In the case when $g=2h$, solutions of the above equation can be found in [10].

We shall use the following lemma.

Lemma 2 (see [9, Corollary 3])**.** *Let $(G,+)$ be a uniquely 2-divisible Abelian group and let $\varepsilon \geq 0$. Let an unbounded function $f\colon G \to \mathbb{C}$ and a function $g\colon G \to \mathbb{C}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 - f(x)g(y)\right| \leq \varepsilon \quad \text{for all} \quad x,y \in G.$$

Then one of the following conditions holds:

(i) *if* $g \neq 0$ *is bounded, then* g *satisfies the sine equation* (2)*;*

(ii) *if* g *is unbounded, then there exists a function* $h\colon G \to \mathbb{C}$ *such that*

$$f(x+y) + f(x-y) = 2f(x)h(y) \quad \text{for all} \quad x, y \in G.$$

For $f = f_e + f_o$*, we have* $f_e(x) = f(0)h(x)$ *for all* $x \in G$ *and* f_o *satisfies the sine equation. Moreover, if* $f(0) = 0$*, then* $g = f = f_o$*.*

2. Main results

Our main result reads as follows.

Theorem 4. *Let* $(G, +)$ *be a uniquely 2-divisible Abelian group and let* $\varepsilon \geq 0$*. If functions* $f, g\colon G \to \mathbb{C}$ *satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[f(y) + g(y)\big] \right| \leq \varepsilon \tag{5}$$

for all $x, y \in G$*, then there exist an exponential function* $m\colon G \to \mathbb{C}$*, additive functions* $a, A\colon G \to \mathbb{C}$*, a bounded function* $B\colon G \to \mathbb{C}$ *and a constant* β *such that one of the following conditions holds:*

(i) *if* $f = 0$*, then* g *is arbitrary;*

(ii) *if* $f \neq 0$ *is bounded, then* g *is bounded, as well;*

(iii) *if the function* f *is unbounded, then*

$$\begin{cases} f(x) = A(x) + B(x) \\ g(x) = a(x) - A(x) - B(x) + 2 \end{cases} \quad \text{for all} \quad x \in G, \tag{6}$$

or

$$\begin{cases} f(x) = f_o(x) + f(0)m_e(x) \\ g(x) = \big(\beta m_o(x) - f_o(x)\big) + \big(2 - f(0)\big)m_e(x) \end{cases} \quad \text{for all} \quad x \in G. \tag{7}$$

Moreover, suppose that $f(0) = 0$*. Then*

$$\begin{cases} f(x) = \beta m_o(x) \\ g(x) = 2m_e(x) \end{cases} \quad \text{for all} \quad x \in G. \tag{8}$$

Proof. Assume that the function f is an unbounded solution of inequality (5). Then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of elements of G such that

$$0 \neq |f(z_n)| \longrightarrow \infty \quad \text{as} \quad n \to \infty. \tag{9}$$

Let us take $x = z_n$ in (5). Then we obtain

$$\left| f\left(\frac{z_n+y}{2}\right)^2 - f\left(\frac{z_n-y}{2}\right)^2 + f(z_n+y) + f(z_n-y) - f(z_n)\big[f(y)+g(y)\big]\right| \le \varepsilon$$

for all $y \in G$ and $n \in \mathbb{N}$, whence

$$\left| \frac{f\left(\frac{z_n+y}{2}\right)^2 - f\left(\frac{z_n-y}{2}\right)^2 + f(z_n+y) + f(z_n-y)}{f(z_n)} - \big[f(y)+g(y)\big]\right| \le \frac{\varepsilon}{|f(z_n)|}$$

for all $y \in G$ and $n \in \mathbb{N}$. Now, taking the limit as $n \to \infty$ and applying (9), we obtain

$$\lim_{n\to\infty} \frac{f\left(\frac{z_n+y}{2}\right)^2 - f\left(\frac{z_n-y}{2}\right)^2 + f(z_n+y) + f(z_n-y)}{f(z_n)} = f(y) + g(y) \tag{10}$$

for all $y \in G$. Hence,

$$f(0) + g(0) = 2. \tag{11}$$

Let us replace x by $z_n + x$ in (5). Then we get

$$\left| f\left(\frac{z_n+x+y}{2}\right)^2 - f\left(\frac{z_n+x-y}{2}\right)^2 + f(z_n+x+y) \right.$$
$$\left. + f(z_n+x-y) - f(z_n+x)\big[f(y)+g(y)\big]\right| \le \varepsilon.$$

Similarly, let us replace x by $z_n - x$ in (5). Then

$$\left| f\left(\frac{z_n-x+y}{2}\right)^2 - f\left(\frac{z_n-x-y}{2}\right)^2 + f(z_n-x+y) \right.$$
$$\left. + f(z_n-x-y) - f(z_n-x)[f(y)+g(y)]\right| \le \varepsilon.$$

From the above inequalities we compute

$$\left| f\left(\frac{z_n+(x+y)}{2}\right)^2 - f\left(\frac{z_n-(x+y)}{2}\right)^2 + f\big(z_n+(x+y)\big) \right.$$
$$+ f\big(z_n-(x+y)\big) + f\left(\frac{z_n+(-x+y)}{2}\right)^2 - f\left(\frac{z_n-(-x+y)}{2}\right)^2$$
$$+ f\big(z_n+(-x+y)\big) + f\big(z_n-(-x+y)\big)$$
$$\left. - \big[f(z_n+x) + f(z_n-x)\big]\cdot\big[f(y)+g(y)\big]\right| \le \varepsilon$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Therefore,

$$\left|\frac{f\left(\frac{z_n+(x+y)}{2}\right)^2 - f\left(\frac{z_n-(x+y)}{2}\right)^2 + f(z_n+(x+y)) + f(z_n-(x+y))}{f(z_n)}\right.$$
$$+\frac{f\left(\frac{z_n+(-x+y)}{2}\right)^2 - f\left(\frac{z_n-(-x+y)}{2}\right)^2 + f(z_n+(-x+y)) + f(z_n-(-x+y))}{f(z_n)}$$
$$\left.-\frac{[f(z_n+x)+f(z_n-x)]}{f(z_n)}\big[f(y)+g(y)\big]\right| \le \varepsilon$$

for all $x, y \in G$ and $n \in \mathbb{N}$. With the use of (9) and (10), we conclude that for every $x \in G$ there exists the following limit as n tends to infinity:

$$\lim_{n\to\infty} \frac{f(z_n+x)+f(z_n-x)}{f(z_n)} =: h(x). \tag{12}$$

Moreover, the so defined function $h\colon G \to \mathbb{C}$ satisfies the equation

$$f(x+y)+g(x+y)+f(-x+y)+g(-x+y)-h(x)\big[f(y)+g(y)\big] = 0, \quad x, y \in G.$$

By interchanging x and y, we obtain

$$f(x+y)+g(x+y)+f(x-y)+g(x-y)-\big[f(x)+g(x)\big]h(y) = 0, \quad x, y \in G.$$

Let $F := f + g$ and $G := \frac{1}{2}h$. Then

$$F(x+y)+F(x-y) = 2F(x)G(y) \quad \text{for all} \quad x, y \in G.$$

On the basis of Lemma 1, we get three possible forms of the function F.

Case 1. Suppose $F = 0$. By putting $f + g = 0$ in (5), we obtain

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y)\right| \le \varepsilon$$

for all $x, y \in G$. Now, setting $y = 0$, we get

$$\big|f(x)\big| \le \frac{\varepsilon}{2} \quad \text{for all} \quad x \in G.$$

This leads to a contradiction since f is unbounded.

Case 2. By Lemma 1 case (ii), let us assume that there exist an additive function a and a constant α such that $F = a + \alpha$. Then

$$f(x)+g(x) = a(x)+\alpha \quad \text{for all} \quad x \in G. \tag{13}$$

Let us take $x = 0$ in (13) and apply (11). We get $\alpha = 2$. By the inequality (5), we obtain

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[a(y)+2\big] \right| \le \varepsilon \tag{14}$$

for all $x, y \in G$. By taking $-y$ instead of y in (14), we infer that

$$\left| f\left(\frac{x-y}{2}\right)^2 - f\left(\frac{x+y}{2}\right)^2 + f(x-y) + f(x+y) - f(x)\big[-a(y)+2\big] \right| \le \varepsilon. \tag{15}$$

By (14) and (15) and the fact that a is odd, we get the following relation:

$$\Bigg| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\big[a(y)+2\big]$$
$$+ f\left(\frac{x-y}{2}\right)^2 - f\left(\frac{x+y}{2}\right)^2 + f(x-y) + f(x+y) - f(x)\big[-a(y)+2\big] \Bigg| \le 2\varepsilon$$

for all $x, y \in G$. Equivalently,

$$\big|f(x+y) + f(x-y) - 2f(x)\big| \le \varepsilon \quad \text{for all} \quad x, y \in G.$$

Applying Theorem 3 to the unbounded function f yields that there exist an additive function A and a constant δ such that

$$\big|f(x) - A(x)\big| \le \delta \quad \text{for all} \quad x, y \in G,$$

hence $f = A + B$, where the function B is bounded by δ. By this fact and by (13) we get $g = a - A - B + 2$. Finally, we obtain (6).

Case 3. By case (iii) of Lemma 1, let us consider $F = \beta m_o + \gamma m_e$, where the function m is exponential and β, γ are constants. Hence, from (5), we obtain

$$f(x) + g(x) = \beta m_o(x) + \gamma m_e(x) \quad \text{for all} \quad x \in G. \tag{16}$$

Applying (16) to $x = 0$ in (11), we get $\gamma m_e(0) = 2$. We know that if $m \neq 0$, then $m(0) = 1$ and $m_e(0) = 1$. In the other words, we have $\gamma = 2$. By (16) and (5), we get the following relation:

$$\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)\left[\beta m_o(y) + 2m_e(y)\right] \right| \le \varepsilon \tag{17}$$

for all $x, y \in G$. Hence, by replacing y by $-y$ in (17), we see that

$$\left| f\left(\frac{x-y}{2}\right)^2 - f\left(\frac{x+y}{2}\right)^2 + f(x-y) + f(x+y) - f(x)\big[-\beta m_o(y) + 2m_e(y)\big] \right| \le \varepsilon \tag{18}$$

for all $x, y \in G$. Summing inequalities (17) and (18) sidewise, we infer that

$$\left|f(x+y) + f(x-y) - 2f(x)m_e(y)\right| \le \varepsilon \quad \text{for all} \quad x, y \in G.$$

If $m_e = 1$, then we obtain Case 2. Therefore, by Theorem 3, functions f and m_e satisfy the following equation:

$$f(x+y) + f(x-y) = 2f(x)m_e(y) \quad \text{for all} \quad x \in G. \tag{19}$$

Applying (19) to (17), we see that

$$\left|f\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 + 2f(x)m_e(y) - f(x)\big[\beta m_o(y) - 2m_e(y)\big]\right| \le \varepsilon,$$

for all $x, y \in G$, which is equivalent to

$$\left|f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x)\beta m_o(y)\right| \le \varepsilon \quad \text{for all} \quad x, y \in G.$$

From Lemma 2 we get what follows.

Subcase 3.1 If the function βm_o is bounded, then from (i) we only conclude that our function satisfies the sine functional equation. We do not get any other interesting information about the function f.

Subcase 3.2 Assume that the function βm_o is unbounded. Therefore, by (ii), there exists a function h such that

$$f(x+y) + f(x-y) = 2f(x)h(y) \quad \text{for all} \quad x \in G. \tag{20}$$

Thus, from equations (19) and (20), we obtain $h = m_e$. Furthermore, we get $f = f(0)m_e + f_o$ and the function f_o satisfies the sine equation (2). It follows from (16) that

$$f(0)m_e(x) + f_o(x) + g(x) = \beta m_o(x) + 2m_e(x) \quad \text{for all} \quad x \in G.$$

Equivalently,

$$g(x) = \beta m_o(x) - f_o(x) + \big(2 - f(0)\big)m_e(x) \quad \text{for all} \quad x \in G. \tag{21}$$

Thus, we have proved that functions f and g are of the form (7).

Moreover, by Lemma 2 applied to a function f for which $f(0) = 0$, we have $\beta m_o = f = f_o$. Hence, the above considerations and the equation (21) imply that $g = 2m_e$. The proof of the theorem is complete. □

Corollary 2. *The equation* (4) *is not always stable.*

Proof. Assume that an unbounded function $f\colon G \to \mathbb{C}$ such that $f(0) = 0$ and a function $g\colon G \to \mathbb{C}$ satisfy the inequality (5) for all $x, y \in G$. We ask whether there exist functions $\widetilde{f}, \widetilde{g}\colon G \to \mathbb{C}$ and a constant δ such that $\widetilde{f}$ and $\widetilde{g}$ satisfy the equation (4) and

$$\left|\widetilde{f}(x) - f(x)\right| \le \delta \quad \text{and} \quad \left|\widetilde{g}(x) - g(x)\right| \le \delta \quad \text{for all} \quad x \in G.$$

By Theorem 4, functions f and g have either of the forms (6) and (8).

Case 1. In the case of the form (8), provided $f(0) = 0$, we define functions $\widetilde{f}, \widetilde{g}\colon G \to \mathbb{C}$ by $\widetilde{f} := f$ and $\widetilde{g} := g$. Therefore, for all $x, y \in G$, we get

$$\beta^2 \left[m_o\left(\frac{x+y}{2}\right)^2 - m_o\left(\frac{x-y}{2}\right)^2 \right] + \beta\left[m_o(x+y) + m_o(x-y)\right] \\ = \beta m_o(x)\big[\beta m_o(y) + 2m_e(y)\big].$$

Case 2. In the case of the form (6), we have two possibilities.

Subcase 2.1. The function g is bounded. Then the function $a - A$ is bounded and additive. Therefore, $a - A = 0$. Let us define functions $\widetilde{f}, \widetilde{g}\colon G \to \mathbb{C}$ by

$$\begin{cases} \widetilde{f}(x) := A(x) \\ \widetilde{g}(x) := 2 \end{cases} \quad \text{for all} \quad x \in G.$$

Then the so defined functions satisfy the equation (4), i.e.,

$$A\left(\frac{x+y}{2}\right)^2 - A\left(\frac{x-y}{2}\right)^2 + A(x+y) + A(x-y) = A(x)\big[A(y) + 2\big]$$

for all $x, y \in G$, and $\left|\widetilde{f} - f\right| = |B| \le \delta$ and $\left|\widetilde{g} - g\right| = |B| \le \delta$ for some δ.

Subcase 2.2. When the function g is unbounded, then functions $\widetilde{f}$ and $\widetilde{g}$ do not exist. □

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