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ON CLASSICAL SYMMETRIES OF ORDINARY DIFFERENTIAL EQUATIONS RELATED TO STATIONARY INTEGRABLE PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We study the relationship between the solutions of stationary integrable partial and ordinary differential equations and coefficients of the second-order ordinary differential equations invariant with respect to one-parameter Lie group. The classical symmetry method is applied. We prove that if the coefficients of ordinary differential equation satisfy the stationary integrable partial differential equation with two independent variables then the ordinary differential equation is integrable by quadratures. If special solutions of integrable partial differential equations are chosen, then the coefficients satisfy the stationary KdV equations. It was shown that the Ermakov equation belong to a class of these equations. In the framework of the approach we obtained the similar results for generalized Riccati equations. By using operator of invariant differentiation we describe a class of higher order ordinary differential equations for which the group-theoretical method enables us to reduce the order of ordinary differential equation.

Keywords: ordinary differential equation, partial differential equation, integrability, symmetry, quadrature, Lie transformation group.

Mathematics Subject Classification: 34A30, 34C14.

1. INTRODUCTION

Second-order ordinary differential equation

$$y_{xx} - Vy = \lambda y, \quad \text{where} \quad V = V(x), \quad y = y(x, \lambda), \tag{1.1}$$

where $\lambda \in R$ is an arbitrary parameter and $V = V(x)$, $y = y(x, \lambda)$ are real-valued functions is related to hierarchy of KdV equations integrable by inverse scattering transform method. The class of stationary (i.e. when $V_t = 0$) higher order KdV-equations or their arbitrary nontrivial linear combinations are called Novikov's equations. It is known that if the potential $V(x)$ satisfies at least one equation from the family of Novikov's equations, which are nonlinear ordinary differential equations then the Schrödinger equation (1.1) is integrable by quadratures [1, 3–5]. It turns out that this result can be obtained by applying group-theoretical methods [1]. In this paper we study the invariance property of the equation

$$y_{xx} - uy = 0, \tag{1.2}$$

where y, and u are functions of independent variable x and parameter variable z. We study the symmetry of (1.2) and its nonlinear generalizations with respect to a one-parameter Lie transformation group of dependent and independent variables. It turns out that the coefficients of the infinitesimal generator of the symmetry group and the coefficients of ordinary differential equations under study are related to solutions of stationary integrable differential equations. Our purpose is to extend the class of differential equations (1.2) and the Riccati equation, which possesses a nontrivial symmetry and therefore can be integrated or simplified via the Lie symmetry method.

In Section 2 we derive the infinitesimal generator of the Lie symmetry group. Then we establish the relationship between the stationary two-dimensional nonlinear partial differential equations integrable by inverse scattering transform method and the integrability by quadratures of equation (1.2) and its nonlinear generalizations. By using operator of invariant differentiation we describe a class of higher order ordinary differentia equations admitting one-parameter Lie transformation group and therefore for which lowering of the order could be realised by group-theoretical methods. Section 3 deals with the Riccati equation and generalized Riccati equations, which are obtained from the Riccati equation by adding nonlinear terms. We study the symmetries and integrability of these equations by quadratures.

As we know the symmetry of equations (2.20), (2.26), (2.28), (2.31), (2.33) when coefficients u and w satisfy stationary integrable equations with two independent variables has not been studied before. We pick out equations (2.31), (2.33) integrable by quadratures from the class (2.26). To our knowledge the proof of the integrability of the equations (2.20), (2.31), (2.33), (3.19) has also not been described in the literature.

Note that the the wide class of differential equations describing the diffusion processes for example neutron diffusion process in nonhomogeneous media [8] leads to equation (1.2). In Section 4 we discuss the obtained results and their applications in solving mathematical physics problems.

2. APPLICATION OF CLASSICAL LIE SYMMETRY METHODS TO DIFFERENTIAL EQUATIONS ASSOCIATED WITH EQUATION (1.2)

Firstly we introduce notions and explanations related to the Lie transformation group. Let V be an open set in $\mathbb{R}^n$ and $N(0, \delta)$, where $\delta > 0$ is a neighborhood of 0, $N(0, \delta) \subset \mathbb{R}$. A local one-parameter Lie group G_1 of local transformations of the space $\mathbb{R}^n$ is the local transformations $f : V \times N(0,\delta) \to \mathbb{R}^n$ possessing the following properties:

1. $f(x, 0) = x$ for any $x \in V$,
2. $f(f(x, a), b) = f(x, a + b)$ for any $a, b, a + b \in N(0, \delta), x \in V$,
3. if$a \in N(0, \delta)$ and $f(x, a) = x$ for all $x \in V,$ then $a = 0$,
4. $f \in C^{\infty}(V \times N(0, \delta))$,

where a is the group parameter. Each transformation $f_a \in G_1$, $a \in N(0,\delta)$ is written in the coordinate form

$$f_a : x'^i = f^i(x, a), \quad i = 1, \ldots, n.$$

The formula

$$\xi^i(x) = \frac{\partial f^i(x, a)}{\partial a}\Big|_{a=0}, \quad i = 1, \ldots, n$$

yields the tangent vector field $\xi : V \to \mathbb{R}^n$ of G_1. We can construct group transformations if we know tangent vector field $\xi(x)$ by using the following Lie equations with initial conditions

$$\frac{df^i}{da} = \xi^i(f^1, \ldots, f^n), \quad f^i(x_1, \ldots, x_n, 0) = x_i, \quad i = 1, \ldots, n.$$

S. Lie established the existence of a one-to-one correspondence between group G_1 and its tangent vector field $\xi(x)$ up to an arbitrary nonzero numerical multiplier. This leads to a one-to-one correspondence between the group G_1 and its infinitesimal generator

$$Q = \sum_{i=1}^{n} \xi^i(x)\partial_{x_i}.$$

Hence, we arrive at the conclusion that the group G_1 can be characterized by its infinitesimal generator. One of the most important applications of Lie group theory is the theory of differential equations. Suppose that the variables under consideration are split into independent variables $x_1 \ldots, x_n$ and dependent variable ψ. In this case the infinitesimal generator Q has the form

$$Q = \sum_{i=1}^{n} \xi^i(x, \psi)\partial_{x_i} + \eta(x, \psi)\partial_{\psi}, \tag{2.1}$$

where

$$\begin{aligned} f_a &: x'^i = f^i(x, \psi, a), \quad i = 1, \ldots, n. \\ g_a &: \psi' = g(x, \psi, a), \end{aligned} \tag{2.2}$$

$$\xi^i(x, \psi) = \frac{\partial f^i(x, \psi, a)}{\partial a}\Big|_{a=0}, \quad \eta(x, \psi) = \frac{\partial g(x, \psi, a)}{\partial a}\Big|_{a=0}, \quad i = 1, \ldots, n.$$

In this case the Lie equations take the following form

$$\begin{aligned} &\frac{df^i}{da} = \xi^i(f^1,\dots,f^n,g), \quad f^i(x_1,\dots,x_n,g,,0) = x_i, \quad i=1,\dots,n,\\ &\frac{dg}{da} = \eta(f^1,\dots,f^n,g), \quad g(x_1,\dots,x_n,g,,0) = \psi. \end{aligned} \tag{2.3}$$

The k-th prolongation $Q^{(k)}$ of infinitesimal generator is used to formulate the criterion of invariance of k-th order differential equations under the group transformations (2.2) regarded as a change of variables. (See [6,7] for more details.) The infinitesimal criterion of invariance of given differential equation results in linear differential equations for the coefficient functions ξ^i, η of the infinitesimal generator (2.1), called the determining equations for the symmetry group of the given equation. Note that the system of determining equations is linear regardless of whether the equation under study is linear or nonlinear. One can obtain group transformations by solving Lie equations (2.3). In this section and Section 3 we show the applications of this method.

For the aims of this article, it is sufficient to work with an infinitesimal generator (2.1) without the description of finite group transformations (2.2).

We study the symmetry of ordinary differential equation (1.2) with respect to the Lie group with infinitesimal generator

$$X = \beta(x,z)y\partial_y - \alpha(x,z)\partial_x \tag{2.4}$$

where $\alpha(x,z)$, $\beta(x,z)$ are unknown smooth functions. X is an infinitesimal generator of symmetry group G_1 of differential equation if and only if its evolutionary representative

$$Q = (\alpha(x,z)y_x + \beta(x,z)y)\,\partial_y, \tag{2.5}$$

is. For the proofs we refer the reader to [6]. We use the infinitesimal generator written in evolutionary form (2.5) in this case.

We work within the local approach.

The infinitesimal criterion of invariance [6]

$$Q^{(2)}\big(y_{xx} - uy)\big)\Big|_{y_{xx}=uy,\,y_{xxx}=u_xy+uy_x} = 0, \tag{2.6}$$

where $Q^{(2)}$ is the second prolongation of Q is used. The second prolongation of Q is given by formulas [6]

$$Q^{(2)} = Q + \eta^1\partial_{y_x} + \eta^2\partial_{y_{xx}}, \tag{2.7}$$

$$\begin{aligned} \eta^1 &= D_x(\eta) = \left(\frac{\partial}{\partial x} + y_x\frac{\partial}{\partial y} + y_{xx}\frac{\partial}{\partial y_x}\right)(\eta)\\ &= \alpha y_{xx} + (\alpha_x+\beta)y_x + \beta_x y,\\ \eta^2 &= D_x(\eta^1) = \left(\frac{\partial}{\partial x} + y_x\frac{\partial}{\partial y} + y_{xx}\frac{\partial}{\partial y_x}\right)(\eta^1)\\ &= \alpha y_{xxx} + (2\alpha_x+\beta)y_{xx} + (\alpha_{xx}+2\beta_x)y_x + \beta_{xx}y. \end{aligned} \tag{2.8}$$

From (2.6), (2.7) we obtain

$$\eta^2 - u(\alpha y_x + \beta y) = 0, \tag{2.9}$$

which must be satisfied whenever $y_{xx} = uy, y_{xxx} = u_x y + uy_x$. Substituting (2.8) into (2.9), and replacing y_{xx} by uy and y_{xxx} by $u_x y + uy_x$ we obtain

$$\begin{aligned} &\alpha(u_x y + uy_x) + (2\alpha_x + \beta)uy + (\alpha_{xx} + 2\beta_x)y_x \\ &+ \beta_{xx} y - u(\alpha y_x + \beta y) = 0. \end{aligned} \tag{2.10}$$

Splitting (2.10) with respect to y_x and y we find the determining equations

$$\alpha_{xx} + 2\beta_x = 0, \tag{2.11}$$

$$\alpha u_x + 2\alpha_x u + \beta_{xx} = 0 \tag{2.12}$$

for $\alpha(x, z)$, $\beta(x, z)$. From (2.11), (2.12) it follows that

$$\alpha u_x + 2\alpha_x u - \frac{1}{2}\alpha_{xxx} = 0, \tag{2.13}$$

and

$$\beta = -\frac{1}{2}\alpha_x + C, \tag{2.14}$$

where $C(z)$ is smooth function. We set $C = 0$ because $Q = C(z)y\partial_y$ generates the obvious trivial symmetry group given by $y' = e^{Ca}y$. Next, we choose

$$\alpha = w_z, \quad u = w_x, \tag{2.15}$$

where $w(x, z)$ is a smooth function. Substituting (2.15) in (2.12) yields

$$4w_{zx}w_x + 2w_z w_{xx} - w_{zxxx} = 0. \tag{2.16}$$

In [2] F. Callogero and A. Degasperis considered the class of partial differential equations with three independent variables t, x, y

$$\begin{aligned} w_{xt} = {} & a_0 w_{xx} + a_1(w_{xxxx} + 6w_{xx}w_x) + \gamma_0 w_{xy} \\ & + \gamma_1(w_{xxxy} + 4w_{xy}w_x + 2w_{xx}w_y) \end{aligned}$$

solvable via the spectral transform based on Schrödinger spectral problem. Putting $a_0 = a_1 = \gamma_0 = 0, \gamma_1 = 1$, and $(w_t = 0)$ one derives the stationary equation

$$w_{xxxy} + 4w_{xy}w_x + 2w_{xx}w_y = 0. \tag{2.17}$$

Replacing $w \to -w$, $y \to z$ in (2.17) we obtain (2.16).

As the next theorem shows, the class of differential equations invariant with respect to Lie group with infinitesimal generator given by (2.4), (2.14), (2.15), (2.16) is not exhausted by (1.2) where $u = w_x$.

Theorem 2.1. *Equation* (1.2) *is invariant with respect to the Lie transformation group with infinitesimal generator*

$$\tilde{Q} = 2w_z\partial_x + w_{zx}y\partial_y,$$

where $w(x,z)$ *is a solution of* (2.16) *if and only if*

$$u = w_x + \frac{f(z)}{w_z^2},$$

where $f(z)$ *is an arbitrary function.*

Proof. Equation (1.2) is invariant with respect to the Lie transformation group with infinitesimal generator $\tilde{Q}$ if and only if it satisfies the infinitesimal invariance criterion (2.6) applied to the twice prolongated operator $\tilde{Q}^{(2)}$

$$\tilde{Q}^{(2)}\big(y'' - uy)\big)\Big|_{y''=Y,y'''=u_x y+uy'} = 0, \tag{2.18}$$

where

$$\tilde{Q} = (2w_z y_x - w_{zx}y)\partial_y.$$

The criterion (2.18) results in

$$2w_{zx}u + w_z u_x - \frac{1}{2}w_{zxxx} = 0,$$

by (2.13). We have to construct the general solution of linear nonhomogeneous ordinary differential equation

$$u_x - 2\frac{w_{zx}}{w_z}u = \frac{w_{zxxx}}{2w_z}$$

to find the general form of u. Since w is a solution of (2.16), we have

$$w_{zxxx} = 4w_{zx}w_x + 2w_z w_{xx}.$$

Hence

$$u_x - 2\frac{w_{zx}}{w_z}u = \frac{2w_{zx}w_x + w_z w_{xx}}{w_z}. \tag{2.19}$$

By using the method of variation of a constant we find the general solution of (2.19) in the form

$$u = \frac{C(x,z)}{w_z^2} = w_x + \frac{f(z)}{w_z^2},$$

where $f(z)$ is an arbitrary function. □

It is obvious that the linear homogeneous equation

$$y_{xx} - \left(w_x + \frac{f(z)}{w_z^2}\right)y = 0 \tag{2.20}$$

is also invariant with respect to one-parameter Lie group with generator

$$X_1 = y\partial_y.$$

Thus, we know at least two-dimensional Lie algebra with basic elements $\tilde{Q}$ and X_1 and commutator relation

$$[\tilde{Q}, X_1] = 0$$

admissible by equation (2.20). Therefore equation (2.20) is integrable by quadratures [6]. Next we give the symmetry integration method for (2.20). We use the property of straightening of vector field also known as a method of canonical variables. Let introduce new variables

$$t = \int \frac{dx}{2w_z}, \quad y = \sqrt{w_z}\Psi$$

satisfying the conditions

$$\tilde{Q}(t) = 1, \quad \tilde{Q}(\Psi) = 0,$$

where $\Psi = \Psi(t)$. Changing to these variables reduces (2.20) to the form

$$\frac{1}{4}\Psi_{tt} + \left(\frac{1}{2}w_z w_{zxx} - \frac{1}{4}w_{zx}^2 - w_x w_z^2 + f(z)\right)\Psi = 0. \tag{2.21}$$

In the (t, ψ)-coordinates the symmetry generator $\tilde{Q}$ has the simple translational form $\tilde{Q} \to \tilde{Q}' = \partial_t$. Thus equation (2.21) must be independent of t. It is easy to check that

$$\frac{d}{dx}\left(\frac{1}{2}w_z w_{zxx} - \frac{1}{4}w_{zx}^2 - w_x w_z^2\right) = \frac{1}{2}w_z\left(w_{zxxx} - 2w_z w_{xx} - 4w_x w_{zx}\right) = 0$$

by virtue of (2.16) so that

$$\frac{1}{2}w_z w_{zxx} - \frac{1}{4}w_{zx}^2 - w_x w_z^2 = K(z),$$

where $K(z)$ is an arbitrary function of z only. Therefore the resulting equation

$$\Psi_{tt} + 4\left(K(z) + f(z)\right)\Psi = 0$$

does not contain the independent variable t and is integrable. Hence, we can easily construct its general solution

$$\begin{aligned} y =& \sqrt{w_z}\left[C_1(z)\exp\left(-\sqrt{-\delta}\int\frac{dx}{2w_z}\right) + C_2(z)\exp\left(\sqrt{-\delta}\int\frac{dx}{2w_z}\right)\right] \\ &\text{if} \quad \delta(z) = K(z) + f(z) < 0, \end{aligned}$$

$$\begin{aligned} y =& \sqrt{w_z}\left[C_3(z)\sin\left(\sqrt{\delta}\int\frac{dx}{2w_z}\right) + C_4(z)\cos\left(\sqrt{\delta}\int\frac{dx}{2w_z}\right)\right] \\ &\text{if} \quad \delta(z) = K(z) + f(z) > 0, \end{aligned}$$

$$y = \sqrt{w_z}\left[C_5(z)\int \frac{dx}{2w_z} + C_6(z)\right] \quad \text{if} \quad \delta(z) = K(z) + f(z) = 0,$$

where $C_1(z), C_2(z), C_3(z), C_4(z), C_5(z), C_6(z)$ are arbitrary functions.

By using differential invariants we can generalize the family of invariant equations (2.20). Taking into account (2.4), (2.14) we obtain the infinitesimal generator of Lie group of point transformations in the form

$$Q_1 = 2\alpha\partial_x + \alpha' y\partial_y \tag{2.22}$$

where $Q_1 = -2X$. Next we construct the second prolongation of Q_1 [7]

$$Q_1^{(2)} = Q_1 + \sigma^1\partial_{y_x} + \sigma^2\partial_{y_{xx}},$$

where

$$\sigma^1 = D_x(\alpha_x y) - 2y_x D_x(\alpha_x y) = \alpha_{xx} y - \alpha_x y_x,$$

$$\sigma^2 = D_x(\alpha_{xx} y - \alpha_x y_x) - 2y_{xx} D_x(\alpha) = \alpha_{xxx} y - 3\alpha_x y_x.$$

We find invariants of the Lie group with infinitesimal generator (2.22)

$$\omega = \frac{y^2}{\alpha}, \quad \omega_1 = \sqrt{\alpha} y_x - \frac{\alpha_x' y}{2\sqrt{\alpha}} \tag{2.23}$$

satisfying conditions

$$Q_1\omega = 0, \quad (Q_1 + \sigma^1\partial_{y_x})\omega_1 = 0.$$

ω_1 is called the differential invariant of the first order. We also constructed the differential invariant of the second order

$$\omega_2 = (y_{xx} - uy)\, y^3 \tag{2.24}$$

which satisfies the condition $Q^{(2)}\omega_2 = 0$ provided that α, u satisfy equation (2.13). One can construct differential invariants of higher order by using operator of invariant differentiation which has the form

$$\delta \cdot \partial = \alpha D_x, \tag{2.25}$$

where $D_x = \partial_x + y_{n+1}\partial_{y_n}$, $n \in \mathbb{N}$, $\partial_x = \partial/\partial x$, $\partial_{y_k} = \partial/\partial y_k$, $y_k = \partial^k y/\partial x^k$ in this case. (See [7] for more details). We assume summation over repeated indices. Recall that the operator (2.25) is said to be an operator of invariant differentiation if for any differential invariant Ω the mapping $(\delta \cdot \partial)\Omega$ is also a differential invariant of this group. Hence we have

$$\omega_k = (\delta \cdot \partial)\omega_{k-1},$$

where $k \geq 3$, ω_k and ω_{k-1} are differential invariants of k-th and $(k-1)$-th order, respectively. Our next claim is that the $(k+2)$-th order ordinary differential equation

$$(\delta \cdot \partial)^k \omega_2 = F(\omega, \omega_1, \cdots, \omega_{k+1}), \tag{2.26}$$

where F is a smooth function is obviously invariant with respect to the Lie group with infinitesimal generator (2.22) provided condition (2.13) holds. Hence it follows that the $(k+2)$-th order differential equation (2.26) reduces to a $(k+1)$-th ordinary differential equation plus a quadrature. For the proofs we refer the reader to [6]. We clarify the statement with an example of a second order ordinary differential equation. Let us consider the special case of (2.26)

$$\omega_2 = \omega^4 H(\omega, \omega_1), \tag{2.27}$$

where H is a smooth function of two variables. Substituting (2.23), (2.24) into (2.27) we have

$$y_{xx} - uy = \frac{y}{\alpha^2} H\left(\frac{y^2}{\alpha}, \sqrt{\alpha} y_x - \frac{\alpha'_x y}{2\sqrt{\alpha}}\right). \tag{2.28}$$

According to the symmetry integration method we introduce new variables

$$s = \int \frac{dx}{2\alpha}, \quad y = \sqrt{\alpha} q \tag{2.29}$$

satisfying the conditions

$$Q_1(s) = 1, \quad Q_1(q) = 0,$$

where $q = q(s)$. In terms of canonical variables $s(x.y), q(x,y)$ equation (2.28) is written in the form

$$\frac{1}{4} q_{ss} + \left(\frac{1}{2}\alpha\alpha_{xx} - \frac{1}{4}\alpha_x^2 - \alpha^2 u\right) q = qH\left(q^2, \frac{1}{2} q_s\right).$$

Since

$$\frac{d}{dx}\left(\frac{1}{2}\alpha\alpha_{xx} - \frac{1}{4}\alpha_x^2 - \alpha^2 u\right) = 0$$

by virtue of (2.13), it follows that

$$\frac{1}{4} q_{ss} + C_1 q = qH\left(q^2, \frac{1}{2} q_s\right), \quad C_1 \in R. \tag{2.30}$$

We see that equation (2.30) does not contain independent variable s and therefore its order can be lowered by one. Moreover choosing $H(\omega, \omega_1) = g(\omega)$, where g is a smooth function of one independent variable, we have differential equation

$$y_{xx} - uy = \frac{y}{\alpha^2} g\left(\frac{y^2}{\alpha}\right) \tag{2.31}$$

which can be written in terms of canonical variables s, q

$$\frac{1}{4} q_{ss} + C_1 q = qH\left(q^2\right), \quad C_1 \in R,$$

and its solution reduces to quadratures

$$\int \frac{dq}{\sqrt{\int (H(q^2) - C_1 q) dq + C_2}} = 2\sqrt{2} s + C_3.$$

In particular, we get the Ermakov equation

$$y_{xx} - uy = \frac{b}{y^3} \quad b \in R \tag{2.32}$$

from (2.31) in the case when $g(\omega) = \dfrac{b}{\omega^2}$. Taking $H(\omega, \omega_1) = p\left(\frac{\omega_1}{\sqrt{\omega}}\right)$, where p is a smooth function, yields

$$y_{xx} - uy = \frac{y}{\alpha^2} p\left(\alpha \frac{y_x}{y} - \frac{1}{2}\alpha_x\right). \tag{2.33}$$

Introducing variables (2.29) we can rewrite (2.33) as

$$q_{ss} + 4C_1 q = qg\left(\frac{q_s}{2q}\right), \quad C_1 \in R. \tag{2.34}$$

Equation (2.34) admits the two-parameter Lie transformation group with generators $Q_1' = \partial_s, Q_2' = q\partial_q$ and therefore is integrable by quadratures. Hence we obtained the two classes of nonlinear second order ordinary differential equations which are integrable by quadratures via the symmetry group method.

Assume $\alpha = R_m[u]$, where the differential polynomial $R_m[u]$ is chosen that

$$D_x(R_m[u]) = K_{m+1}[u] = 0$$

gives the $(m+1)$-th order stationary KdV equation. Then (2.13) is written as a hierarchy of higher order stationary KdV equations

$$R_{2m}(u)u_x + 2D_x(R_{2m}(u))u - \frac{1}{2}D_x^3(R_{2m}(u)) = 0, \quad m = 0, 1, 2, \ldots$$

constructed by virtue of recursion operator [6]. Since the set of infinitesimal generators form a Lie algebra, we see that α can be chosen as a linear combination of $R_1, R_2, \ldots, R_p$ with arbitrary constant coefficients $C_1, \ldots, C_p \in R$

$$\alpha = \sum_{i=1}^{p} C_i R_i[u].$$

It is easy to see that the same conclusion can be drawn for hierarchie of the Harry–Dym equations. Take for example $\alpha = R_0[u] = u$ and $\alpha = H_0[u] = u^{-1/2}$. We then obtain the stationary KdV equation

$$u_{xxx} - 6uu_x = 0$$

and the Harry–Dym equation

$$(u^{-1/2})_{xxx} = 0$$

respectively from (2.13). From this it follows that if $u(x)$ is a solution of an equation of the KdV hierarchy of the form

$$\sum_{i=1}^{p} \tilde{C}_j K_{2j+1}[u] = 0, \quad \tilde{C}_j \in R, j = 0, \ldots, p, \tag{2.35}$$

then equation (2.26) is invariant with respect to the Lie transformation group with infinitesimal generator (2.22). This property of invariance of (2.26) is also fulfilled for $u(x)$ satisfying the stationary Harry–Dym equation

$$\sum_{i=1}^{l} \tilde{C}_j' H_{2j+1}[u] = 0, \quad \tilde{C}_j \in R, j = 0, \ldots, l. \tag{2.36}$$

Therefore the order of equation (2.26) can be lowered by one via symmetry method. Moreover, equations (2.31) and (2.33) are integrable by quadratures in this case.

3. SYMMETRY OF GENERALIZED RICCATI EQUATIONS

Generalized Riccati differential equations are obtained from standard Riccati equations by adding a nonlinear term that preserves the symmetry of the equation. We study the invariance of Riccati equation

$$y_x - y^2 + v(x) = 0, \tag{3.1}$$

where $y = y(x)$ with respect to the Lie group with infinitesimal generator

$$X = a(x)\partial_x + [b(x)y + \gamma(x)]\partial_y, \tag{3.2}$$

where $a(x)$, $b(x)$, $\gamma(x)$ are smooth functions. We find $a(x)$, $b(x)$, $\gamma(x)$ so that the one-parameter Lie group with infinitesimal generator (3.2) is a symmetry group of (3.1). In this case the infinitesimal criterion of invariance [6,7] takes the form

$$X^{(1)}\big(y_x - y^2 + v(x)\big)\Big|_{y_x = y^2 - v(x)} = 0, \tag{3.3}$$

where $X^{(1)}$ is given by formulae

$$X^{(1)} = X + \eta^x \partial_{y_x}, \tag{3.4}$$

and

$$\eta^x = D_x\,(b(x)y + \gamma(x)) - y_x D_x(a(x)) = b_x y + b y_x + \gamma_x - a_x y_x. \tag{3.5}$$

From (3.3) we have

$$\big(\eta^x - 2y(b(x)y + \gamma(x)) + a(x)v_x\big)\Big|_{y_x = y^2 - v(x)} = 0. \tag{3.6}$$

Condition (3.6) now leads to

$$b_x y + (b - a_x)(y^2 - v) + \gamma_x - 2y(by + \gamma) + a v_x = 0 \tag{3.7}$$

Since a, b, γ, v depend only on x, (3.7) decomposes into three equations

$$b - a_x - 2b = 0, \tag{3.8}$$

$$b_x - 2\gamma = 0, \tag{3.9}$$

$$-bv + a_x v + \gamma_x + a v_x = 0. \tag{3.10}$$

This follows by the same method as in Section 2.

From (3.8), (3.9), (3.10) it follows that

$$a v_x + 2a_x v - \frac{1}{2}a_{xxx} = 0, \tag{3.11}$$

$$b = -a_x,$$
$$\gamma = -\frac{1}{2}a_{xx}.$$

From this we obtain the infinitesimal generator

$$X = 2a(x)\partial_x - [2a_x y + a_{xx}]\partial_y \tag{3.12}$$

of the Lie transformation group, which is a symmetry group of equation (3.1), where $a(x)$, $v(x)$ satisfy (3.11).

Let us consider the generalised Riccati equation

$$y_x - y^2 + v(x) = F(x, y), \tag{3.13}$$

where $F(x, y)$ is a smooth function. The following is an application of the Lie method to equation (3.13).

Theorem 3.1. *Equation* (3.13) *is invariant with respect to the Lie transformation group with infinitesimal generator* (3.12)*, where* v *is a solution of* (3.11) *if and only if*

$$F = \frac{1}{a^2}h(\omega),$$

where $\omega = 2ay + a_x$*,* $a \neq 0$*,* $h(\omega)$ *is a smooth function.*

Proof. Equation (3.13) is invariant with respect to the Lie transformation group with infinitesimal generator X (3.12) if and only if

$$X^{(1)}\big(y_x - y^2 + v(x) - F(x, y)\big)\Big|_{y_x = y^2 - v(x) + F(x,y)} = 0. \tag{3.14}$$

From (3.4), (3.5), (3.12) we have

$$X^{(1)} = 2a(x)\partial_x - [2a_x y + a_{xx}]\partial_y - (a_{xxx} + 2a_{xx}y + 4a_x y_x)\partial_{y_x}. \tag{3.15}$$

(3.14) now becomes

$$\begin{aligned} &- a_{xxx} - 2a_{xx}y - 4a_x(y^2 - v + F) + 2y(a_{xx} + 2a_x y) + 2av_x \\ &= 2aF'_x - (a_{xx} + 2a_x y)F'_y. \end{aligned} \tag{3.16}$$

From this we obtain

$$-4a_xF = 2aF'_x - (a_{xx} + 2a_xy)F'_y \tag{3.17}$$

taking into account (3.11). The general solution of (3.17) can be obtain by integrating characteristic system of ordinary differential equations

$$\frac{dx}{2a} = \frac{dy}{-a_{xx} - 2a_xy} = \frac{dF}{-4a_x}. \tag{3.18}$$

The general solution of (3.18) can be written in the form

$$F = \frac{1}{a^2}h(\omega(x,y)),$$

where $\omega = 2ay + a_x$, $h(\omega)$ is a smooth function. □

Therefore this proves that the equation

$$y_x - y^2 + v(x) = \frac{1}{a^2}h(\omega(x,y)), \tag{3.19}$$

where h is an arbitrary smooth function is invariant under one-parameter Lie group of transformations with infinitesimal generator (3.12). It is well known that if the first order ordinary differential equation is invariant with respect to one-parameter Lie group, then it can be integrated by quadrature [6]. By using the symmetry integration method we introduce new variables

$$t = \int \frac{dx}{2a(x)}, \quad \varphi = 2a(x)y + a_x(x) \tag{3.20}$$

satisfying the conditions

$$X(t) = 1, \quad X(\varphi) = 0,$$

where $\varphi = \varphi(t)$. We next find y from (3.20)

$$y(x) = \frac{1}{2a(x)}\varphi(t) - \frac{a_x(x)}{2a(x)}, \tag{3.21}$$

and calculate

$$y_x = -\frac{a_x}{2a^2}\varphi + \frac{1}{4a^2}\varphi_t - \frac{a_{xx}}{2a} + \frac{a_x^2}{2a^2}. \tag{3.22}$$

Substituting (3.21), (3.22) into (3.19) yields

$$\varphi_t - \varphi^2 + a_x^2 - 2aa_{xx} + 4a^2v = h(\varphi).$$

Note that

$$\begin{aligned}\frac{d}{dx}\left(a_x^2 - 2aa_{xx} + 4a^2v\right) &= 2a_xa_{xx} - 2a_xa_{xx} - 2aa_{xxx} + 8aa_xv + 4a^2v \\ &= 2a(-a_{xxx} + 4a_xv + 2av_x) = 0,\end{aligned}$$

by (3.11). Therefore

$$a_x^2 - 2aa_{xx} + 4a^2 v = M, \quad M \in R.$$

This gives

$$\varphi_t - \varphi^2 + M = h(\varphi).$$

This equation is integrable by quadratures, with

$$\int \frac{d\varphi}{h(\varphi) + \varphi^2 - M} = t + C_1$$

for some constant C_1. Analysis similar to that in solving equation (2.20) shows that if $c(x)$ satisfies partial differential equation (2.16), where $v(x) = c_x$ or if $v(x)$ satisfies ordinary differential equations (2.35) or (2.36), then equation (3.19) is integrable by quadratures. One can also obtain higher order differential equations from (3.19) by using operator of invariant differentiation just as in the case of equation (2.26). Moreover, the method of lowering of the order of differential equations is also applicable in this case.

4. CONCLUSIONS

We used the symmetry method for finding functions u or v for which equation (2.20), (2.31), (2.33), and (3.19) are integrable by quadratures. We proved that if we can determine solutions u or v of stationary partial differential equation (2.16) or (2.35), (2.36) integrable by inverse scattering transform method explicitly, then the computation of solutions to differential equations (2.20), (2.31), (2.33), and (3.19) reduces to quadratures. In particular, we have obtained integrable Ermakov equation (2.32). We believe that these results are novel.

Note that choosing $f(z) = \lambda = const$ in (2.20) we obtain the equation with arbitrary parameter λ. We also present the symmetry integration method for these equations and obtain the general solutions in explicit form. Note that if we construct the solution of equation in the form $w = \varphi(x + z)$, then we obtain the solutions for which w_x satisfies the stationary KdV equation. By using operator of invariant differentiation we have also constructed higher order differential equations (2.26) admitting one-parameter Lie group. Therefore their order can be lowered by one. In the framework of the approach we obtained the similar results for generalized Riccati equations.

A wide class of the second-order linear ordinary differential equations which appear in the Mathematical Physics problems in nonhomogeneous media can be reduced to the form (1.2). Therefore, our results can be used in solving these equations. Moreover, equations of the form (2.20) are used for constructing the Lax pairs for integrable nonlinear partial differential equations.

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