Opuscula Math. 43, no. 3 (2023), 429–453
https://doi.org/10.7494/OpMath.2023.43.3.429 Opuscula Mathematica

ON LOCAL ANTIMAGIC TOTAL LABELING OF COMPLETE GRAPHS AMALGAMATION

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Communicated by Andrzej Żak

Abstract. Let $G = (V, E)$ be a connected simple graph of order p and size q. A graph G is called local antimagic (total) if G admits a local antimagic (total) labeling. A bijection $g : E \to \{1, 2, \dots, q\}$ is called a local antimagic labeling of G if for any two adjacent vertices u and v, we have $g^+(u) \neq g^+(v)$, where $g^+(u) = \sum_{e\in E(u)} g(e)$, and $E(u)$ is the set of edges incident to u. Similarly, a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ is called a local antimagic total labeling of G if for any two adjacent vertices u and v, we have $w_f(u) \neq w_f(v)$, where $w_f(u) = f(u) + \sum_{e\in E(u)} f(e)$. Thus, any local antimagic (total) labeling induces a proper vertex coloring of G if vertex v is assigned the color $g^+(v)$ (respectively, $w_f(u)$). The local antimagic (total) chromatic number, denoted $\chi_{la}(G)$ (respectively $\chi_{lat}(G)$), is the minimum number of induced colors taken over local antimagic (total) labeling of G. In this paper, we determined $\chi_{lat}(G)$ where G is the amalgamation of complete graphs. Consequently, we also obtained the local antimagic (total) chromatic number of the disjoint union of complete graphs, and the join of K_1 and amalgamation of complete graphs under various conditions. An application of local antimagic total chromatic number is also given.

Keywords: local antimagic (total) chromatic number, amalgamation, complete graphs.

Mathematics Subject Classification: 05C78, 05C15.

1. INTRODUCTION

Consider a (p, q)-graph $G = (V, E)$ of order p and size q. In this paper, all graphs are simple. For positive integers $a < b$, let $[a, b] = \{a, a + 1, \dots, b\}$. Let $g : E(G) \to [1, q]$ be a bijective edge labeling that induces a vertex labeling $g^+ : V(G) \to \mathbb{N}$ such that $g^+(v) = \sum_{uv\in E(G)} g(uv)$. We say g is a *local antimagic labeling* of G if $g^+(u) \neq g^+(v)$ for each $uv \in E(G)$ [1, 2]. The number of distinct colors induced by g is called the *color number* of g and is denoted by $c(g)$. The number

$$\chi_{la}(G) = \min\{c(g) \mid g \text{ is a local antimagic labeling of } G\}$$

is called the *local antimagic chromatic number* of G [1]. Clearly, $\chi_{la}(G) \geq \chi(G)$.

Let $f : V(G) \cup E(G) \to [1, p+q]$ be a bijective total labeling that induces a vertex labeling $w_f : V(G) \to \mathbb{N}$, where

$$w_f(u) = f(u) + \sum_{uv \in E(G)} f(uv)$$

and is called the *weight* of u for each vertex $u \in V(G)$. We say f is a *local antimagic total labeling* of G (and G is *local antimagic total*) if $w_f(u) \neq w_f(v)$ for each $uv \in E(G)$. Clearly, w_f corresponds to a proper vertex coloring of G if each vertex v is assigned the color $w_f(v)$. If no ambiguity, we shall drop the subscript f. Let $w(f)$ be the number of distinct vertex weights induced by f. The number

$$\min\{w(f) \mid f \text{ is a local antimagic total labeling of } G\}$$

is called the *local antimagic total chromatic number* of G, denoted $\chi_{lat}(G)$. Clearly, $\chi_{lat}(G) \geq \chi(G)$. It is well known that determining the chromatic number of a graph G is NP-hard [14]. Thus, in general, it is also very difficult to determine $\chi_{la}(G)$ and $\chi_{lat}(G)$.

For a graph G, the graph $H = G \vee K_1$ is obtained from G by joining a new vertex to every vertex of G. We refer to [3] for notation not defined in this paper.

In [5], the author proved that every connected graph of order at least 3 is local antimagic.

Theorem 1.1. *Let G be a graph of order $p \geq 2$ and size q with $V(G) = \{v_i \mid 1 \leq i \leq p\}$.*

(a) $\chi(G) \leq \chi_{lat}(G) \leq \chi_{la}(G \vee K_1) - 1$.
(b) *Suppose f is local antimagic total $\chi_{lat}(G)$-coloring. If $\sum_{i=1}^{p} f(v_i) \neq w_f(v_j)$, $1 \leq j \leq p$, then $\chi_{la}(G \vee K_1) = \chi_{lat}(G) + 1$.*

Proof. Suppose G is a (p, q)-graph. Let the vertex sets of G and K_1 be $V(G) = \{v_i \mid 1 \leq i \leq p\}$ and $V(K_1) = \{v\}$, respectively. Since the order of $G \vee K_1$ is greater than 2, it is local antimagic. Let g be a local antimagic labeling of $G \vee K_1$ with color number $\chi_{la}(G \vee K_1)$.

Define a total labeling $f : V(G) \cup E(G) \to [1, p+q]$ of G by $f(e) = g(e)$ for each edge $e \in E(G)$ and $f(v_i) = g(vv_i)$. Clearly, $w_f(v_i) = g^+(v_i)$. Thus, $w_f(v_i) = w_f(v_j)$ if and only if $g^+(v_i) = g^+(v_j)$. Therefore, f is a local antimagic total labeling of G that induces $\chi_{la}(G \vee K_1) - 1$ vertex weights. Hence $\chi_{lat}(G) \leq \chi_{la}(G \vee K_1) - 1$. Moreover, $\chi_{lat}(G) \geq \chi(G)$ is obvious. So we have (a).

Let $\chi_{lat}(G) = a$. Define $g : E(G \vee K_1) \to [1, p+q]$ by $g(e) = f(e)$ if $e \in E(G)$, and $g(vv_i) = f(v_i)$ for each $v_i \in V(G)$. Clearly, $g^+(v) = \sum_{i=1}^{p} f(v_i)$ and $g^+(v_i) = w_f(v_i)$. Since $w_f(v_i) \neq w_f(v_j)$ if $v_iv_j \in E(G)$ and $g^+(v) \neq w_f(v_j)$ for $1 \leq j \leq p$, g is a local antimagic labeling and g^+ is an $(a+1)$-coloring of G. Hence, $\chi_{la}(G \vee K_1) \leq a + 1$. By (a), $\chi_{la}(G \vee K_1) \geq \chi_{lat}(G) + 1$. Thus we have $\chi_{la}(G \vee K_1) = \chi_{lat}(G) + 1$. We have (b). □

For $m \geq 2$ and $1 \leq i \leq m$, let G_i be a simple graph with an induced subgraph H. An *amalgamation* of $G_1, \ldots, G_m$ over H is the simple graph obtained by identifying the vertices of H of each G_i so that the new obtained graph contains a subgraph H induced by the identified vertices. Suppose G is a graph with a proper subgraph K_r, $r \geq 1$. Let $A(mG, K_r)$ be the amalgamation of $m \geq 2$ copies of G over K_r. Note that there may be many non-isomorphic $A(mG, K_r)$ graphs. For example, $A(2P_3, K_2)$ may be either $K_{1,3}$ or P_4. When $r = 1$, the graph is also known as one-point union of graphs. Note that $A(mK_2, K_1) \cong K_{1,m}$ and $A(mK_3, K_1)$ is the friendship graph $f_m, m \geq 2$. In [9, Theorem 2.4], the authors completely determined $\chi_{la}(A(mC_n, K_1))$ for $m \geq 2, n \geq 3$, where $2 \leq \chi(A(mC_n, K_1)) \leq 3$. Motivated by this, in this paper, we determine $\chi_{lat}(A(mK_n, K_r))$ and $\chi_{la}(A(mK_n, K_r) \vee K_1)$ for $m \geq 2$, $n \geq 2$, $n > r \geq 0$ (except $m \geq 4$, odd $n \geq 3$ and $r = 2$). Consequently, we determine $\chi_{lat}(mK_n)$ for even n and $\chi_{la}(mK_{4k+1})$, $k \geq 1$. We also obtain sharp upper bounds on $\chi_{lat}(mK_n)$ for odd $n \geq 3$. Moreover, we give an application of the number of local antimagic total chromatic number.

2. AMALGAMATIONS OF COMPLETE GRAPHS

Let f be a total labeling of a simple (p, q)-graph G. Let $V(G) = \{u_1, \ldots, u_p\}$. We define a total labeling matrix which is similar to the labeling matrix of an edge labeling introduced in [13].

Suppose $f : V(G) \cup E(G) \to S$ is a mapping, where S is a set of labels. A *total labeling matrix* M of f for G is a $p \times p$ symmetric matrix in which the (i, i)-entry of M is $f(u_i)$; the (i, j)-entry of M is $f(u_i u_j)$ if $u_i u_j \in E$ and is $*$ otherwise. If f is a local antimagic total labeling of G, then a total labeling matrix of f is called a *local antimagic total labeling matrix* of G. Clearly the i-th row sum (and i-th column sum) is $w_f(u_i)$, where $*$'s are treated as zero. Thus the condition of a total labeling matrix being a local antimagic total labeling matrix is the i-th row sum different from the j-th row sum when $u_i u_j \in E$.

For $m \geq 2$ and $n > r \geq 1$, let

$$V(A(mK_n, K_r)) = \{v_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

and

$$E(A(mK_n, K_r)) = \{v_{i,j} v_{i,k} \mid 1 \leq i \leq m, 1 \leq j < k \leq n\},$$

where $v_{1,j} = \cdots = v_{m,j}$ for each $n - r + 1 \leq j \leq n$. For convenience, let $u_j = v_{1,j}$ for $n - r + 1 \leq j \leq n$. Note that

$$A(mK_n, K_r) \equiv mK_{n-r} \vee K_r.$$

We first list the vertices of the m copies of K_{n-r} in lexicographic order followed by $u_{n-r+1}, \ldots, u_n$. Now let us show the structure of a total labeling matrix M of the graph $A(mK_n, K_r)$ under this list of vertices as a block matrix. Namely,

$$M = \begin{pmatrix} L_1 & \bigstar & \bigstar & \cdots & \cdots & \bigstar & B_1 \\ \bigstar & L_2 & \bigstar & \ddots & \cdots & \bigstar & B_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \bigstar & \cdots & \bigstar & L_i & \ddots & \bigstar & B_i \\ \vdots & \ddots & \cdots & \ddots & \ddots & \vdots & \vdots \\ \bigstar & \bigstar & \bigstar & \cdots & \bigstar & L_m & B_m \\ B_1^T & B_2^T & \cdots & B_i^T & \cdots & B_m^T & A \end{pmatrix}, \tag{2.1}$$

where L_i is an $(n-r)\times(n-r)$ symmetric matrix, B_i is an $(n-r)\times r$ matrix, $1 \le i \le m$, and A is an $r \times r$ symmetric matrix. Here $\bigstar$ denotes an $(n-r)\times(n-r)$ matrix whose entries are $*$'s. Thus, the corresponding total labeling matrix of the i-th K_n is

$$M_i = \begin{pmatrix} L_i & B_i \\ B_i^T & A \end{pmatrix}. \tag{2.2}$$

Now, a local antimagic total labeling for the graph $A(mK_n, K_r)$ is obtained if we use the integers in $[1, N]$ for all entries of the upper triangular part of all L_i's and A, and all entries of all B_i's such that the row sums of each matrix M_i are distinct, where

$$N = \frac{mn(n+1)}{2} - \frac{(m-1)r(r+1)}{2}.$$

We may extend the case to $r = 0$. We let $A(mK_n, K_0) = mK_n$ by convention. For this case, all of B_i's and A in (2.1) and (2.2) do not exist.

For a given matrix B, we shall use $\mathcal{R}_i(B)$ and $\mathcal{C}_j(B)$ to denote the i-th row sum and the j-th column sum of B, respectively. Also we shall use $\mathcal{D}(B)$ to denote the sum of the main diagonal of B if B is a square matrix. Suppose S is a finite subset of $\mathbb{Z}$. Let S^- and S^+ be a decreasing sequence and an increasing sequence of S, respectively.

We shall keep the notation defined above in this section.

Lemma 2.1. *Suppose $m \le n$. Let $M = (m_{j,k})$ be an $m \times n$ matrix with the following properties:*

(a) $m_{k,j} = m_{j,k}$ *for all* j, k, *where* $1 \le j < k \le m$,
(b) $m_{j,j} < m_{k,k}$ *if* $1 \le j < k \le m$,
(c) *for* $j_1 < k_1$ *and* $j_2 < k_2$, $(j_1, k_1) < (j_2, k_2)$ *in lexicographic order implies that* $m_{j_1,k_1} < m_{j_2,k_2}$.

Then $\mathcal{R}_j(M)$ is a strictly increasing function of j.

Proof.

$$\begin{aligned}\mathcal{R}_{j+1}(M)-\mathcal{R}_j(M) &= \sum_{k=1}^{n}(m_{j+1,k}-m_{j,k})\\ &= \sum_{k=1}^{j-1}(m_{j+1,k}-m_{j,k}) + (m_{j+1,j}-m_{j,j}+m_{j+1,j+1}-m_{j,j+1})\\ &\quad + \sum_{k=j+2}^{n}(m_{j+1,k}-m_{j,k})\\ &= \sum_{k=1}^{j-1}(m_{k,j+1}-m_{k,j}) + (m_{j+1,j+1}-m_{j,j})\\ &\quad + \sum_{k=j+2}^{n}(m_{j+1,k}-m_{j,k}) > 0.\end{aligned}$$

Note that the empty sum is treated as 0. This completes the proof. □

Lemma 2.2. *For positive integers t and m, let $S(a) = [m(a-1)+1, ma]$, $1 \le a \le t$. Then the following assertions hold.*

(i) *$\{S(a) \mid 1 \le a \le t\}$ is a partition of $[1, mt]$.*
(ii) *If $a < b$, then every term of $S(a)$ is less than every term of $S(b)$.*
(iii) *For any $1 \le a, b \le t$, the sum of the i-th term of $S^+(a)$ and that of $S^-(b)$ is independent of the choice of i, $1 \le i \le m$.*
(iv) *For any $1 \le a_l, b_l \le t$, $\sum_{l=1}^{k}$ (i-th term of $S^+(a_l)$) $+ \sum_{l=1}^{k}$ (i-th term of $S^-(b_l)$) is independent of the choice of i, $1 \le i \le m$.*

Proof. The first two parts are obvious. For (iii), the i-th terms of $S^+(a)$ and $S^-(b)$ are $m(a-1)+i$ and $m(b-1)+(m+1-i)$, respectively. So the sum is $m(a+b-1)+1$ which is independent of i. The last part follows from (iii). □

Before providing results about $\chi_{lat}(A(mK_n, K_r))$ for some m, n, r, we define a "sign matrix" $\mathcal{S}_n$ for even n.

Let $\mathcal{S}_2 = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}$ be a 2×2 matrix and $\mathcal{S}_4 = \begin{pmatrix} \mathcal{S}_2 & \mathcal{S}_2 \\ \mathcal{S}_2 & -\mathcal{S}_2 \end{pmatrix}$ be a 4×4 matrix. Let $\mathcal{S}_{4k}$ be a $(4k) \times (4k)$ matrix given by the following block matrix, where $k \ge 2$:

$$\mathcal{S}_{4k} = \begin{pmatrix} \mathcal{S}_4 & \cdots & \mathcal{S}_4 \\ \vdots & \ddots & \vdots \\ \mathcal{S}_4 & \cdots & \mathcal{S}_4 \end{pmatrix}.$$

Let $\mathcal{S}_{4k+2}$ be a $(4k+2)\times(4k+2)$ matrix as the following block matrix, where $k \geq 1$:

$$\mathcal{S}_{4k+2} = \left(\begin{array}{ccc|c} & & & \mathcal{S}_2 \\ & \mathcal{S}_{4k} & & \vdots \\ & & & \mathcal{S}_2 \\ \hline \mathcal{S}_2 & \cdots & \mathcal{S}_2 & \mathcal{S}_2 \end{array}\right).$$

We shall keep these notation in this section.

Remark 2.3. It is easy to see that each row and column sum of $\mathcal{S}_n$ are zero. Moreover, the diagonal sum of $\mathcal{S}_{4k}$ is zero.

Theorem 2.4. *For $m \geq 2$, even n and $n > r \geq 0$, $\chi_{lat}(A(mK_n, K_r)) = n$.*

Proof. Let $\mathcal{S}$ be the $(n-r)\times n$ matrix obtained from $\mathcal{S}_n$ by removing the last r rows of $\mathcal{S}_n$. First, define an $(n-r)\times n$ matrix $\mathcal{M}'$ in which $(\mathcal{M}')_{j,k} = (\mathcal{M}')_{k,j}$ for $1 \leq j < k \leq n-r$. Assign the increasing sequence $[1, (n-r)(n+r-1)/2]$ in lexicographic order to the upper part of the off-diagonal entries of $\mathcal{M}'$, denoted (j,k) if in row j and column k for $1 \leq j < k \leq n$. Next, assign $[(n-r)(n+r-1)/2+1, (n-r)(n+r-1)/2+(n-r)]$ to the entries of the main diagonal of $\mathcal{M}'$ in natural order.

Now, define an $(n-r)\times n$ 'guide matrix' $\mathcal{M}$ whose (j,k)-th entry is $(\mathcal{S})_{j,k}(\mathcal{M}')_{j,k}$, $1 \leq j \leq n-r$ and $1 \leq k \leq n$.

Stage 1. We shall assign labels to the upper triangular entries of L_i's and all the entries of B_i's. Note that if $r = 0$, all of B_i's and A do not exist. There are

$$N_1 = (n-r)(n+r-1)/2 + (n-r) = \frac{(n+r+1)(n-r)}{2}$$

entries needed to be filled for each i.

Now we shall use labels in $[1, mN_1]$ to fill in the m submatrices $\begin{pmatrix} L_i & B_i \end{pmatrix}$, $1 \leq i \leq m$. We use the sequences $S(a)$ defined in Lemma 2.2, where $t = N_1$. The (j,k)-entry of M_i is the i-th term of $S^+(a)$ or $S^-(a)$ if the corresponding (j,k)-entry of $\mathcal{M}$ is $+a$ or $-a$ respectively, where $1 \leq j \leq n-r$.

By Lemma 2.2 (iv), $\mathcal{R}_j(M_i)$ are the same for all i, $1 \leq i \leq m$. In other words, $w(v_{i,j})$ is a constant function for a fixed j, $1 \leq j \leq n-r$.

Stage 2. Note that when $r = 0$, the total labeling matrix M does not have the last row and column of block matrices so that we only need to perform Stage 1 above. Thus, we now assume $r \neq 0$. Also note that all integers in $[1, mN_1]$ are used up in Stage 1. Use the increasing sequence $[mN_1 + 1, mN_1 + r(r-1)/2]$ in lexicographic order for the off-diagonal entries of A. Lastly, use $[mN_1 + r(r-1)/2 + 1, N]$ in natural order for the diagonals of A. The lower triangular part duplicates the upper triangular part.

Consider the matrix M_m. Clearly it satisfies the conditions of Lemma 2.1. Hence $\mathcal{R}_j(M_m)$ is a strictly increasing function of j, $1 \le j \le n$. Thus $w(v_{i,j}) = w(v_{m,j})$ is a strictly increasing function of j, $1 \le j \le n - r$, for $1 \le i \le m$.

By the structure of B_i and A, and by Lemma 2.2(i), we have

$$\begin{aligned} w(u_{n-r+j}) &= \mathcal{R}_j(A) + \sum_{i=1}^{m} \mathcal{R}_j(B_i^T) = \mathcal{R}_j(A) + \sum_{i=1}^{m} \mathcal{C}_j(B_i) \\ &< \mathcal{R}_{j+1}(A) + \sum_{i=1}^{m} \mathcal{C}_{j+1}(B_i) = w(u_{n-r+j+1}), \end{aligned}$$

for $1 \le j \le r - 1$.

Thus $\chi_{lat}(A(mK_n, K_r)) = n$ since $\chi(A(mK_n, K_r)) = n$. □

Remark 2.5. Suppose f is a local antimagic total labeling of a graph G and M is the corresponding total labeling matrix. From the proof of Theorem 1.1(b) we can see that, if $\mathcal{D}(M)$ does not equal to every row (also column) sum of M, then f induces a local antimagic labeling of $G \vee K_1$ and

$$\chi(G \vee K_1) \le \chi_{la}(G \vee K_1) = \chi_{lat}(G) + 1.$$

Corollary 2.6. *For $m \ge 2$, even n and $n > r \ge 0$,*

$$\chi_{la}(A(mK_{n+1}, K_{r+1})) = \chi_{la}(A(mK_n, K_r) \vee K_1) = n + 1.$$

Proof. Note that $A(mK_{n+1}, K_{r+1}) \cong A(mK_n, K_r) \vee K_1$. Since

$$\chi_{la}(A(mK_n, K_r) \vee K_1) \ge \chi(A(mK_n, K_r) \vee K_1) = n + 1,$$

we only need to show

$$\chi_{la}(A(mK_n, K_r) \vee K_1) \le n + 1.$$

Keep the total labeling matrix M of $A(mK_n, K_r)$ in the proof of Theorem 2.4.

Since each diagonal of M is the largest entry in the corresponding column, $\mathcal{D}(M)$ is larger than each row sum of M. Thus, $\mathcal{D}(M)$ is greater than all vertex weights of $A(mK_n, K_r)$. By Remark 2.5, we get that $\chi_{la}(A(mK_n, K_r) \vee K_1) \le n + 1$. □

Example 2.7. We take $m = 3$, $n = 6$ and $r = 0$. The guide matrix is

$$\mathcal{M} = \begin{pmatrix} +16 & -1 & +2 & -3 & +4 & -5 \\ -1 & +17 & -6 & +7 & -8 & +9 \\ +2 & -6 & -18 & +10 & +11 & -12 \\ -3 & +7 & +10 & -19 & -13 & +14 \\ +4 & -8 & +11 & -13 & +20 & -15 \\ -5 & +9 & -12 & +14 & -15 & +21 \end{pmatrix}$$

M_1	$v_{1,1}$	$v_{1,2}$	$v_{1,3}$	$v_{1,4}$	$v_{1,5}$	$v_{1,6}$	sum
$v_{1,1}$	46	3	4	9	10	15	87
$v_{1,2}$	3	49	18	19	24	25	138
$v_{1,3}$	4	18	54	28	31	36	171
$v_{1,4}$	9	19	28	57	39	40	192
$v_{1,5}$	10	24	31	39	58	45	207
$v_{1,6}$	15	25	36	40	45	61	222

M_2	$v_{2,1}$	$v_{2,2}$	$v_{2,3}$	$v_{2,4}$	$v_{2,5}$	$v_{2,6}$	sum
$v_{2,1}$	47	2	5	8	11	14	87
$v_{2,2}$	2	50	17	20	23	26	138
$v_{2,3}$	5	17	53	29	32	35	171
$v_{2,4}$	8	20	29	56	38	41	192
$v_{2,5}$	11	23	32	38	59	44	207
$v_{2,6}$	14	26	38	41	44	62	222

M_3	$v_{3,1}$	$v_{3,2}$	$v_{3,3}$	$v_{3,4}$	$v_{3,5}$	$v_{3,6}$	sum
$v_{3,1}$	48	1	6	7	12	13	87
$v_{3,2}$	1	51	16	21	22	27	138
$v_{3,3}$	6	16	52	30	33	34	171
$v_{3,4}$	7	21	30	55	37	42	192
$v_{3,5}$	12	22	33	37	60	43	207
$v_{3,6}$	13	27	34	42	43	63	222

The above matrices give $\chi_{lat}(3K_6) = 6$. Let v be the vertex of K_1. If the main diagonal labels are the edge labels of $3K_6 \vee K_1$ incident with v, then the induced label of v is 981. Thus, the matrices give $\chi_{la}(3K_6 \vee K_1) = 7$.

Deleting edge $vv_{3,6}$ of label 63 from $3K_6 \vee K_1$, we get a local antimagic labeling of $(3K_6 \vee K_1) - vv_{3,6}$. Thus, by symmetry, $\chi_{la}((3K_6 \vee K_1) - e) = 7$ for e not belonging to any K_6.

Delete the edge $v_{3,1}v_{3,2}$ that has label 1 and reduce all other labels by 1. We get $\chi_{lat}(3K_6 - e) = 6$ by symmetry for e that belongs to any K_6. Now, if the main diagonal labels are the edge labels of $(3K_6 - e) \vee K_1$ incident with v, then we have $\chi_{la}(3K_6 \vee K_1) - e) = 7$ for e that belongs to any K_6.

Example 2.8. We take $m = 3$, $n = 6$ and $r = 1$. The guide matrix is obtained from the guide matrix of Example 2.7 by deleting the last row. So we have

$$M_1 = \left(\begin{array}{ccccc|c||c} 46 & 3 & 4 & 9 & 10 & 15 & 87 \\ 3 & 49 & 18 & 19 & 24 & 25 & 138 \\ 4 & 18 & 54 & 28 & 31 & 36 & 171 \\ 9 & 19 & 28 & 57 & 39 & 40 & 192 \\ 10 & 24 & 31 & 39 & 58 & 45 & 207 \\ \hline 15 & 25 & 36 & 40 & 45 & 61 & 222 \end{array}\right),$$

$$M_2 = \left(\begin{array}{ccccc|c||c} 47 & 2 & 5 & 8 & 11 & 14 & 87 \\ 2 & 50 & 17 & 20 & 23 & 26 & 138 \\ 5 & 17 & 53 & 29 & 32 & 35 & 171 \\ 8 & 20 & 29 & 56 & 38 & 41 & 192 \\ 11 & 23 & 32 & 38 & 59 & 44 & 207 \\ \hline 14 & 26 & 38 & 41 & 44 & 61 & 221 \end{array}\right),$$

$$M_3 = \left(\begin{array}{ccccc|c||c} 48 & 1 & 6 & 7 & 12 & 13 & 87 \\ 1 & 51 & 16 & 21 & 22 & 27 & 138 \\ 6 & 16 & 52 & 30 & 33 & 34 & 171 \\ 7 & 21 & 30 & 55 & 37 & 42 & 192 \\ 12 & 22 & 33 & 37 & 60 & 43 & 207 \\ \hline 13 & 27 & 34 & 42 & 43 & 61 & 220 \end{array}\right).$$

The last column of each matrix is the corresponding row sum. Now $w(u_6) = 222+221+220-2\times 61 = 541$. Hence $\chi_{lat}(A(3K_6, K_1)) = 6$. Since $\mathcal{D}(M) = 856$, $\chi_{la}(A(3K_6, K_1) \vee K_1) = \chi_{la}(A(3K_7, K_2)) = 7$.

Corollary 2.9. *For $m \geq 2$, $\chi_{la}(mK_{4k+1}) = 4k + 1$.*

Proof. Consider the total labeling matrix of mK_{4k} defined in the proof of Theorem 2.4. For each matrix $M_i = L_i$, $1 \leq i \leq m$, we add the $(n+1)$-st extra column at the right of M_i with entry $*$. For each row of this matrix, swap the diagonal entry with the entry of the $(n+1)$-st column. Add the $(n+1)$-st extra row to this matrix and let the $(n+1, n+1)$-entry be $*$ and then make the resulting matrix Q_i to be symmetric. Then Q_i is a labeling matrix of the i-th copy of K_{4k+1}.

By Remark 2.3 and Lemma 2.2 (iv), all the diagonal sums of M_i's are the same, $1 \leq i \leq n$. Thus the j-th row sum of Q_i is independent of i, $1 \leq j \leq n+1$. Hence we have $\chi_{la}(mK_{4k+1}) \leq 4k+1$. Since $\chi(mK_{4k+1}) = 4k+1$, $\chi_{la}(mK_{4k+1}) = 4k+1$. □

Example 2.10. We take $m = 3$, $n = 4$. So

$$\mathcal{M} = \begin{pmatrix} +7 & -1 & +2 & -3 \\ -1 & +8 & -4 & +5 \\ +2 & -4 & -9 & +6 \\ -3 & +5 & +6 & -10 \end{pmatrix},$$

and

$$M_1 = \left(\begin{array}{cccc||c} 19 & 3 & 4 & 9 & 35 \\ 3 & 22 & 12 & 13 & 50 \\ 4 & 12 & 27 & 16 & 59 \\ 9 & 13 & 16 & 30 & 68 \end{array}\right), \quad M_2 = \left(\begin{array}{cccc||c} 20 & 2 & 5 & 8 & 35 \\ 2 & 23 & 11 & 14 & 50 \\ 5 & 11 & 26 & 17 & 59 \\ 8 & 14 & 17 & 29 & 68 \end{array}\right),$$

$$M_3 = \left(\begin{array}{cccc||c} 21 & 1 & 6 & 7 & 35 \\ 1 & 24 & 10 & 15 & 50 \\ 6 & 10 & 25 & 18 & 59 \\ 7 & 15 & 18 & 28 & 68 \end{array}\right).$$

Thus $\chi_{lat}(3K_4) = 4$.

Let

$$Q_1 = \left(\begin{array}{ccccc||c} * & 3 & 4 & 9 & 19 & 35 \\ 3 & * & 12 & 13 & 22 & 50 \\ 4 & 12 & * & 16 & 27 & 59 \\ 9 & 13 & 16 & * & 30 & 68 \\ 19 & 22 & 27 & 30 & * & 98 \end{array}\right), \quad Q_2 = \left(\begin{array}{ccccc||c} * & 2 & 5 & 8 & 20 & 35 \\ 2 & * & 11 & 14 & 23 & 50 \\ 5 & 11 & * & 17 & 26 & 59 \\ 8 & 14 & 17 & * & 29 & 68 \\ 20 & 23 & 26 & 29 & * & 98 \end{array}\right),$$

$$Q_3 = \left(\begin{array}{ccccc||c} * & 1 & 6 & 7 & 21 & 35 \\ 1 & * & 10 & 15 & 24 & 50 \\ 6 & 10 & * & 18 & 25 & 59 \\ 7 & 15 & 18 & * & 28 & 68 \\ 21 & 24 & 25 & 28 & * & 98 \end{array}\right).$$

Thus $\chi_{la}(3K_5) = 5$.

Theorem 2.11. *For $m \geq 2$, odd $n \geq 5$ and $n > r \geq 3$,*

$$\chi_{lat}(A(mK_n, K_r)) = n.$$

Proof. Suppose r is odd so that $n - r$ is even.

Stage 1. Using the same approach of the proof of Theorem 2.4, we construct an $(n-r) \times (n-r)$ guide matrix $\mathcal{M}$.

Similar to the proof of Theorem 2.4, we use the guide matrix $\mathcal{M}$ for all entries of L_i, $1 \leq i \leq m$, using labels in $[1, mN_2]$, where $N_2 = (n-r)(n-r+1)/2$. Thus, $\mathcal{R}_j(L_i)$ is a function only depending on j and is strictly increasing, for $1 \leq i \leq m$ and $1 \leq j \leq n-r$.

Stage 2. Use $[mN_2+1, mN_2+(m(n-r)+1)r]$ to form an $(m(n-r)+1) \times r$ magic rectangle Ω. Note that the existence of this magic rectangle is referred to in [4]. Let

$$\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \\ \alpha \end{pmatrix} = \Omega,$$

where $\alpha = (A_{1,1}, A_{2,2}, \ldots, A_{r,r})$. Now, for a fixed i, $w(v_{i,j}) = \mathcal{R}_j(L_i) + \mathcal{R}_j(B_i)$. So $w(v_{i,j})$ is a function only depending on j and is strictly increasing, for $1 \le i \le m$ and $1 \le j \le n-r$.

Stage 3. Use the increasing sequence $[mN_2 + (m(n-r)+1)r+1, N]$ in lexicographic order for the remaining entries of the upper triangular part of A. For $1 \le j \le r$,

$$\begin{aligned}
w(u_{n-r+j}) = \sum_{i=1}^{m} \mathcal{R}_j(B_i^T) + \mathcal{R}_j(A) &= \sum_{i=1}^{m} \mathcal{C}_j(B_i) + A_{j,j} + \sum_{\substack{l=1 \\ l \neq j}}^{r} A_{j,l} \\
&= \mathcal{C}_j(\Omega) + \sum_{l=1}^{j-1} A_{j,l} + A_{j,j+1} + \sum_{l=j+2}^{r} A_{j,l} \\
&< \mathcal{C}_j(\Omega) + \sum_{l=1}^{j-1} A_{j+1,l} + A_{j+1,j} + \sum_{l=j+2}^{r} A_{j+1,l} \\
&\quad \text{(since } r \ge 3\text{, there is at least one non-empty sum)} \\
&= \mathcal{C}_{j+1}(\Omega) + \sum_{\substack{l=1 \\ l \neq j+1}}^{r} A_{j+1,l} = w(u_{n-r+j+1}).
\end{aligned}$$

So $w(u_{n-r+j})$ is a strictly increasing function of j for $1 \le j \le r$.

$$\begin{aligned}
w(v_{m,n-r}) &= \mathcal{R}_{n-r}(L_m) + \mathcal{R}_{n-r}(B_m) = \mathcal{C}_{n-r}(L_m) + \mathcal{R}_1(\alpha) \\
&\quad \text{(since } \Omega \text{ is a magic rectangle)} \\
&< \mathcal{C}_1(B_m) + \mathcal{R}_1(\alpha) < \mathcal{R}_1(B_m^T) + \sum_{k=1}^{r} A_{1,k} < w(u_{n-r+1}).
\end{aligned}$$

Thus, we have $\chi_{lat}A(mK_n, K_r) = n$.

Suppose r is even so that $n-r+1$ is even.

(a) Suppose m is odd so that $m(n-r)$ and $r-1$ are odd and at least 3.

Stage 1(a). We use a modification of the proof of Theorem 2.4. Firstly we use the $(n-r) \times (n-r+1)$ sign matrix $\mathcal{S}$. Next, we define an $(n-r) \times (n-r+1)$ matrix $\mathcal{M}'$ by assigning the increasing sequence $[1, (n-r)(n-r+1)/2]$ in lexicographic order to the off-diagonal entries of the upper triangular part of $\mathcal{M}'$. Now, assign $[(n-r)(n+r+1)/2 - (n-r-1), (n-r)(n+r+1)/2]$ to the entries of the main diagonal of $\mathcal{M}'$ in natural order. The $(n-r) \times (n-r+1)$ guide matrix $\mathcal{M}$ is defined the same way as in the proof of Theorem 2.4.

Write $B_i = \begin{pmatrix} \beta_i & X_i \end{pmatrix}$, where β_i and X_i are $(n-r) \times 1$ and $(n-r) \times (r-1)$ matrices, respectively. Similar to Stage 1 of the odd r case, for $\begin{pmatrix} L_i & \beta_i \end{pmatrix}$ use the labels in $[1, mZ_1] \cup [m[Z_2 - (n-r)] + 1, mZ_2]$, $1 \le i \le m$, where $Z_1 = (n-r)(n-r+1)/2$ and $Z_2 = Z_1 + (n-r)r$.

Thus, $\mathcal{R}_j \begin{pmatrix} L_i & \beta_i \end{pmatrix}$ is a function only depending on j and is a strictly increasing function of j, $1 \le i \le m$ and $1 \le j \le n-r$.

Stage 2(a). Use $[mZ_1+1, m[Z_1+(n-r)(r-1)]]$ to form an $m(n-r)\times(r-1)$ magic rectangle Ω. Let

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{pmatrix} = \Omega.$$

Now, for a fixed i, $w(v_{i,j}) = \mathcal{R}_j\begin{pmatrix} L_i & \beta_i\end{pmatrix} + \mathcal{R}_j(X_i)$. So $w(v_{i,j})$ is a function only depending on j and is a strictly increasing function of j for $1 \le j \le n-r$ and $1 \le i \le m$.

Stage 3(a). Use the increasing sequence $[mZ_2+1, mZ_2+(r-1)r/2]$ in lexicographic order for the off-diagonal entries of the upper triangular part of A and then use $[mZ_2+(r-1)r/2+1, N]$ in natural order for the diagonals of A. By Lemma 2.1, $\mathcal{R}_j(A)$ is a strictly increasing function of j for $1 \le j \le r$.

Now, for $2 \le j \le r$,

$$\begin{aligned} w(u_{n-r+j}) &= \sum_{i=1}^{m}\sum_{k=1}^{n-r}(B_i^T)_{j,k} + \sum_{l=1}^{r} A_{j,l} = \sum_{i=1}^{m}\sum_{k=1}^{n-r}(B_i)_{k,j} + \sum_{l=1}^{r} A_{j,l} \\ &= \mathcal{C}_{j-1}(\Omega) + \mathcal{R}_j(A). \end{aligned}$$

So $w(u_{n-r+j})$ is a strictly increasing function of j for $2 \le j \le r$.

Next

$$\begin{aligned} w(u_{n-r+1}) &= \sum_{i=1}^{m}\sum_{k=1}^{n-r}(B_i^T)_{1,k} + \sum_{l=1}^{r} A_{1,l} = \sum_{i=1}^{m}\sum_{k=1}^{n-r}(B_i)_{k,1} + \mathcal{R}_1(A) \\ &< \sum_{i=1}^{m}\sum_{k=1}^{n-r}(B_i)_{k,2} + \mathcal{R}_2(A) = w(u_{n-r+2}). \end{aligned}$$

Now

$$\begin{aligned} w(v_{m,n-r}) &= \mathcal{R}_{n-r}(L_m) + \mathcal{R}_{n-r}(B_m) = \mathcal{C}_{n-r}(L_m) + \mathcal{R}_{n-r}(B_m) \\ &< \mathcal{C}_1(B_m) + \sum_{k=1}^{r} A_{1,k} = \mathcal{R}_1(B_m^T) + \sum_{k=1}^{r} A_{1,k} < w(u_{n-r+1}). \end{aligned}$$

Thus we have $\chi_{lat}(A(mK_n, K_r)) = n$.

(b) Suppose m is even so that $m(n-r)$ is even.

Stage 1(b). Similar to Stage 1 of the odd r case we define an $(n-r)\times(n-r+1)$ guide matrix $\mathcal{M}$.

Write $B_i = \begin{pmatrix}\beta_i & X_i\end{pmatrix}$, where β_i and X_i are $(n-r)\times 1$ and $(n-r)\times(r-1)$ matrices, respectively. Similar to Stage 1 of the odd r case, for $\begin{pmatrix} L_i & \beta_i\end{pmatrix}$ use labels in $[1, mN_3]$, $1 \le i \le m$, where $N_3 = (n-r+3)(n-r)/2$. Thus, $\mathcal{R}_j\begin{pmatrix} L_i & \beta_i\end{pmatrix}$ is a function only depending on j and is a strictly increasing function of j, $1 \le i \le m$ and $1 \le j \le n-r$.

Stage 2(b). $[mN_3+2, mN_3+(m(n-r)+1)(r-1)+1]$ to form an $(m(n-r)+1)\times(r-1)$ magic rectangle Ω. We will assign mN_3+1 to $A_{1,1}$ in the next stage.

Let

$$\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \\ \alpha \end{pmatrix} = \Omega,$$

where α is a $1\times(r-1)$ matrix.

Same as Stage 2(a), we have $w(v_{i,j})$ is a function only depending on j and is a strictly increasing function for $1 \le j \le n-r$ and $1 \le i \le m$.

Stage 2(c). Let $A_{1,1} = mN_3+1$. Use the increasing sequence $[mN_3+(m(n-r)+1)\cdot(r-1)+2, N]$ in lexicographic order for the remaining entries of the upper triangular part of A.

By the same proof of Stage 3 of the odd r case, we have $w(u_{n-r+j})$ is a strictly increasing function of j, $2 \le j \le r$. By a similar proof of Stage 3(a) we have $w(u_{n-r+1}) < w(u_{n-r+2})$.

Now

$$\begin{aligned} w(v_{m,n-r}) &= \mathcal{R}_{n-r}(L_m) + \mathcal{R}_{n-r}(B_m) \\ &= \sum_{k=1}^{n-r}(L_m)_{n-r,k} + (\beta_m)_{n-r,1} + \sum_{k=1}^{r-1}(X_m)_{n-r,k} \\ &= \sum_{k=1}^{n-r-1}(L_m)_{k,n-r} + (L_m)_{n-r,n-r} + (\beta_m)_{n-r,1} + \sum_{k=1}^{r-1}(X_m)_{n-r,k} \\ &< \sum_{k=1}^{n-r-1}(\beta_m)_{k,1} + A_{1,1} + (\beta_m)_{n-r,1} + \sum_{k=2}^{r} A_{1,k} < w(u_{n-r+1}). \end{aligned}$$

Thus we have $\chi_{lat}(A(mK_n, K_r)) = n$. □

Corollary 2.12. *For $m \ge 2$, odd $n \ge 5$ and $n > r \ge 3$,*

$$\chi_{la}(A(mK_{n+1}, K_{r+1})) = \chi_{la}(A(mK_n, K_r) \vee K_1) = n+1.$$

Proof. Similarly to Corollary 2.6, we only need to show that

$$\chi_{la}(A(mK_n, K_r) \vee K_1) \le n+1.$$

Keep the construction in the proof of Theorem 2.11.

(a) Suppose r is odd so that $n-r \geq 2$ is even.

$$\begin{aligned} w(v_{m,n-r}) &= \sum_{i=1}^{n-r}(L_m)_{n-r,i} + \sum_{i=1}^{r}(B_m)_{n-r,i} \\ &= \sum_{i=1}^{n-r}(L_m)_{i,n-r} + \sum_{i=1}^{r} A_{i,i} \qquad \text{(since } \Omega \text{ is a magic rectangle)} \\ &< \sum_{i=1}^{n-r}(L_m)_{i,i} + \sum_{i=1}^{r} A_{i,i} \leq \mathcal{D}(M). \end{aligned}$$

Next

$$\begin{aligned} \mathcal{D}(M) &= \sum_{i=1}^{m}\sum_{j=1}^{n-r}(L_i)_{j,j} + A_{1,1} + \sum_{k=2}^{r} A_{k,k} < \sum_{i=1}^{m}\sum_{j=1}^{n-r}(B_i)_{j,1} + A_{1,1} + \sum_{k=2}^{r} A_{k,k} \\ &< \sum_{i=1}^{m}\sum_{j=1}^{n-r}(B_i)_{j,1} + A_{1,1} + \sum_{k=2}^{r} A_{1,k} = w(u_{n-r+1}). \end{aligned}$$

Thus we have $w(v_{m,n-r}) < \mathcal{D}(M) < w(u_{n-r+1})$.

(b) Suppose r is even and m is odd. Since each diagonal of M is the largest entry in the corresponding column, $\mathcal{D}(M)$ is larger than all the vertex weights.

(c) Suppose r is even and m is even.

$$\begin{aligned} \mathcal{D}(M) &= \sum_{i=1}^{m}\sum_{j=1}^{n-r}(L_i)_{j,j} + \sum_{k=1}^{r} A_{k,k} < \sum_{i=1}^{m}\sum_{j=1}^{n-r}(X_i)_{j,1} + \sum_{k=1}^{r} A_{2,k} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n-r}(X_i^T)_{1,j} + \sum_{k=1}^{r} A_{2,k} = w(u_{n-r+2}). \end{aligned}$$

$$\begin{aligned} w(u_{n-r+1}) &= \sum_{i=1}^{m}\sum_{k=1}^{n-r}(\beta_i)_{k,1} + A_{1,1} + \sum_{k=2}^{r} A_{1,k} \\ &< \sum_{i=1}^{m}\sum_{k=1}^{n-r}(L_i)_{k,k} + A_{1,1} + \sum_{k=2}^{r} A_{1,k} \\ &< \sum_{i=1}^{m}\sum_{k=1}^{n-r}(L_i)_{k,k} + A_{1,1} + \sum_{k=2}^{r} A_{k,k} = \mathcal{D}(M). \end{aligned}$$

From Remark 2.5, we have $\chi_{la}(A(mK_n, K_r) \vee K_1) \leq n+1$. □

Example 2.13. We take $m=2$, $n=7$ and $r=3$. The guide matrix is

$$\mathcal{M} = \begin{pmatrix} +7 & -1 & +2 & -3 \\ -1 & +8 & -4 & +5 \\ +2 & -4 & -9 & +6 \\ -3 & +9 & +6 & -10 \end{pmatrix}.$$

$$L_1 = \begin{pmatrix} 13 & 2 & 3 & 6 \\ 2 & 15 & 8 & 9 \\ 3 & 8 & 18 & 11 \\ 6 & 9 & 11 & 20 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 14 & 1 & 4 & 5 \\ 1 & 16 & 7 & 10 \\ 4 & 7 & 17 & 12 \\ 5 & 10 & 12 & 19 \end{pmatrix}.$$

$$\Omega = \begin{pmatrix} B_1 \\ B_2 \\ \alpha \end{pmatrix} = \left(\begin{array}{ccc} 25 & 30 & 47 \\ 34 & 39 & 29 \\ 36 & 44 & 22 \\ 41 & 28 & 33 \\ \hline 43 & 21 & 38 \\ 45 & 26 & 31 \\ 23 & 37 & 42 \\ 27 & 35 & 40 \\ \hline 32 & 46 & 24 \end{array}\right), \quad A = \begin{pmatrix} 32 & 48 & 49 \\ 48 & 46 & 50 \\ 49 & 50 & 24 \end{pmatrix}.$$

$$M = \left(\begin{array}{cccc|cccc|ccc||c} 13 & 2 & 3 & 6 & * & * & * & * & 25 & 30 & 47 & 126 \\ 2 & 15 & 8 & 9 & * & * & * & * & 34 & 39 & 29 & 136 \\ 3 & 8 & 18 & 11 & * & * & * & * & 36 & 44 & 22 & 142 \\ 6 & 9 & 11 & 20 & * & * & * & * & 41 & 28 & 33 & 148 \\ \hline * & * & * & * & 14 & 1 & 4 & 5 & 43 & 21 & 38 & 126 \\ * & * & * & * & 1 & 16 & 7 & 10 & 45 & 26 & 31 & 136 \\ * & * & * & * & 4 & 7 & 17 & 12 & 23 & 37 & 42 & 142 \\ * & * & * & * & 5 & 10 & 12 & 19 & 27 & 35 & 40 & 148 \\ \hline 25 & 34 & 36 & 41 & 43 & 45 & 23 & 27 & 32 & 48 & 49 & 403 \\ 30 & 39 & 44 & 28 & 21 & 26 & 37 & 35 & 48 & 46 & 50 & 404 \\ 47 & 29 & 22 & 33 & 38 & 31 & 42 & 40 & 49 & 50 & 24 & 405 \end{array}\right).$$

Thus $\chi_{lat}(A(2K_7, K_3)) = 7$. Since $\mathcal{D}(M) = 234$, $\chi_{la}(A(2K_7, K_3) \vee K_1) = 8$.

Example 2.14. We take $m = 3$, $n = 7$ and $r = 4$. The guide matrix is

$$\mathcal{M} = \begin{pmatrix} +16 & -1 & +2 & -3 \\ -1 & +17 & -4 & +5 \\ +2 & -4 & -18 & +6 \end{pmatrix}.$$

$$(L_1|\beta_1) = \left(\begin{array}{ccc|c} 46 & 3 & 4 & 9 \\ 3 & 49 & 12 & 13 \\ 4 & 12 & 54 & 16 \end{array}\right), \quad (L_2|\beta_2) = \left(\begin{array}{ccc|c} 47 & 2 & 5 & 8 \\ 2 & 50 & 11 & 14 \\ 5 & 11 & 53 & 17 \end{array}\right),$$

$$(L_3|\beta_3) = \left(\begin{array}{ccc|c} 48 & 1 & 6 & 7 \\ 1 & 51 & 10 & 15 \\ 6 & 10 & 52 & 18 \end{array}\right).$$

$$\Omega = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \left(\begin{array}{ccc} 27 & 32 & 37 \\ 20 & 34 & 42 \\ 31 & 39 & 26 \\ \hline 36 & 41 & 19 \\ 29 & 43 & 24 \\ 40 & 21 & 35 \\ \hline 45 & 23 & 28 \\ 38 & 25 & 33 \\ 22 & 30 & 44 \end{array}\right), \quad A = \begin{pmatrix} 61 & 55 & 56 & 57 \\ 55 & 62 & 58 & 59 \\ 56 & 58 & 63 & 60 \\ 57 & 59 & 60 & 64 \end{pmatrix}.$$

$$M = \left(\begin{array}{ccc|ccc|ccc|cccc||c} 46 & 3 & 4 & * & * & * & * & * & * & 9 & 27 & 32 & 37 & 158 \\ 3 & 49 & 12 & * & * & * & * & * & * & 13 & 20 & 34 & 42 & 173 \\ 4 & 12 & 54 & * & * & * & * & * & * & 16 & 31 & 39 & 26 & 182 \\ \hline * & * & * & 47 & 2 & 5 & * & * & * & 8 & 36 & 41 & 19 & 158 \\ * & * & * & 2 & 50 & 11 & * & * & * & 14 & 29 & 43 & 24 & 173 \\ * & * & * & 5 & 11 & 53 & * & * & * & 17 & 40 & 21 & 35 & 182 \\ \hline * & * & * & * & * & * & 48 & 1 & 6 & 7 & 45 & 23 & 28 & 158 \\ * & * & * & * & * & * & 1 & 51 & 10 & 15 & 38 & 25 & 33 & 173 \\ * & * & * & * & * & * & 6 & 10 & 52 & 18 & 22 & 30 & 44 & 182 \\ \hline 9 & 13 & 16 & 8 & 14 & 17 & 7 & 15 & 18 & 61 & 55 & 56 & 57 & 346 \\ 27 & 20 & 31 & 36 & 29 & 40 & 45 & 38 & 22 & 55 & 62 & 58 & 59 & 522 \\ 32 & 34 & 39 & 41 & 43 & 21 & 23 & 25 & 30 & 56 & 58 & 63 & 60 & 525 \\ 37 & 42 & 26 & 19 & 24 & 35 & 28 & 33 & 44 & 57 & 59 & 60 & 64 & 528 \end{array}\right).$$

Thus $\chi_{lat}(A(3K_7, K_4)) = 7$. Since $\mathcal{D}(M) = 700$,

$$\chi_{la}(A(3K_7, K_4) \vee K_1) = \chi_{la}(A(3K_8, K_5)) = 8.$$

Example 2.15. We take $m = 2$, $n = 7$ and $r = 4$. The guide matrix is

$$\mathcal{M} = \begin{pmatrix} +7 & -1 & +2 & -3 \\ -1 & +8 & -4 & +5 \\ +2 & -4 & -9 & +6 \end{pmatrix}.$$

$$(L_1|\beta_1) = \left(\begin{array}{ccc|c} 13 & 2 & 3 & 6 \\ 2 & 15 & 8 & 9 \\ 3 & 8 & 18 & 11 \end{array}\right), \quad (L_2|\beta_2) = \left(\begin{array}{ccc|c} 14 & 1 & 4 & 5 \\ 1 & 16 & 7 & 10 \\ 4 & 7 & 17 & 12 \end{array}\right).$$

$$\Omega = \begin{pmatrix} X_1 \\ X_2 \\ \alpha \end{pmatrix} = \left(\begin{array}{ccc} 27 & 25 & 38 \\ 24 & 31 & 35 \\ 21 & 32 & 37 \\ \hline 39 & 22 & 29 \\ 36 & 26 & 28 \\ 33 & 34 & 23 \\ \hline 30 & 40 & 20 \end{array}\right), \quad A = \begin{pmatrix} 19 & 30 & 40 & 20 \\ 30 & 41 & 42 & 43 \\ 40 & 42 & 44 & 45 \\ 20 & 43 & 45 & 46 \end{pmatrix}.$$

$$M = \left(\begin{array}{ccc|ccc|cccc||c} 13 & 2 & 3 & * & * & * & 6 & 27 & 25 & 38 & 114 \\ 2 & 15 & 8 & * & * & * & 9 & 24 & 31 & 35 & 124 \\ 3 & 8 & 18 & * & * & * & 11 & 21 & 32 & 37 & 130 \\ \hline * & * & * & 14 & 1 & 4 & 5 & 39 & 22 & 29 & 114 \\ * & * & * & 1 & 16 & 7 & 10 & 36 & 26 & 28 & 124 \\ * & * & * & 4 & 7 & 17 & 12 & 33 & 34 & 23 & 130 \\ \hline 6 & 9 & 11 & 5 & 10 & 12 & 19 & 30 & 40 & 20 & 162 \\ 27 & 24 & 21 & 39 & 36 & 33 & 30 & 41 & 42 & 43 & 336 \\ 25 & 31 & 32 & 22 & 26 & 34 & 40 & 42 & 44 & 45 & 341 \\ 38 & 35 & 37 & 29 & 28 & 23 & 20 & 43 & 45 & 46 & 344 \end{array}\right).$$

Thus $\chi_{lat}(A(2K_7, K_4)) = 7$. Since $\mathcal{D}(M) = 243$, $\chi_{la}(A(2K_7, K_4) \vee K_1) = 8$.

Theorem 2.16. *For $m \geq 2$ and $n = 4k + 3 \geq 3$, $\chi_{lat}(mK_n) \leq n + 1$.*

Proof. Consider $\mathcal{S}_{4k+2}$. Change each diagonal entry of $\mathcal{S}_{4k+2}$ from ± 1 to ± 2. Let this new sign matrix be $\mathcal{S}'$. Now define $\mathcal{S}_{4k+3}$ by extending $\mathcal{S}'$ to a $4k+3$ square matrix by adding the last column and row. Use $+1$, -1 or $+2$ for each entry of this new column and row such that the row sum and the column sums of $\mathcal{S}_{4k+3}$ are zero.

Define a symmetric matrix $\mathcal{M}'$ of order $4k + 3$ by using the increasing sequence $[1, (2k + 1)(4k + 3)]$ in lexicographic order for the upper triangular entries of $\mathcal{M}'$, and use $*$ for all diagonal entries.

Now let $\mathcal{M}$ be the guide matrix whose (j, l)-th entry is $*$ if $j = l$; and $(\mathcal{S}_{4k+3})_{j,l}(\mathcal{M}')_{j,l}$, $1 \leq j, l \leq 4k + 3$ if $j \neq l$.

Stage 1. Using the same procedure as in Stage 1 in the proof of Theorem 2.4 fill the off-diagonals of the m submatrices L_i, $1 \leq i \leq m$, with labels in $[1, mN_4]$, where $N_4 = \frac{n(n-1)}{2}$.

Stage 2. Let

$$T(2a - 1) = \{mN_4 + 2l - 1 + 2m(a - 1) \mid 1 \leq l \leq m\}$$

and

$$T(2a) = \{mN_4 + 2l + 2m(a - 1) \mid 1 \leq l \leq m\},$$

$1 \leq a \leq 2k + 1$. For $1 \leq j \leq 4k + 2$, the (j, j)-entry of L_i is the i-th term of $T^-(j)$ or $T^+(j)$, if the corresponding (j, j)-entry of $\mathcal{M}$ is -2 or $+2$, respectively.

Stage 3. Let

$$U(1) = \{mN_4 + (4k + 2)m + 2l - 1 \mid 1 \leq l \leq \lceil m/2 \rceil\}$$

and

$$U(2) = \{mN_4 + (4k + 2)m + 2l \mid 1 \leq l \leq \lfloor m/2 \rfloor\}.$$

Let U be the compound sequence $U^+(1)U^+(2)$, i.e., list the terms of $U^+(1)$ first and then follow with the terms of $U^+(2)$. The $(4k + 3, 4k + 3)$-entry of L_i is the i-th term of U.

For a fixed j, $1 \le j \le 4k+2$, according to the structures of $S(a)$'s and $T(a)$'s, $(L_{i+1})_{j,l} - (L_i)_{j,l}$ is $(\mathcal{S}_{4k+3})_{j,l}$ for $1 \le i \le m-1$, $1 \le l \le 4k+3$. Thus, $w(v_{i,j})$ is a constant for a fixed j.

Similarly, $(L_{i+1})_{4k+3,l} - (L_i)_{4k+3,l}$ is $(\mathcal{S}_{4k+3})_{4k+3,l}$ for $1 \le i \le m-1$, $1 \le l \le 4k+2$. Finally, for $1 \le i \le m-1$,

$$(L_{i+1})_{4k+3,4k+3} - (L_i)_{4k+3,4k+3} = +2 \quad \text{if } i \ne \lceil m/2 \rceil.$$

Thus, $\{w(v_{i,4k+3}) \mid 1 \le i \le m\}$ consists of two different values.

Since each L_i satisfies the condition of Lemma 2.1, $\mathcal{R}_j(L_i)$ is a strictly increasing function of j. Hence $\chi_{lat}(mK_n) \le n+1$. □

Example 2.17. Take $n = 7$ and $m = 2$. So

$$\mathcal{S}_6 = \begin{pmatrix} +1 & -1 & +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 & +1 & -1 \\ -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & -1 & +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 & -1 & +1 \end{pmatrix} \to \mathcal{S}_7 = \left(\begin{array}{cccccc|c} +2 & -1 & +1 & -1 & +1 & -1 & -1 \\ -1 & +2 & -1 & +1 & -1 & +1 & -1 \\ +1 & -1 & -2 & +1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -2 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & +2 & -1 & -1 \\ -1 & +1 & -1 & +1 & -1 & +2 & -1 \\ \hline -1 & -1 & +1 & +1 & -1 & -1 & +2 \end{array}\right),$$

$$\mathcal{M} = \begin{pmatrix} * & -1 & +2 & -3 & +4 & -5 & -6 \\ -1 & * & -7 & +8 & -9 & +10 & -11 \\ +2 & -7 & * & +12 & +13 & -14 & +15 \\ -3 & +8 & +12 & * & -16 & +17 & +18 \\ +4 & -9 & +13 & -16 & * & -19 & -20 \\ -5 & +10 & -14 & +17 & -19 & * & -21 \\ -6 & -11 & +15 & +18 & -20 & -21 & * \end{pmatrix},$$

$$L_1 = \left(\begin{array}{ccccccc||c} 43 & 2 & 3 & 6 & 7 & 10 & 12 & 83 \\ 2 & 44 & 14 & 15 & 18 & 19 & 22 & 134 \\ 3 & 14 & 49 & 23 & 25 & 28 & 29 & 171 \\ 6 & 15 & 23 & 50 & 32 & 33 & 35 & 194 \\ 7 & 18 & 25 & 32 & 51 & 38 & 40 & 211 \\ 10 & 19 & 28 & 33 & 38 & 52 & 42 & 222 \\ 12 & 22 & 29 & 35 & 40 & 42 & 55 & 235 \end{array}\right),$$

$$L_2 = \left(\begin{array}{ccccccc||c} 45 & 1 & 4 & 5 & 8 & 9 & 11 & 83 \\ 1 & 46 & 13 & 16 & 17 & 20 & 21 & 134 \\ 4 & 13 & 47 & 24 & 26 & 27 & 30 & 171 \\ 5 & 16 & 24 & 48 & 31 & 34 & 36 & 194 \\ 8 & 17 & 26 & 31 & 53 & 37 & 39 & 211 \\ 9 & 20 & 27 & 34 & 37 & 54 & 41 & 222 \\ 11 & 21 & 30 & 36 & 39 & 41 & 56 & 234 \end{array}\right).$$

So $\chi_{lat}(2K_7) \le 8$.

Theorem 2.18. *For $m \geq 2$ and $n = 4k+1 \geq 5$,*

$$\chi_{lat}(mK_n) \leq \min\{n+3, n-1+m\}.$$

Proof. Similar to the proof of Theorem 2.16 we will define a sign matrix $\mathcal{S}$ and a guide matrix $\mathcal{M}$ of order $4k+1$. We define a $(4k) \times (4k)$ matrix $\mathcal{S}'_{4k}$ first.

Let

$$\mathcal{S}'_4 = \begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{pmatrix} \quad \text{and} \quad \mathcal{S}'_{4k} = \left(\begin{array}{ccc|c} & \mathcal{S}'_{4k-4} & & \begin{matrix} \mathcal{S}'_4 \\ \vdots \\ \mathcal{S}'_4 \end{matrix} \\ \hline \mathcal{S}'_4 & \cdots & \mathcal{S}'_4 & \mathcal{S}_4 \end{array}\right)$$

when $k \geq 2$.

Now, we define a symmetric sign matrix $\mathcal{S}_{4k+1}$ of order $4k+1$ using the same method as in the proof of Theorem 2.16. Note that the $(4k+1, 4k+1)$-entry of $\mathcal{S}_{4k+1}$ is $+4$. The definition of a guide matrix $\mathcal{M}$ and Stage 1 are similar to the proof of Theorem 2.16.

Stage 2. Using a similar procedure to Stage 2 in the proof of Theorem 2.16, fill all diagonals of each L_i except $(L_i)_{4k+1,4k+1}$ by using $T(j)$ defined in the proof of Theorem 2.16, $1 \leq j \leq 4k$.

Now we have to fill $[mN_4 + 4mk + 1, mN_4 + 4mk + m]$ in the $(L_i)_{4k+1,4k+1}$, here $N_4 = \frac{n(n-1)}{2}$. Suppose $m = 4s + m_0$, for some $s \geq 0$ and $0 \leq m_0 < 4$.

Suppose $s \geq 1$. Let

$$U(a) = \{mN_4 + 4km + a + 4l \mid 0 \leq l \leq s\}$$

for $1 \leq a \leq m_0$; and

$$U(a) = \{mN_4 + 4km + a + 4l \mid 0 \leq l \leq s-1\}$$

for $m_0 < a \leq 4$. Let U be the compound sequence $U^+(1)U^+(2)U^+(3)U^+(4)$. Let the $(4k+1, 4k+1)$-entry of L_i be the i-th term of U.

Using a proof similar to Theorem 2.16, we have $\chi_{lat}(mK_n) \leq n+3$.

Suppose $s = 0$. That means $2 \leq m \leq 4$. Then fill $[mN_4 + 4km + 1, mN_4 + 4km + m]$ to the $(4k+1, 4k+1)$-entry of L_i, respectively.

Using a proof similar to that above, $\mathcal{R}_j(L_i)$ is a strictly increasing function of j and hence we have $\chi_{lat}(mK_n) \leq n-1+m$.

Combining these two cases, we conclude that $\chi_{lat}(mK_n) \leq \min\{n+3, n-1+m\}$. □

Corollary 2.19. *Let $m \geq 2$ and odd $n \geq 3$,*

$$\chi_{lat}(A(mK_n, K_1)) = n.$$

Proof. From the proofs of Theorems 2.16 and 2.18, we see that the (n,n)-th entries of all L_i's are the largest m labels. If we change $(L_i)_{n,n}$ to N and denote this new matrix by M_i, then the matrix M becomes a total labeling matrix of $A(mK_n, K_1)$ with n difference row sums. Note that, $(L_i)_{n,n}$'s are identified to $A_{1,1}$. Thus $\chi_{lat}(A(mK_n, K_1)) = n$. □

Example 2.20. Let $n = 9$, $m = 2$. Then

$$\mathcal{S}_8' = \left(\begin{array}{cccc|cccc}
+1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
-1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
-1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\
\hline
+1 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
-1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\
+1 & -1 & +1 & -1 & +1 & -1 & -1 & +1 \\
-1 & +1 & -1 & +1 & -1 & +1 & +1 & -1
\end{array}\right)$$

$$\to \mathcal{S}_9 = \left(\begin{array}{cccc|cccc|c}
+2 & -1 & +1 & -1 & +1 & -1 & +1 & -1 & -1 \\
-1 & +2 & -1 & +1 & -1 & +1 & -1 & +1 & -1 \\
+1 & -1 & +2 & -1 & +1 & -1 & +1 & -1 & -1 \\
-1 & +1 & -1 & +2 & -1 & +1 & -1 & +1 & -1 \\
\hline
+1 & -1 & +1 & -1 & +2 & -1 & +1 & -1 & -1 \\
-1 & +1 & -1 & +1 & -1 & +2 & -1 & +1 & -1 \\
+1 & -1 & +1 & -1 & +1 & -1 & -2 & +1 & +1 \\
-1 & +1 & -1 & +1 & -1 & +1 & +1 & -2 & +1 \\
\hline
-1 & -1 & -1 & -1 & -1 & -1 & +1 & +1 & +4
\end{array}\right).$$

Hence

$$\mathcal{M} = \left(\begin{array}{cccc|cccc|c}
* & -1 & +2 & -3 & +4 & -5 & +6 & -7 & -8 \\
-1 & * & -9 & +10 & -11 & +12 & -13 & +14 & -15 \\
+2 & -9 & * & -16 & +17 & -18 & +19 & -20 & -21 \\
-3 & +10 & -16 & * & -22 & +23 & -24 & +25 & -26 \\
\hline
+4 & -11 & +17 & -22 & * & -27 & +28 & -29 & -30 \\
-5 & +12 & -18 & +23 & -27 & * & -31 & +32 & -33 \\
+6 & -13 & +19 & -24 & +28 & -31 & * & +34 & +35 \\
-7 & +14 & -20 & +25 & -29 & +32 & +34 & * & +36 \\
\hline
-8 & -15 & -21 & -26 & -30 & -33 & +35 & +36 & *
\end{array}\right).$$

We have

$$L_1 = \left(\begin{array}{ccccccccc||c}
73 & 2 & 3 & 6 & 7 & 10 & 11 & 14 & 16 & 142 \\
2 & 74 & 18 & 19 & 22 & 23 & 26 & 27 & 30 & 241 \\
3 & 18 & 77 & 32 & 33 & 36 & 37 & 40 & 42 & 318 \\
6 & 19 & 32 & 78 & 44 & 45 & 48 & 49 & 52 & 373 \\
7 & 22 & 33 & 44 & 81 & 54 & 55 & 58 & 60 & 414 \\
10 & 23 & 36 & 45 & 54 & 82 & 62 & 63 & 66 & 441 \\
11 & 26 & 37 & 48 & 55 & 62 & 87 & 67 & 69 & 462 \\
14 & 27 & 40 & 49 & 58 & 63 & 67 & 88 & 71 & 477 \\
16 & 30 & 42 & 52 & 60 & 66 & 69 & 71 & 89 & 495
\end{array}\right),$$

$$L_2 = \left(\begin{array}{ccccccccc||c} 75 & 1 & 4 & 5 & 8 & 9 & 12 & 13 & 15 & 142 \\ 1 & 76 & 17 & 20 & 21 & 24 & 25 & 28 & 29 & 241 \\ 4 & 17 & 79 & 31 & 34 & 35 & 38 & 39 & 41 & 318 \\ 5 & 20 & 31 & 80 & 43 & 46 & 47 & 50 & 51 & 373 \\ 8 & 21 & 34 & 43 & 83 & 53 & 56 & 57 & 59 & 414 \\ 9 & 24 & 35 & 46 & 53 & 84 & 61 & 64 & 65 & 441 \\ 12 & 25 & 38 & 47 & 56 & 61 & 85 & 68 & 70 & 462 \\ 13 & 28 & 39 & 50 & 57 & 64 & 68 & 86 & 72 & 477 \\ 15 & 29 & 41 & 51 & 59 & 65 & 70 & 72 & 90 & 492 \end{array}\right).$$

Hence $\chi_{lat}(2K_9) \leq 10$. Clearly $\mathcal{D}(M)$ is larger than all vertex weights, so $\chi_{la}((2K_9) \vee K_1) \leq 11$.

If we change $(L_2)_{9,9}$ to 89 and identify the vertices $v_{1,9}$ with $v_{2,9}$, then we have a local antimagic total labeling for $A(2K_9, K_1)$ and hence $\chi_{lat}(A(2K_9, K_1)) = 9$. Clearly $\mathcal{D}(M)$ is larger than all vertex weights, so $\chi_{la}(A(2K_9, K_1) \vee K_1) = 10$.

Corollary 2.21. *Let $m \geq 2$ and odd $n \geq 3$,*

$$\chi_{la}(mK_{n+1}) = \chi_{la}((mK_n) \vee K_1) \leq \begin{cases} \min\{n+4, n+m\} & \text{if } n \equiv 1 \pmod 4, \\ n+2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. From the proofs of Theorems 2.16 and 2.18, we see that the diagonals of M are the largest mn labels. Thus, $\mathcal{D}(M) > w(v_{m,n})$. So we have the corollary. □

By Corollary 2.19 and the same argument above, we have

Corollary 2.22. *Let $m \geq 2$ and odd $n \geq 3$,*

$$\chi_{la}(A(mK_{n+1}, K_2)) = \chi_{la}(A(mK_n, K_1) \vee K_1) = n + 1.$$

We have an ad hoc result for $A(mK_n, K_2)$ for n odd as follows.

Theorem 2.23. *For $2 \leq m \leq 3$ and odd $n \geq 3$,*

$$\chi_{lat}(A(mK_n, K_2)) = n \quad \text{and} \quad \chi_{la}(A(mK_n, K_2) \vee K_1) = n + 1.$$

Proof. The guide matrix is obtained from the guide matrix $\mathcal{M}$ of order n defined in the proof of Theorem 2.16 or Theorem 2.18 by deleting the last two rows. Apply the same procedure as Stage 1 in the proof of Theorem 2.16 or Theorem 2.18. It is equivalent to defining the matrix $\begin{pmatrix} L_i & B_i \end{pmatrix}$ of order $(n-2) \times n$ by deleting the last two rows of L_i defined in the proof of Theorem 2.16 or Theorem 2.18. Note that $(\mathcal{M})_{n-1,n} = -\frac{n(n-1)}{2}$ is not used in this stage. So the labels used are $[1, mN_5]$, where $N_5 = \frac{(n-2)(n+1)}{2}$.

Stage 2. Similar to Stage 2 of the proof of Theorem 2.16 or Theorem 2.18, we define the set $T(j)$ by labels $[mN_5+1, mN_5+m(n-2)+m]$ for $1 \le j \le n-1$ and fill the diagonals of L_i's. Note that, $mN_5+m(n-2)+m \le N$ and $T(n-1)$ is not used in this stage.

By Lemma 2.1, each j-th row sum of each $\begin{pmatrix} L_i & B_i \end{pmatrix}$ is a constant, and each j-th row sum of a fixed matrix $\begin{pmatrix} L_i & B_i \end{pmatrix}$ is a strictly increasing function of j, $1 \le j \le n-2$.

Stage 3. Use the remaining 3 labels to fill in the matrix $A_{1,2}, A_{1,1}, A_{2,2}$ in natural order. It is easy to see that the last two row sums of M are distinct and larger than the other row sums of M.

It is easy to see that the diagonal entries of M are the largest entries in the corresponding columns, and $\mathcal{D}(M)$ is greater than all weights of vertices.

Thus we have $\chi_{lat}(A(mK_n, K_2)) = n$ and $\chi_{la}(A(mK_n, K_2) \vee K_1) = n+1$. □

Example 2.24. Take $n = 7$ and $m = 2$.

$$\begin{pmatrix} L_1 & B_1 \end{pmatrix} = \left(\begin{array}{ccccc|cc||c} 41 & 2 & 3 & 6 & 7 & 10 & 12 & 82 \\ 2 & 42 & 14 & 15 & 18 & 19 & 22 & 132 \\ 3 & 14 & 47 & 23 & 25 & 28 & 29 & 169 \\ 6 & 15 & 23 & 48 & 32 & 33 & 35 & 192 \\ 7 & 18 & 25 & 32 & 49 & 38 & 40 & 209 \\ \hline 10 & 19 & 28 & 33 & 38 & 52 & 50 & 230 \\ 12 & 22 & 29 & 35 & 40 & 50 & 53 & 241 \end{array}\right),$$

$$\begin{pmatrix} L_2 & B_2 \end{pmatrix} = \left(\begin{array}{ccccc|cc||c} 43 & 1 & 4 & 5 & 8 & 9 & 11 & 81 \\ 1 & 44 & 13 & 16 & 17 & 20 & 21 & 132 \\ 4 & 13 & 45 & 24 & 26 & 27 & 30 & 169 \\ 5 & 16 & 24 & 46 & 31 & 34 & 36 & 192 \\ 8 & 17 & 26 & 31 & 51 & 37 & 39 & 209 \\ \hline 9 & 20 & 27 & 34 & 37 & 52 & 50 & 229 \\ 11 & 21 & 30 & 36 & 39 & 50 & 53 & 240 \end{array}\right).$$

$w(u_6) = 230 + 229 - (50 + 52) = 357$ and $w(u_7) = 241 + 240 - (50 + 53) = 378$. So $\chi_{lat}(A(2K_7, K_2)) = 7$.

For $m \ge 2$, $n > r \ge 0$, let us summarize our results in this section.

According to Tables 1 and 2 below, there are still some open problems for further study.

Table 1

r	n		$\chi_{lat}(mK_n, K_r)$
≥ 0	even	≥ 2	n
0	$\equiv 3 \pmod 4$	≥ 3	$\leq n+1$
0	$\equiv 1 \pmod 4$	≥ 5	$\leq \min\{n+3, n-1+m\}$
1	odd	≥ 3	n
2	odd	≥ 3	n, where $2 \leq m \leq 3$
≥ 3	odd	≥ 5	n

Table 2

r	n		$\chi_{la}(mK_n, K_r)$
≥ 1	odd	≥ 3	n
0	$\equiv 0 \pmod 4$	≥ 4	$\leq n+1$
0	$\equiv 1 \pmod 4$	≥ 5	n
0	$\equiv 2 \pmod 4$	≥ 6	$\leq \min\{n+3, n-1+m\}$
2	even	≥ 4	n
3	even	≥ 3	n, where $2 \leq m \leq 3$
≥ 4	even	≥ 6	n

3. AN APPLICATION

Suppose a township has p junctions and q streets. Each junction has a lamp post, and each street also has at least a lamp post in between the two end junctions. To beautify the lamp posts along each street and at each junction, the town council decided to paint all the lamp posts by a color with a specific code. The following conditions are decided:

(a) At first, each street and all lamp posts at junctions are given a specific integer from 1 to $p+q$ bijectively.
(b) Only 2 different colors of paints are available, code 1 and 2. All other colors must be a combination of code 1 and 2 in certain proportions.
(c) Every integer assigned to a street is the color code for lamp posts along that street. The lamp post at a junction is painted with color code given by the sum of the integers assigned to adjacent street lamp posts and the lamp post at the junction itself.
(d) For maximum attractiveness, no two lamp posts that belong to the ends of a street are given the same color code.
(e) The number of colors used for lamp posts at junctions must be minimized.
(f) For costs effectiveness, we would like to maximize the purchase quantity of a color and minimize the purchase quantity of another color, and so restrict the purchase types to only 2.

Thus, the minimum number of colors for lamp posts at junction is given by the local antimagic total chromatic number of the corresponding street graph that has p vertices and q edges.

Claim 3.1. *For a color code given by* $c = n_1 + n_2 + \cdots + n_k = 1 + (c-1)$, $k \geq 2, c \geq 3$, *the proportion of code 1 is* $1/c$ *and the proportion of code* $(c-1)$ *is* $(c-1)/c$. *More generally, we can also use* $(c-2)/c$ *of code 1 and* $2/c$ *of code 2. In this way, very small amount of a color and a much larger amount of another color are needed.*

Proof. We see that

1. Code 3 (= 1 + 2) uses a proportion of 1/3 of code 1 and 2/3 of code 2.
2. Code 4 (= 1 + 3) uses a proportion of 1/4 of code 1 and 3/4 of code 3 which is equivalent to 2/4 of code 1 and code 2 each.
3. Code 5 (= 1 + 4) uses a proportion of 1/5 of code 1 and 4/5 of code 4 that is equivalent to 1/5 + 2/5 = 3/5 of code 1 and 2/5 of code 2.
4. Code 6 (= 1 + 5) uses a proportion of 1/6 of code 1 and 5/6 of code 5 that is equivalent to 1/6 + 3/6 = 4/6 of code 1 and 2/6 of code 2.

In general, by induction, one can prove that code c uses $1/c$ of code 1 and $(c-1)/c$ of code $(c-1)$ that is equivalent to $1/c + (c-3)/c = (c-2)/c$ of code 1 and $2/c$ of code 2.

Thus, no two colors of distinct code are the same color and two lamp posts of same code must get the same color. □

In [8], the authors determined that $\chi_{lat}(P_n) = 2$ for $n \geq 2$, and that $\chi_{lat}(mC_{4k+2}) = 2$ for $m, k \geq 1$. Thus, any streets of p junctions or any circular arrangement of lamp posts with $4k + 2 \geq 6$ junctions may apply this idea accordingly.

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Received: May 4, 2022.
Revised: March 27, 2023.
Accepted: March 28, 2023.