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ABOUT ONE METHOD OF FINDING EXPECTED INCOMES IN HM-QUEUEING NETWORK WITH POSITIVE CUSTOMERS AND SIGNALS

Mikhail Matalytski

Institute of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
m.matalytski@gmail.com

Abstract. In the paper an open Markov HM(Howard-Matalytski)-Queueing Network (QN) with incomes, positive customers and signals (G(Gelenbe)-QN with signals) is investigated. The case is researched, when incomes from the transitions between the states of the network are random variables (RV) with given mean values. In the main part of the paper a description is given of G-network with signals and incomes, all kinds of transition probabilities and incomes from the transitions between the states of the network. The method of finding expected incomes of the researched network was proposed, which is based on using of found approximate and exact expressions for the mean values of random incomes. The variances of incomes of queueing systems (QS) was also found. A calculation example, which illustrates the differences of expected incomes of HM-networks with negative customers and QN without them and also with signals, has been given. The practical significance of these results consist of that they can be used at forecasting incomes in computer systems and networks (CSN) taking into account virus penetration into it and also at load control in such networks.

Keywords: *HM-queueing network, positive and negative customers, signals, expected incomes, transient regime*

1. Introduction

For the first time Markov nets with positive customers and signals were introduced and investigated at the non-stationary behavior by E. Gelenbe, see [1]. The action of the signal consists of instantaneous movement of positive customers of this system to some other network system. The signal may work as a trigger, which doesn't destroy the customers, but only moves them instantly with a given probability of a given system to another network system.

In developing models of computer viruses we can use negative customers. And for load control in the network can be inputted signals (triggers). When viruses penetrate into computers in the information system, it suffers costs or losses due to the loss of information or CSN distortion. The accounting of the losses in CSN can be realized with the help of a Markov QN model with incomes (HM-networks), positive and negative customers and signals.

In this paper, an open Markov HM-network with positive and negative customers, signals, when the incomes from the transitions between the states of the network are random variables (RV) with time-dependent customer servicing in the systems has been carried out. The expressions for the variances of incomes of queueing systems (QS) was also obtained.

A description of the network is given in [2, 3]. Signal, coming in an empty system S_i (in which there are no positive customers), does not have any impact on the network and immediately disappeared from it. Otherwise, if the system S_i is not empty, when it receives a signal, the following events may occur: incoming signal instantly moves the positive customer from the system S_i into the system S_j with probability q_{ij}, in this case, signal is referred to as a trigger; or with probability $q_{i0} = 1 - \sum_{j=1}^{n} q_{ij}$ signal is triggered by a negative customer and destroys in QS S_i positive customer. The state of the network meaning the vector $k(t) = (k, t) = (k_1, k_2, \ldots, k_n, t)$, where k_i - the number of customers at the moment of time t at the system S_i, $i = \overline{1, n}$.

2. Description of incomes from the transitions between the states of the network

Let ξ_i - time of customers service in the system S_i with the distribution function (DF) $F_{\xi_i}(t) = 1 - e^{-\mu_i t}$, $i = \overline{1, n}$. Consider the dynamics of income changes of a network system S_i, $i = \overline{1, n}$. Let at the initial moment of time the income of this QS be equal to v_{i0}. We are interested in income $V_i(t)$ at time *t*. The income of its QS at moment time $t + \Delta t$ can be represented in the form $V_i(t + \Delta t) = V_i(t) + \Delta V_i(t, \Delta t)$, where $\Delta V_i(t, \Delta t)$ - income changes of the system S_i at the time interval $[t, t + \Delta t)$, $i = \overline{1, n}$.

To find the income of the system S_i we write the conditional probabilities of the events that may occur during Δt. The following cases are possible:

1) with probability $\lambda_{0i}^{+} \Delta t + o(\Delta t)$ to the system S_i from the external environment a positive customer will arrive, which will bring an income in the amount of r_{0i}, where r_{0i} - RV with expectation (*E*) of which equals $E\{r_{0i}\} = a_{0i}$, $i = \overline{1, n}$;
2) with probability $\lambda_{0i}^{-} q_{i0} \Delta t + o(\Delta t)$ in the QS S_i from the external environment a signal will arrive, which will bring an income (loss) in the amount of signal is

triggered by a negative customer and destroys in QS S_i positive customer, which will bring a loss in the amount of $-\tilde{r}_{0i}$, where $\tilde{r}_{0i}$ - RV with E $E\{\tilde{r}_{0i}\} = \tilde{a}_{0i}$, $i = \overline{1,n}$;

3) the incoming signal instantly moves the positive customer from the system S_i into the system S_j; probability of this event equals $\lambda_{0i}^{-} q_{ij} u(k_i(t))\Delta t + o(\Delta t)$, $i, j = \overline{1,n}$, $i \neq j$; by this transition the income of S_i is reduced by the amount $\bar{r}_{0i}$, and income of S_j is increased by this amount, where, $\bar{r}_{0i}$ - RV with E $E\{\bar{r}_{0i}\} = \bar{a}_{0i}$, $i = \overline{1,n}$;

4) with probability $\mu_i p_{i0} u(k_i(t))\Delta t + o(\Delta t)$ a positive customer will depart from the network to the external environment, while the total amount of income of QS S_i is reduced by an amount which is equal to R_{i0}, where R_{i0} - RV with E $E\{R_{i0}\} = b_{i0}$, $i = \overline{1,n}$;

5) with probability $\mu_i (1 - u(k_i(t))) p_{ij}^{-} \Delta t + o(\Delta t)$ a customer from the system S_i transit to the system S_j as a signal, if in it there were no customers, and the income of S_i is reduced by the amount $\bar{R}_{ij}$, where $\bar{R}_{ij}$ - RV with E $E\{\bar{R}_{ij}\} = \bar{a}_{ij}$, $i = \overline{1,n}$;

6) a customer from the QS S_i transit to the system S_j with probability $\mu_i p_{ij}^{+} u(k_i(t))\Delta t + o(\Delta t)$, $i, j = \overline{1,n}$, $i \neq j$; by such a transition the income of system S_i is reduced by the amount $R_{ij}(\xi_i)$, and the income of system S_j is increased by this amount, $E\{R_{ij}(\xi_i)\} = \mu_i \int_0^{\infty} R_{ij}(t) e^{-\mu_i t} dt = a_{ij}$, $i = \overline{1,n}$, $j = \overline{1,n}$, $i \neq j$;

7) with probability $\mu_j p_{ji}^{+} u(k_i(t))\Delta t + o(\Delta t)$ positive customer transit from the system S_j to the system S_i, wherein the income of the QS S_i will increase by the value of $R_{ji}(\xi_j)$, and the income of S_j is reduced by this amount, $E\{R_{ji}(\xi_j)\} = a_{ji}$, $j = \overline{1,n}$, $j \neq i$;

8) after finishing servicing of a positive customer in QS S_i it is sent to S_j as a signal, which is triggered by a negative customer and destroyed in QS S_j positive customer; the probability of this event equals $\mu_i p_{ij}^{-} q_{j0} \Delta t + o(\Delta t)$; wherein the income of the QS S_i will reduce by value $\tilde{R}_{ji}(\xi_j)$, and the income of S_i is reduced also by the amount, $E\{\tilde{R}_{ji}(\xi_j)\} = \tilde{a}_{ji}$, $j = \overline{1,n}$, $j \neq i$;

9) after finishing servicing of positive customer in QS S_i, it is sent to S_j as a signal, which instantly moves the positive customer from the system S_j into the system S_s; the probability of this event equals $\mu_i p_{ij}^- q_{js} u(k_j(t))\Delta t + o(\Delta t)$, $i=\overline{1,n}$, $j=\overline{1,n}$, $i \neq j$; by such a transition the income of system S_i and S_j reduced by the amount R_{is}, and income of system S_s is increased by this amount respectively, where R_{ijs} - RV with the E $E\{R_{ijs}\}=c_{ijs}$, $i,j,s=\overline{1,n}$, $i \neq j$, $s \neq i$;
10) with probability $1-(\lambda_{0j}^+ + \lambda_{0j}^- + \mu_j)\, u(k_j(t))\Delta t + o(\Delta t)$ on time interval Δt network state will not change;
11) for every small time interval Δt system S_i because of customers' presence in it increases its income by the amount of $r_i \Delta t$, where r_i - RV with the E $E\{r_i\}=d_i$, $i=\overline{1,n}$.

3. Finding the expected incomes of the network systems

Income changes of the QS S_i on interval $[t, t+\Delta t)$ can be written as:

$$\Delta V_i(t,\Delta t) = \begin{cases} r_{0i} + r_i\Delta t \text{ with probability } \lambda_{0i}^+\Delta t + o(\Delta t), \\ -\widetilde{r}_{0i} + r_i\Delta t \text{ with probability } \lambda_{0i}^- q_{i0}\Delta t + o(\Delta t), \\ -\overline{r}_{0i} + r_i\Delta t \text{ with probability } \lambda_{0i}^- q_{ij} u(k_i(t))\,\Delta t + o(\Delta t), \\ -R_{i0} + r_i\Delta t \text{ with probability } \mu_i p_{i0} u(k_i(t))\,\Delta t + o(\Delta t), \\ -\overline{R}_{ij} + r_i\Delta t \text{ with probability } \mu_i (1-u(k_i(t))) p_{ij}^-\Delta t + o(\Delta t), \\ -R_{ij}(\xi_i) + r_i\Delta t \text{ with probability } \mu_i p_{ij}^+ u(k_i(t))\Delta t + o(\Delta t), \\ R_{ji}(\xi_j) + r_i\Delta t \text{ with probability } \mu_j p_{ji}^+ u(k_j(t))\Delta t + o(\Delta t), \\ -\widetilde{R}_{ji}(\xi_j) + r_i\Delta t \text{ with probability } \mu_i p_{ij}^- q_{j0}\Delta t + o(\Delta t), \\ -R_{ijs} + r_i\Delta t \text{ with probability } \mu_i p_{ij}^- q_{js} u(k_j(t))\Delta t + o(\Delta t), \\ r_i\Delta t \text{ with probability } 1-\left[\lambda_{0j}^+ + \lambda_{0j}^- + \mu_j\right] u(k_j(t))\Delta t + o(\Delta t), \\ j,s=\overline{1,n},\ j \neq i, s \neq i\,. \end{cases} \quad (1)$$

Let us find the expression for the expected income of the system S_i in time t, suppose, all the network systems operate under heavy-traffic regime, i.e. $k_i(t)>0$, $\forall t>0$, $i=\overline{1,n}$. Taking into account (1) for the E or income changes we can write:

$$
\begin{aligned}
E\{\varDelta V_i(t,\Delta t)\} &= \left(a_{0i}+d_i\Delta t\right)\left(\lambda_{0i}^{+}\Delta t+o(\Delta t)\right)+\\
&+\left(-\widetilde{a}_{0i}+d_i\Delta t\right)\left(\lambda_{0i}^{-}q_{i0}\Delta t+o(\Delta t)\right)+\\
&+\sum_{j=1}^{n}\left[\left(-\overline{a}_{0i}+d_i\Delta t\right)\left(\lambda_{0i}^{-}q_{ij}\Delta t+o(\Delta t)\right)\right]+\\
&+\left(-b_{i0}+d_i\Delta t\right)\left(\mu_i p_{i0}\Delta t+o(\Delta t)\right)+\\
&+\sum_{j=1}^{n}\left[\left(-a_{ij}+d_i\Delta t\right)\left(\mu_i p_{ij}^{+}\Delta t+o(\Delta t)\right)\right]+\\
&+\sum_{j=1}^{n}\left[\left(a_{ji}+d_i\Delta t\right)\left(\mu_j p_{ji}^{+}\Delta t+o(\Delta t)\right)\right]+\\
&+\sum_{j=1}^{n}\left[\left(-\widetilde{a}_{ji}+d_i\Delta t\right)\left(\mu_i p_{ij}^{-}q_{j0}\Delta t+o(\Delta t)\right)\right]+\\
&+\sum_{j=1}^{n}\sum_{s=1}^{n}\left[\left(c_{ijs}+d_i\Delta t\right)\left(\mu_i p_{ij}^{-}q_{js}\Delta t+o(\Delta t)\right)\right]+\\
&+d_i\Delta t\left(1-\sum_{j=1}^{n}\left[\lambda_{0j}^{+}+\left(\lambda_{0j}^{-}+\mu_j\right)\right]\Delta t+o(\Delta t)\right)=\\
&=\left(a_{0i}\lambda_{0i}^{+}-\widetilde{a}_{0i}\lambda_{0i}^{-}q_{i0}-b_{i0}\mu_i p_{i0}+d_i\right)\Delta t+\\
&+\sum_{j=1}^{n}\left[-\overline{a}_{0i}\lambda_{0i}^{-}q_{ij}-a_{ij}\mu_i p_{ij}^{+}+a_{ji}\mu_j p_{ji}^{+}-\right.\\
&\left.-\widetilde{a}_{ji}\mu_i p_{ij}^{-}q_{j0}-\sum_{s=1}^{n}c_{ijs}\mu_i p_{ij}^{-}q_{js}\right]\Delta t+o(\Delta t),\ i=\overline{1,n}.
\end{aligned}
$$

Then, similarly as in [3], we obtain

$$
\begin{aligned}
v_i(t) &= E\{V_i(t)\}=v_{i0}+\\
&+\left(a_{0i}\lambda_{0i}^{+}-\widetilde{a}_{0i}\lambda_{0i}^{-}q_{i0}-b_{i0}\mu_i p_{i0}+d_i-\overline{a}_{0i}\lambda_{0i}^{-}\sum_{j=1}^{n}q_{ij}-\right.\\
&\left.-\mu_i\sum_{j=1}^{n}a_{ij}p_{ij}^{+}+\sum_{j=1}^{n}a_{ji}\mu_j p_{ji}^{+}-\mu_i\sum_{j=1}^{n}\widetilde{a}_{ji}p_{ij}^{-}q_{j0}-\mu_i\sum_{j=1}^{n}p_{ij}^{-}\sum_{s=1}^{n}c_{ijs}q_{js}\right)\Delta t=\\
&=v_{i0}+\left\{\lambda_{0i}^{+}a_{0i}-\lambda_{0i}^{-}\widetilde{a}_{0i}q_{i0}-\mu_i b_{i0}p_{i0}+d_i+\right.\\
&+\sum_{j=1}^{n}\left[-\lambda_{0i}^{-}\overline{a}_{0i}q_{ij}+\mu_j a_{ji}p_{ji}^{+}-\right.\\
&\left.\left.-\mu_i\left(a_{ij}p_{ij}^{+}+\widetilde{a}_{ji}p_{ij}^{-}q_{j0}-\sum_{s=1}^{n}c_{ijs}p_{ij}^{-}q_{js}\right)\right]\right\}t,\ i=\overline{1,n}
\end{aligned}
\qquad (2)
$$

4. Finding variances of the incomes of the network systems

As in [4], system income S_i can be presented in form $V_i(t) = v_{i0} + \sum_{l=1}^{m} \Delta V_{il}(t)$, where m - count of partitions of the interval $[0, t]$ by equal parts, $\Delta t = \frac{t}{m}$; $\Delta V_{il}(t) = V_{il}(t + \Delta t) - V_{il}(t)$ - income changes i-th QS on l-th time interval, $i = \overline{1, n}$, $l = \overline{1, m}$. To calculate the variance of the system income in the network, we introduce the following designations:

$$E\{r_{0i}^2\} = a_{20i},\ E\{\widetilde{r}_{0i}^2\} = \widetilde{a}_{20i},\ E\{\overline{r}_{0i}^2\} = \overline{a}_{20i},\ E\{R_{i0}^2\} = b_{2i0},$$
$$E\{R_{ij}^2(\xi_i)\} = a_{2ij},\ E\{R_{ji}^2(\xi_j)\} = a_{2ji},\ E\{\widetilde{R}_{ji}^2(\xi_j)\} = \widetilde{a}_{2ji},$$
$$E\{R_{ijs}^2\} = c_{2ijs},\ E\{r_i^2\} = d_{2i},\ i, j = \overline{1, n}.$$

Let us consider the square of the difference $(V_i(t) - v_{i0})^2$:

$$(V_i(t) - v_{i0})^2 = \left(v_{i0} + \sum_{l=1}^{m} \Delta V_{il}(t, \Delta t) - v_{i0}\right)^2 = \left(\sum_{l=1}^{m} \Delta V_{il}(t, \Delta t)\right)^2 =$$
$$= \sum_{l=1}^{m} \Delta V_{il}^2(t, \Delta t) + \sum_{l=1}^{m} \sum_{\substack{j=1 \\ l \neq j}}^{m} \Delta V_{il}(t, \Delta t) \Delta V_{ij}(t, \Delta t),\ i = \overline{1, n}. \tag{3}$$

Let us find the expectations of summands in the right side of the last equality. For this we write support equalities considering that RV and functions of RV r_{0i}, $\widetilde{r}_{0i}$, $\overline{r}_{0i}$, R_{i0}, $\overline{R}_{ij}$, $R_{ij}(\xi_i)$, $R_{ji}(\xi_j)$, $\widetilde{R}_{ji}(\xi_j)$, R_{ijs} pairwise independent from r_i, $i, j = \overline{1, n}$. Then

$$E\{(r_{0i} + r_i \Delta t)^2\} = a_{20i} + 2a_{0i} d_i \Delta t + d_{2i} (\Delta t)^2, \tag{4}$$

$$E\{(-\widetilde{r}_{0i} + r_i \Delta t)^2\} = \widetilde{a}_{20i} - 2\widetilde{a}_{0i} d_i \Delta t + d_{2i} (\Delta t)^2, \tag{5}$$

$$E\{(-\overline{r}_{0i} + r_i \Delta t)^2\} = \overline{a}_{20i} - 2\overline{a}_{0i} d_i \Delta t + d_{2i} (\Delta t)^2, \tag{6}$$

$$E\{(-R_{i0} + r_i \Delta t)^2\} = b_{2i0} - 2b_{i0} d_i \Delta t + d_{2i} (\Delta t)^2, \tag{7}$$

$$E\{(-R_{ij}(\xi_i) + r_i \Delta t)^2\} = a_{2ij} - 2a_{ij} d_i \Delta t + d_{2i} (\Delta t)^2, \tag{8}$$

$$E\{(R_{ji}(\xi_j) + r_i \Delta t)^2\} = a_{2ji} + 2a_{ji} d_i \Delta t + d_{2i} (\Delta t)^2, \tag{9}$$

$$E\left\{\left(-\widetilde{R}_{ji}\left(\xi_j\right)+r_i\Delta t\right)^2\right\}=\widetilde{a}_{2ji}-2\widetilde{a}_{ji}d_i\Delta t+d_{2i}(\Delta t)^2, \tag{10}$$

$$E\left\{\left(-R_{ijs}+r_i\Delta t\right)^2\right\}=c_{2ijs}-2c_{ijs}d_i\Delta t+d_{2i}(\Delta t)^2. \tag{11}$$

Considering (4)-(11), we have:

$$\begin{gathered}
E\left\{\varDelta V_{il}^2(t,\Delta t)\right\}=\left(a_{20i}+2a_{0i}d_i\Delta t+d_{2i}(\Delta t)^2\right)\lambda_{0i}^{+}\Delta t+\\
+\left(\widetilde{a}_{20i}-2\widetilde{a}_{0i}d_i\Delta t+d_{2i}(\Delta t)^2\right)\lambda_{0i}^{-}q_{i0}\Delta t+\\
+\left(\overline{a}_{20i}-2\overline{a}_{0i}d_i\Delta t+d_{2i}(\Delta t)^2\right)\lambda_{0i}^{-}\sum_{j=1}^{n}q_{ij}\Delta t+\\
+\left(b_{2i0}-2b_{i0}d_i\Delta t+d_{2i}(\Delta t)^2\right)\mu_i p_{i0}\Delta t+\\
+\sum_{j=1}^{n}\left[\left(a_{2ij}-2a_{ij}d_i\Delta t+d_{2i}(\Delta t)^2\right)\mu_i p_{ij}^{+}\right]\Delta t+\\
+\sum_{j=1}^{n}\left[\left(a_{2ji}+2a_{ji}d_i\Delta t+d_{2i}(\Delta t)^2\right)\mu_j p_{ji}^{+}\right]\Delta t+\\
+\sum_{j=1}^{n}\left[\left(\widetilde{a}_{2ji}-2\widetilde{a}_{ji}d_i\Delta t+d_{2i}(\Delta t)^2\right)\mu_i p_{ij}^{-}q_{j0}\right]\Delta t+\\
+\sum_{j=1}^{n}\sum_{s=1}^{n}\left[\left(c_{2ijs}-2c_{ijs}d_i\Delta t+d_{2i}(\Delta t)^2\right)\mu_i p_{ij}^{-}q_{js}\right]\Delta t+\\
+d_{2i}(\Delta t)^2\left(1-\sum_{j=1}^{n}\left[\lambda_{0j}^{+}+\left(\lambda_{0j}^{-}+\mu_j\right)\right]\Delta t\right)+o(\Delta t)=\\
=\left(a_{20i}\lambda_{0i}^{+}+\widetilde{a}_{20i}\lambda_{0i}^{-}q_{i0}+b_{2i0}\mu_i p_{i0}+d_{2i}\right)\Delta t+\\
+\sum_{j=1}^{n}\left[\overline{a}_{20i}\lambda_{0i}^{-}q_{ij}+a_{2ij}\mu_i p_{ij}^{+}+a_{2ji}\mu_j p_{ji}^{+}+\widetilde{a}_{2ji}\mu_i p_{ij}^{-}q_{j0}+\right.\\
\left.+\sum_{s=1}^{n}c_{2jis}\mu_i p_{ij}^{-}q_{js}\right]\Delta t+o(\Delta t),\ i=\overline{1,n}
\end{gathered} \tag{12}$$

Insofar as the values $\Delta V_{il}(t,\Delta t)$ and $\Delta V_{ij}(t,\Delta t)$ are independent at $l\neq j$, using (12) one can find

$$\begin{gathered}
E\left\{\Delta V_{il}(t,\Delta t)\Delta V_{ij}(t,\Delta t)\right\}=\left\{\left(\lambda_{0i}^{+}a_{0i}-\lambda_{0i}^{-}\widetilde{a}_{0i}q_{i0}-\mu_i b_{i0}p_{i0}+d_i+\right.\right.\\
+\sum_{j=1}^{n}\left[-\lambda_{0i}^{-}\overline{a}_{0i}q_{ij}+\mu_j a_{ji}p_{ji}^{+}-\right.\\
\left.\left.\left.-\mu_i\left(a_{ij}p_{ij}^{+}+\widetilde{a}_{ji}p_{ij}^{-}q_{j0}-\sum_{s=1}^{n}c_{is}p_{ij}^{-}q_{js}\right)\right]\right)\Delta t+o(\Delta t)\right\}^2=o(\Delta t).
\end{gathered} \tag{13}$$

Then, further, passing to the limit $\Delta t \to 0$, from (3), (12), (13) and also, that $m\Delta t = t$, obtain

$$E\left\{(V_i(t)-v_{i0})^2\right\}=\sum_{l=1}^{m} E\left\{\Delta V_{il}^{\ 2}(t,\Delta t)/k(t)\right\}+$$
$$+\sum_{l=1}^{m}\sum_{\substack{q=1\\ q\neq l}}^{m} E\left\{\Delta V_{il}(t,\Delta t)\Delta V_{iq}(\Delta t)\right\}(t,\Delta t)=$$
$$=\Bigg(a_{20i}\lambda_{0i}^{+}+\tilde{a}_{20i}\lambda_{0i}^{-}q_{i0}+b_{2i0}\mu_i p_{i0}+d_{2i}+\sum_{j=1}^{n}\Big[\overline{a}_{20i}\lambda_{0i}^{-}q_{ij}+a_{2ij}\mu_i p_{ij}^{+}+ \tag{14}$$
$$+a_{2ji}\mu_j p_{ji}^{+}+\tilde{a}_{2ji}\mu_i p_{ij}^{-}q_{j0}+\sum_{s=1}^{n}c_{2is}\mu_i p_{ij}^{-}q_{js}\Big]\Bigg)t,\ i=\overline{1,n}$$

Now let us find an expression for $E^2\{(V_i(t)-v_{i0})\}$, using (2):

$$E^2\{(V_i(t)-v_{i0})\}=\Big\{\Big(\lambda_{0i}^{+}a_{0i}-\lambda_{0i}^{-}\tilde{a}_{0i}q_{i0}-\mu_i b_{i0}p_{i0}+d_i+$$
$$+\sum_{j=1}^{n}\Big[-\overline{a}_{0i}\lambda_{0i}^{-}q_{ij}-a_{ij}\mu_i p_{ij}^{+}+a_{2ji}\mu_j p_{ji}^{+}-\tilde{a}_{ji}\mu_i p_{ij}^{-}q_{j0}+ \tag{15}$$
$$+\sum_{s=1}^{n}c_{is}\mu_i p_{ij}^{-}q_{js}\Big]\Big)t\Big\}^2,\ i=\overline{1,n}.$$

In this way, the variance of income of i-th QS, considering (14), (15), can be written in the form

$$VarV_i(t)=Var\{V_i(t)-v_{i0}\}=$$
$$=Var\left\{(V_i(t)-v_{i0})^2\right\}-M^2\{(V_i(t)-v_{i0})\}=$$
$$=\Big(a_{20i}\lambda_{0i}^{+}+\tilde{a}_{20i}\lambda_{0i}^{-}q_{i0}+b_{2i0}\mu_i p_{i0}+d_{2i}+$$
$$+\sum_{j=1}^{n}\Big[\overline{a}_{2,0i}\lambda_{0i}^{-}q_{ij}+a_{2ij}\mu_i p_{ij}^{+}+a_{2ji}\mu_j p_{ji}^{+}+\tilde{a}_{2ji}\mu_i p_{ij}^{-}q_{j0}+\sum_{s=1}^{n}c_{2is}\mu_i p_{ij}^{-}q_{js}\Big]\Big)t- \tag{16}$$
$$-\Big\{\Big(\lambda_{0i}^{+}a_{0i}-\lambda_{0i}^{-}\tilde{a}_{0i}q_{i0}-\mu_i b_{i0}p_{i0}+d_i+$$
$$+\sum_{j=1}^{n}\Big[-\overline{a}_{0i}\lambda_{0i}^{-}q_{ij}-a_{ij}\mu_i p_{ij}^{+}+a_{2ji}\mu_j p_{ji}^{+}-\tilde{a}_{ji}\mu_i p_{ij}^{-}q_{j0}+$$
$$+\sum_{s=1}^{n}c_{is}\mu_i p_{ij}^{-}q_{js}\Big]\Big)t\Big\}^2,\ i=\overline{1,n}$$

5. Numerical example

As is known (see [5]), relation for the expected income of the system S_i in the case, where there are no negative customers in the network and all systems operate in a heavy-traffic regime, has the form:

$$\tilde{v}_i(t) = v_{i0} + \left[\lambda a_{0i}\tilde{p}_{0i} - \mu_i b_{i0}\tilde{p}_{i0} + \sum_{j=1}^{n} \mu_j a_{ji}\tilde{p}_{ji} - \right. \\ \left. - \mu_i \sum_{j=1}^{n} a_{ij}\tilde{p}_{ij} + d_i \right] t,\ i = \overline{1,n} \tag{17}$$

where: λ - input rate of customers; $\tilde{p}_{0i}$ - probability of a customer arriving to the system S_i, $i = \overline{1,n}$, $\sum_{i=1}^{n} \tilde{p}_{0i} = 1$; $\tilde{p}_{ij}$ - probability that the customer which finished servicing in i-th QS move to j-th QS, $\sum_{j=0}^{n} \tilde{p}_{ij} = 1$, $i = \overline{1,n}$; $\tilde{p}_{i0}$ - probability of the departure of a customer, $\tilde{p}_{i0} = 1 - \sum_{j=1}^{n} \tilde{p}_{ij}$, $i = \overline{1,n}$. Relation for the expected income of the system S_i with negative customers but without considering input signals, can be written as [6]:

$$\bar{v}_i(t) = v_{i0} + \left[\lambda_{0i}^{+} a_{0i} - \lambda_{0i}^{-}\bar{a}_{0i} - \mu_i \left(b_{i0}p_{i0} + \sum_{j=1}^{n} \left(a_{ij}p_{ij}^{+} + c_{ij}p_{ij}^{-} \right) \right) + \right. \\ \left. + \sum_{j=1}^{n} \mu_j a_{ji} p_{ji}^{+} + d_i \right] t,\ i = \overline{1,n}. \tag{18}$$

Let $n = 10$; input rates of positive customers and signals λ_{0i}^{+} and λ_{0i}^{-} respectively equal $\lambda_{0i}^{+} = 1$, $\lambda_{0i}^{-} = 0{,}5$, $i = \overline{1,n}$. Service rates of customers μ_i equal $\mu_1 = \mu_2 = \mu_3 = 1$, $\mu_4 = 2$, $\mu_5 = 3$, $\mu_6 = 4$, $\mu_7 = 7$, $\mu_8 = 13$, $\mu_9 = 7$, $\mu_{10} = 0{,}5$. Let us transition probabilities of positive customers p_{ij}^{+} respectively equal: $p_{12}^{+} = 1/8$, $p_{13}^{+} = 1/8$, $p_{21}^{+} = 1/10$, $p_{23}^{+} = 1/10$, $p_{24}^{+} = 1/10$, $p_{25}^{+} = 1/10$, $p_{31}^{+} = 1/10$, $p_{32}^{+} = 1/10$, $p_{36}^{+} = 1/10$, $p_{37}^{+} = 1/10$, $p_{42}^{+} = 1/8$, $p_{45}^{+} = 1/8$, $p_{48}^{+} = 1/8$, $p_{52}^{+} = p_{54}^{+} = p_{56}^{+} = p_{58}^{+} = 1/8$, $p_{63}^{+} = p_{65}^{+} = p_{67}^{+} = p_{69}^{+} = 1/8$, $p_{73}^{+} = p_{76}^{+} = p_{79}^{+} = 1/8$, $p_{84}^{+} = p_{85}^{+} = p_{89}^{+} = p_{8,10}^{+} = 1/8$, $p_{96}^{+} = p_{97}^{+} = p_{98}^{+} = p_{9,10}^{+} = 1/8$, $p_{10,8}^{+} = p_{10,9}^{+} = 1/5$, other equal zero. Probabilities, that were serviced in S_i, move to S_j as negative, equal: $p_{12}^{-} = 1/9$, $p_{13}^{-} = 1/9$,

$p_{21}^{-}=1/11$, $p_{23}^{-}=1/11$, $p_{24}^{-}=1/11$, $p_{25}^{-}=1/11$, $p_{31}^{-}=1/11$, $p_{32}^{-}=1/11$, $p_{36}^{-}=1/11$, $p_{37}^{-}=1/11$, $p_{42}^{-}=p_{45}^{-}=p_{48}^{-}=1/9$, $p_{52}^{-}=p_{54}^{-}=p_{56}^{-}=p_{58}^{-}=1/9$, $p_{63}^{-}=1/9$, $p_{65}^{-}=1/9$, $p_{67}^{-}=1/9$, $p_{69}^{-}=1/9$, $p_{73}^{-}=p_{76}^{-}=p_{79}^{-}=1/9$, $p_{84}^{-}=1/9$, $p_{85}^{-}=1/9$, $p_{89}^{-}=1/9$, $p_{8,10}^{-}=1/9$, $p_{96}^{-}=p_{97}^{-}=p_{98}^{-}=p_{9,10}^{-}=1/9$, $p_{10,8}^{-}=p_{10,9}^{-}=1/6$, other equal zero. Departure probabilities equal $p_{40}=p_{70}=7/24$, $p_{10,0}=4/15$. Probabilities of a signal arriving, which are instantly moves the positive customer from the system S_i to the system S_j respectively equal: $q_{12}=1/10$, $q_{13}=1/10$, $q_{21}=q_{23}=q_{24}=q_{25}=1/12$, $q_{31}=q_{32}=q_{36}=q_{37}=1/12$, $q_{42}=q_{45}=q_{48}=1/10$, $q_{52}=q_{54}=q_{56}=q_{58}=1/10$, $q_{63}=q_{65}=q_{67}=q_{69}=1/10$, $q_{73}=1/10$, $q_{76}=1/10$, $q_{79}=1/10$, $q_{84}=1/10$, $q_{85}=q_{89}=q_{8,10}=1/10$, $q_{96}=q_{97}=q_{98}=q_{9,10}=1/10$, $q_{10,8}=q_{10,9}=1/7$, other equal zero. Probabilities that signal is triggered as a negative customer and destroys in QS positive customer equal: $q_{10}=4/5$, $q_{20}=2/3$, $q_{30}=2/3$, $q_{40}=7/10$, $q_{50}=3/5$, $q_{60}=3/5$, $q_{70}=7/10$, $q_{80}=3/5$, $q_{90}=3/5$, $q_{10,0}=5/7$.

Let us set values for the required expectations: $a_{01}=a_{02}=15000$, $a_{03}=25000$, $a_{04}=500000$, $a_{05}=a_{06}=15000$, $a_{07}=41000$, $a_{08}=35000$, $a_{09}=55000$, $a_{0,10}=50000$; $\tilde{a}_{01}=50000$, $\tilde{a}_{02}=\tilde{a}_{03}=15000$, $\tilde{a}_{04}=50000$, $\tilde{a}_{05}=80000$, $\tilde{a}_{06}=30000$, $\tilde{a}_{07}=14000$, $\tilde{a}_{08}=15000$, $\tilde{a}_{09}=10000$, $\tilde{a}_{0,10}=80000$; $\bar{a}_{01}=10000$, $\bar{a}_{02}=30000$, $\bar{a}_{03}=15000$, $\bar{a}_{04}=40000$, $\bar{a}_{05}=60000$, $\bar{a}_{06}=90000$, $\bar{a}_{07}=15000$, $\bar{a}_{08}=15020$, $\bar{a}_{09}=900$, $\bar{a}_{0,10}=300$; $b_{10}=100$, $b_{20}=7000$, $b_{30}=8000$, $b_{40}=15000$, $b_{50}=b_{60}=b_{70}=2000$, $b_{80}=8000$, $b_{90}=3000$, $b_{10,0}=1000$; $c_{12}=2000$, $c_{13}=15000$, $c_{21}=20000$, $c_{23}=20000$, $c_{24}=c_{25}=20000$, $c_{31}=c_{32}=c_{36}=c_{37}=4000$, $c_{42}=c_{45}=c_{48}=20000$, $c_{52}=c_{54}=c_{56}=c_{58}=15000$, $c_{63}=c_{65}=c_{67}=c_{69}=17000$, $c_{73}=c_{76}=c_{79}=23000$, $c_{84}=c_{85}=20000$, $c_{89}=c_{8,10}=20000$, $c_{96}=c_{97}=c_{98}=c_{9,10}=30000$, $c_{10,8}=c_{10,9}=13000$; $d_1=1000$, $d_2=1200$, $d_3=1300$, $d_4=1200$, $d_5=2000$, $d_6=1000$, $d_7=8000$, $d_8=1000$, $d_9=1200$, $d_{10}=3000$, other equal zero.

Expectations for the random system of incomes has been calculated. Their values have the form: $a_{12}=15000$, $a_{13}=5000{,}1$, $a_{21}=2570{,}5$, $a_{23}=5000$, $a_{24}=1000$, $a_{25}=10000$, $a_{31}=1000$, $a_{32}=10000$, $a_{36}=4500$, $a_{37}=5000$, $a_{42}=-900$, $a_{45}=1020$, $a_{48}=20000$, $a_{52}=3230$, $a_{54}=4330$, $a_{56}=6330$, $a_{58}=3330$, $a_{63}=10000$, $a_{65}=10010$, $a_{67}=1009$, $a_{69}=10000$, $a_{73}=2330$, $a_{76}=a_{79}=3330$, $a_{84}=7678$, $a_{85}=7780$, $a_{89}=-900$, $a_{8,10}=1500$, $a_{97}=1000$, $a_{98}=1000$, $a_{9,10}=-990$, $a_{10,8}=396$, $a_{10,9}=978$; $\tilde{a}_{12}=a_{12}$, $\tilde{a}_{13}=a_{13}$, $\tilde{a}_{21}=2000$, $\tilde{a}_{23}=5000$, $\tilde{a}_{24}=10500$, $\tilde{a}_{25}=5000$, $\tilde{a}_{31}=\tilde{a}_{32}=1000$, $\tilde{a}_{36}=4500$, $\tilde{a}_{37}=100$, $\tilde{a}_{42}=-1900$, $\tilde{a}_{45}=5020$, $\tilde{a}_{48}=10000$,

$\tilde{a}_{52} = 2300$, $a_{54} = 120.333$, $\tilde{a}_{56} = 33.333$, $\tilde{a}_{54} = 33.7$, $\tilde{a}_{63} = 1000$, $\tilde{a}_{65} = 1200$, $\tilde{a}_{67} = 10000$, $\tilde{a}_{69} = 1000$, $\tilde{a}_{73} = a_{73}$, $\tilde{a}_{76} = a_{76}$, $\tilde{a}_{79} = a_{79}$, $\tilde{a}_{84} = 7.692$, $\tilde{a}_{85} = 7.704$, $\tilde{a}_{89} = 7.692$, $\tilde{a}_{8,10} = a_{8,10}$, $\tilde{a}_{96} = 143$, $\tilde{a}_{97} = a_{97}$, $\tilde{a}_{98} = 1000$, $\tilde{a}_{9,10} = -990$, $\tilde{a}_{10,8} = 3700$, $\tilde{a}_{10,9} = -620$.

Let us suppose that income at the initial time equals $v_{i0} = 0$, $i = \overline{1,n}$. Consider the length of the time interval of 10 hours, $t \in [0,T]$, $T = 10$. Then using formulas (2), (17), (18) analytical expressions have been found for the expected system incomes of the networks.

In Figure 1, income changes is shown of the QS S_2 for HM-network with negative customers and without them, and also with signals. One can see that the negative customers reduce expected income of the system S_2. Signals inputting also influence the income changes, reducing it.

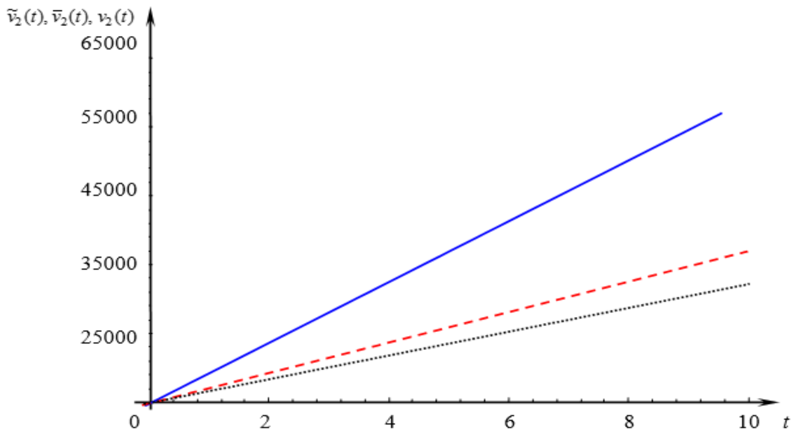

Fig. 1. Income changes of the S_2 QS, solid line - a case without negative customers, dashed - a case with negative customers, dotted - taking into account the signals

6. Conclusions

In this paper a method was proposed for finding expected incomes in HM-network systems with positive and negative customers and also with signals. Incomes from the transitions between the states of the network are RV with given mean values. This method is based on the using of found approximate and exact expressions for the mean values of the random incomes. An example was calculated. The expressions for the variances of incomes of QS was obtained. The obtained results can be used in modeling income changes in various CSN, the virus penetration into it, and also to load control in CSN.

References

[1] Gelenbe E., G-networks with triggered customer movement, Journal of Applied Probability 1993, 30, 742-748.

[2] Matalytski M., Naumenko V., Investigation of G-network with signals at transient behavior, Journal of Applied Mathematics and Computational Mechanics 2014, 13(1), 75-86.

[3] Naumenko V., Matalytski M., Finding the expected incomes in the Markov G-network with signals, Vestnik of GrSU. A Series of Mathematics, Physics, Computer Science, Computer Facilities and Management 2014, 2, 134-143.

[4] Naumenko V., Matalytski M., Investigation and application of G-networks with incomes and signals with random time activation Vestnik of GrSU. A Series of Mathematics, Physics, Computer Science, Computer Facilities and Management 2014, 3, 142-152.

[5] Matalytski M., On some results in analysis and optimization of Markov networks with incomes and their application, Automation and Remote Control 2009, 70(10), 1683-1697.

[6] Naumenko V., Matalytski M., Analysis of Markov net with incomes, positive and negative customers, Informatics 2014, 1, 5-14.