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ON $F(p,n)$-FIBONACCI BICOMPLEX NUMBERS

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Abstract

In this paper we introduce $F(p,n)$-Fibonacci bicomplex numbers and $L(p,n)$-Lucas bicomplex numbers as a special type of bicomplex numbers. We give some their properties and describe relations between them.

1. Introduction

Let consider the set $\mathbb{C}$ of complex numbers $a+bi$, where $a,b \in \mathbb{R}$, with the imaginary unit i. Let $\mathbb{B}$ be the set of bicomplex numbers w of the form

$$w = z_1 + z_2 j, \tag{1}$$

where $z_1, z_2 \in \mathbb{C}$. Then i and j are commuting imaginary units, i.e.

$$ij = ji,\ i^2 = j^2 = -1. \tag{2}$$

Let $w_1 = (a_1 + b_1 i) + (c_1 + d_1 i)j$ and $w_2 = (a_2 + b_2 i) + (c_2 + d_2 i)j$ be arbitrary two bicomplex numbers. Then the equality, the addition, the substraction, the multiplication and the multiplication by scalar are defined in the following way.
Equality: $w_1 = w_2$ only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$,
addition: $w_1 + w_2 = ((a_1+a_2) + (b_1+b_2)i) + ((c_1+c_2) + (d_1+d_2)i)j$,
substraction: $w_1 - w_2 = ((a_1-a_2) + (b_1-b_2)i) + ((c_1-c_2) + (d_1-d_2)i)j$,
multiplication by scalar $s \in \mathbb{R}$: $sw_1 = (sa_1 + sb_1 i) + (sc_1 + sd_1 i)j$,
multiplication:
$w_1 \cdot w_2 = ((a_1a_2 - b_1b_2 - c_1c_2 + d_1d_2) + (a_1b_2 + a_2b_1 - c_1d_2 - c_2d_1)i) +$
$+ ((a_1c_2 + a_2c_1 - b_1d_2 - b_2d_1) + (a_1d_2 + a_2d_1 + b_1c_2 + b_2c_1)i)\, j.$

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The bicomplex numbers were introduced in 1892 by Segre, see [5]. The theory of bicomplex numbers is developed, many of papers concerning this topic are published quite recently, see for example [2], [3], [4].

The Fibonacci numbers F_n are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ with $F_0 = F_1 = 1$. The nth Lucas number L_n is defined recursively by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with the initial terms $L_0 = 2$, $L_1 = 1$.

In this paper we recall some generalizations of Fibonacci numbers and Lucas numbers and we introduce the bicomplex numbers related with these generalizations.

2. The $F(p,n)$-Fibonacci numbers

The Fibonacci sequence has been generalized in many ways but a very natural is firstly to use one-parameter generalization of the Fibonacci sequence. A generalization uses one parameter p, $p \geq 2$ was introduced and studied by Kwaśnik and I. Włoch in the context of the number of p-independent sets in graphs, see [1]. We recall this definition.

Let $p \geq 2$ be integer. Then

$$\begin{aligned} &F(p,n) = n+1, \text{ for } n = 0,1,\ldots,p-1,\\ &F(p,n) = F(p,n-1) + F(p,n-p), \text{ for } n \geq p, \end{aligned} \tag{3}$$

is the $F(p,n)$-Fibonacci number.

Moreover $L(p,n)$-Lucas number is a cyclic version of $F(p,n)$ defined in the following way

$$\begin{aligned} &L(p,n) = n+1, \text{ for } n = 0,1,\ldots,2p-1,\\ &L(p,n) = L(p,n-1) + L(p,n-p), \text{ for } n \geq 2p, \end{aligned} \tag{4}$$

where $p \geq 2$, $n \geq 0$.
Note that for $n \geq 0$ we have that $F(2,n) = F_{n+1}$ and for $n \geq 2$ $L(2,n) = L_n$.

The following Tables present the initial words of the generalized Fibonacci numbers and the generalized Lucas numbers for special case of n and p.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55	89
$F(2,n)$	1	2	3	5	8	13	21	34	55	89	144
$F(3,n)$	1	2	3	4	6	9	13	19	28	41	60
$F(4,n)$	1	2	3	4	5	7	10	14	19	26	36
$F(5,n)$	1	2	3	4	5	6	8	11	15	20	26

Table 1. The values of $F(p,n)$ and F_n.

n	0	1	2	3	4	5	6	7	8	9	10
L_n	2	1	3	4	7	11	18	29	47	76	123
$L(2,n)$	1	2	3	4	7	11	18	29	47	76	123
$L(3,n)$	1	2	3	4	5	6	10	15	21	31	46
$L(4,n)$	1	2	3	4	5	6	7	8	13	19	26

Table 2. The values of $L(p,n)$ and L_n.

Generalized Fibonacci numbers $F(p,n)$ and generalized Lucas numbers $L(p,n)$ have been studied recently, mainly with respect to their graph and combinatorial properties, see for example [7], [8], [9], [10]. Among other some identities for $F(p,n)$ and $L(p,n)$ were given. We recall some of them.

Theorem 1 ([8]). *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

$$\sum_{l=0}^{n-p} F(p,l) = F(p,n) - p. \tag{5}$$

Theorem 2 ([8]). *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp-1) + 1 = F(p,np). \tag{6}$$

Theorem 3 ([6]). *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp) = F(p,np+1) - F(p,1), \tag{7}$$

$$\sum_{l=1}^{n} F(p,lp+1) = F(p,np+2) - F(p,2), \tag{8}$$

$$\sum_{l=1}^{n} F(p,lp+2) = F(p,np+3) - F(p,3). \tag{9}$$

Theorem 4 ([8]). *Let $p \geq 2$, $n \geq 2p-2$ be integers. Then*

$$F(p,n) = \sum_{l=0}^{p-1} F(p,n-(p-1)-l). \tag{10}$$

Theorem 5 ([8]). *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p,pl) = L(p,np+1) - (p+2). \tag{11}$$

Theorem 6 ([6])**.** *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p, pl+1) = L(p, np+2) - L(p, p+2). \tag{12}$$

$$\sum_{l=2}^{n} L(p, pl+2) = L(p, np+3) - L(p, p+3). \tag{13}$$

$$\sum_{l=2}^{n} L(p, pl+3) = L(p, np+4) - L(p, p+4). \tag{14}$$

Theorem 7 ([8])**.** *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$L(p,n) = pF(p, n-(2p-1)) + F(p, n-p). \tag{15}$$

3. The $F(p,n)$-Fibonacci bicomplex numbers

Let $n \geq 0$ be an integer. The nth $F(p,n)$-Fibonacci bicomplex number BF_n^p and the nth $L(p,n)$-Lucas bicomplex number BL_n^p are defined as

$$BF_n^p = (F(p,n) + F(p,n+1)i) + (F(p,n+2) + F(p,n+3)i)j, \tag{16}$$

$$BL_n^p = (L(p,n) + L(p,n+1)i) + (L(p,n+2) + L(p,n+3)i)j, \tag{17}$$

respectively.

Using the above definitions we can write selected $F(p,n)$-Fibonacci bicomplex numbers, i.e.

$$\begin{aligned}
&BF_0^3 = (1+2i) + (3+4i)j,\\
&BF_1^3 = (2+3i) + (4+6i)j,\\
&BF_2^3 = (3+4i) + (6+9i)j,\\
&\ldots\\
&BF_0^4 = (1+2i) + (3+4i)j,\\
&BF_1^4 = (2+3i) + (4+5i)j,\\
&BF_2^4 = (3+4i) + (5+7i)j,\\
&\ldots\\
&BF_0^5 = (1+2i) + (3+4i)j,\\
&BF_1^5 = (2+3i) + (4+5i)j,\\
&BF_2^5 = (3+4i) + (5+6i)j,\\
&\ldots
\end{aligned}$$

In the same way one can easily write selected $L(p,n)$-Lucas bicomplex numbers.

The addition, the subtraction and the multiplication of $F(p,n)$-Fibonacci bicomplex numbers and $L(p,n)$-Lucas bicomplex numbers are defined in the same way as for bicomplex numbers.

In the set $\mathbb{C}$, the complex conjugate of $x+yi$ is $\overline{x+yi}=x-yi$. In the set $\mathbb{B}$, for a bicomplex number $w=(a+bi)+(c+di)j$, there are three distinct conjugations.
Let BF_n^p be the nth $F(p,n)$-Fibonacci bicomplex number, i.e.
$BF_n^p=(F(p,n)+F(p,n+1)i)+(F(p,n+2)+F(p,n+3)i)j$,
The bicomplex conjugation of BF_n^p with respect to i has the form

$$\begin{aligned}\overline{BF_n^p}^i &= \overline{(F(p,n)+F(p,n+1)i)}+\overline{(F(p,n+2)+F(p,n+3)i)}j= \\ &= (F(p,n)-F(p,n+1)i)+(F(p,n+2)-F(p,n+3)i)j.\end{aligned}$$

The bicomplex conjugation of BF_n^p with respect to j has the form

$$\begin{aligned}\overline{BF_n^p}^j &= (F(p,n)+F(p,n+1)i)-(F(p,n+2)+F(p,n+3)i)j= \\ &= (F(p,n)+F(p,n+1)i)+(-F(p,n+2)-F(p,n+3)i)j.\end{aligned}$$

The third kind of conjugation is a composition of the above two conjugations. Putting $k:=ji=ij$ we can define the bicomplex conjugation of BF_n^p with respect to k as follows

$$\begin{aligned}\overline{BF_n^p}^k &= \overline{(F(p,n)+F(p,n+1)i)}-\overline{(F(p,n+2)+F(p,n+3)i)}j= \\ &= (F(p,n)-F(p,n+1)i)+(-F(p,n+2)+F(p,n+3)i)j.\end{aligned}$$

Using the bicomplex conjugation of BF_n^p with respect to i, j, k respectively and (16) we can write

$$\begin{aligned}& BF_n^p\cdot\overline{BF_n^p}^i= \\ &=\left(|F(p,n)+F(p,n+1)i|^2-|F(p,n+2)+F(p,n+3)i|^2\right)+ \\ &\quad +2\Re\left((F(p,n)+F(p,n+1)i)\cdot\overline{(F(p,n+2)+F(p,n+3)i)}\right)j= \\ &=(F(p,n))^2+(F(p,n+1))^2-(F(p,n+2))^2-(F(p,n+3))^2+ \\ &\quad +2\left(F(p,n)F(p,n+2)+F(p,n+1)F(p,n+3)\right)j.\end{aligned}$$

$$\begin{aligned}& BF_n^p\cdot\overline{BF_n^p}^j= \\ &=(F(p,n)+F(p,n+1)i)^2+(F(p,n+2)+F(p,n+3)i)^2= \\ &=(F(p,n))^2-(F(p,n+1))^2+(F(p,n+2))^2-(F(p,n+3))^2+ \\ &\quad +2\left(F(p,n)F(p,n+1)+F(p,n+2)F(p,n+3)\right)i.\end{aligned}$$

$$\begin{aligned}& BF_n^p\cdot\overline{BF_n^p}^k= \\ &=\left(|F(p,n)+F(p,n+1)i|^2+|F(p,n+2)+F(p,n+3)i|^2\right)+ \\ &\quad -2\Im\left((F(p,n)+F(p,n+1)i)\cdot\overline{(F(p,n+2)+F(p,n+3)i)}\right)k= \\ &=(F(p,n))^2+(F(p,n+1))^2+(F(p,n+2))^2+(F(p,n+3))^2+ \\ &\quad -2\left(F(p,n+1)F(p,n+2)-F(p,n)F(p,n+3)\right)k.\end{aligned}$$

In the set $\mathbb{C}$, the modulus of $x+yi$ is $|x+yi|=\sqrt{(x+yi)\cdot\overline{(x+yi)}}=\sqrt{x^2+y^2}$. In the set $\mathbb{B}$ there are four different moduli, named: real modulus

$|BF_n^p|$, $i-$modulus $|BF_n^p|_i$, $j-$modulus $|BF_n^p|_j$ and $k-$modulus $|BF_n^p|_k$. We give the formulae of the squares of these modules:

$$|BF_n^p|^2 = |F(p,n) + F(p,n+1)i|^2 + |F(p,n+2) + F(p,n+3)i|^2 = \\ = (F(p,n))^2 + (F(p,n+1))^2 + (F(p,n+2))^2 + (F(p,n+3)^2,$$

$$|BF_n^p|_i^2 = BF_n^p \cdot \overline{BF_n^p}^i,$$

$$|BF_n^p|_j^2 = BF_n^p \cdot \overline{BF_n^p}^j,$$

$$|BF_n^p|_k^2 = BF_n^p \cdot \overline{BF_n^p}^k.$$

The different conjugations and squares of modules for $L(p,n)$-Lucas bicomplex number BL_n^p are presented as follows

$$\overline{BL_n^p}^i = (L(p,n) - L(p,n+1)i) + (L(p,n+2) - L(p,n+3)i)j,$$

$$\overline{BL_n^p}^j = (L(p,n) + L(p,n+1)i) + (-L(p,n+2) - L(p,n+3)i)j,$$

$$\overline{BL_n^p}^k = (L(p,n) - L(p,n+1)i) + (-L(p,n+2) + L(p,n+3)i)j.$$

$$|BL_n^p|^2 = (L(p,n))^2 + (L(p,n+1))^2 + (L(p,n+2))^2 + (L(p,n+3)^2,$$

$$|BL_n^p|_i^2 = (L(p,n))^2 + (L(p,n+1))^2 - (L(p,n+2))^2 - (L(p,n+3))^2 + \\ +2\left(L(p,n)L(p,n+2) + L(p,n+1)L(p,n+3)\right)j.$$

$$|BL_n^p|_j^2 = (L(p,n))^2 - (L(p,n+1))^2 + (L(p,n+2))^2 - (L(p,n+3))^2 + \\ +2\left(L(p,n)L(p,n+1) + L(p,n+2)L(p,n+3)\right)i.$$

$$|BL_n^p|_k^2 = (L(p,n))^2 + (L(p,n+1))^2 + (L(p,n+2))^2 + (L(p,n+3))^2 + \\ -2\left(L(p,n+1)L(p,n+2) - L(p,n)L(p,n+3)\right)k.$$

4. Properties of $F(p,n)$-Fibonacci bicomplex numbers

We will give some properties of $F(p,n)$-Fibonacci bicomplex numbers and $L(p,n)$-Lucas bicomplex numbers.

Theorem 8. *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

(18)

$$\sum_{l=0}^{n-p} BF_l^p = BF_n^p - [p + (p + F(p,0))i + \\ + ((p + F(p,0) + F(p,1)) + (p + F(p,0) + F(p,1) + F(p,2))i)\,j]\,.$$

Proof. Using (5) and (16) we have

$$\begin{aligned}
&\sum_{l=0}^{n-p} BF_l^p = BF_0^p + BF_1^p + \ldots + BF_{n-p}^p = \\
&= (F(p,0) + F(p,1)i) + (F(p,2) + F(p,3)i)j + \\
&+(F(p,1) + F(p,2)i) + (F(p,3) + F(p,4)i)j + \ldots + \\
&+(F(p,n-p) + F(p,n-p+1)i) + \\
&+(F(p,n-p+2) + F(p,n-p+3)i)j = \\
&= F(p,0) + F(p,1) + \ldots + F(p,n-p) + \\
&+ (F(p,1) + \ldots + F(p,n-p+1) + F(p,0) - F(p,0))\, i + \\
&+ [F(p,2) + \ldots + F(p,n-p+2) + F(p,0) + F(p,1) - F(p,0) + \\
&-F(p,1) + (F(p,3) + \ldots + F(p,n-p+3) + F(p,0) + F(p,1) + \\
&+F(p,2) - F(p,0) - F(p,1) - F(p,2))\, i]\, j = \\
&= (F(p,n) - p + (F(p,n+1) - p - F(p,0))i) + \\
&+ [(F(p,n+2) - p - F(p,0) - F(p,1)) + \\
&+ (F(p,n+3) - p - F(p,0) - F(p,1) - F(p,2))\, i]\, j = \\
&= BF_n^p - (p + (p + F(p,0))\, i) - [(p + F(p,0) + F(p,1)) + \\
&+ (p + F(p,0) + F(p,1) + F(p,2))\, i]\, j,
\end{aligned}$$

which ends the proof. □

Theorem 9. *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} BF_{lp-1}^p = BF_{np}^p - [(F(p,0) + F(p,1)i) + (F(p,2) + F(p,3)i)\, j]\,. \tag{19}$$

Proof. Using (16) we have

$$\begin{aligned}
&\sum_{l=1}^{n} BF_{lp-1}^p = BF_{p-1}^p + BF_{2p-1}^p + \ldots + BF_{np-1}^p = \\
&= (F(p,p-1) + F(p,p)i) + (F(p,p+1) + F(p,p+2)i)j + \\
&+(F(p,2p-1) + F(p,2p)i) + (F(p,2p+1) + F(p,2p+2)i)j + \ldots + \\
&+(F(p,np-1) + F(p,np)i) + (F(p,np+1) + F(p,np+2)i)j = \\
&= F(p,p-1) + F(p,2p-1) + \ldots + F(p,np-1) + \\
&+ (F(p,p) + F(p,2p) + \ldots + F(p,np))\, i + \\
&+ [(F(p,p+1) + F(p,2p+1) + \ldots + F(p,np+1)) + \\
&+ (F(p,p+2) + F(p,2p+2) + \ldots + F(p,np+2))\, i]\, j.
\end{aligned}$$

Writing (6) as $\sum_{l=1}^{n} F(p,lp-1) = F(p,np) - 1 = F(p,np) - F(p,0)$ and using (7)–(9) we obtain (19). □

Theorem 10. *Let $p \geq 2$, $n \geq 2p-2$ be integers. Then*

$$BF_n^p = \sum_{l=0}^{p-1} BF_{n-(p-1)-l}^p. \tag{20}$$

Proof. Using (10) and (16) we have

$$\begin{aligned}
&\sum_{l=0}^{p-1} BF^p_{n-(p-1)-l} = BF^p_{n-(p-1)} + BF^p_{n-(p-1)-1} + \ldots + BF^p_{n-(p-1)-(p-1)} = \\
&= (F(p, n-(p-1)) + F(p, n-(p-1)+1)i) + \\
&+ [F(p, n-(p-1)+2) + F(p, n-(p-1)+3)i]\, j + \\
&+ (F(p, n-(p-1)-1) + F(p, n-(p-1))i) + \\
&+ [F(p, n-(p-1)+1) + F(p, n-(p-1)+2)i]\, j + \ldots + \\
&+ (F(p, n-(p-1)-(p-1)) + F(p, n-(p-1)-(p-1)+1)i) + \\
&+ [F(p, n-(p-1)-(p-1)+2) + F(p, n-(p-1)-(p-1)+3)i]\, j = \\
&= (F(p,n) + F(p,n+1)i) + (F(p,n+2) + F(p,n+3)i)j = BF^p_n,
\end{aligned}$$

which ends the proof. □

Theorem 11. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} BL^p_{pl} = BL^p_{np+1} - BL^p_{p+1}. \tag{21}$$

Proof. Using (17) we have

$$\begin{aligned}
&\sum_{l=2}^{n} BL^p_{pl} = BL^p_{2p} + BL^p_{3l} + \ldots + BL^p_{nl} = \\
&= (L(p,2p) + L(p,2p+1)i) + (L(p,2p+2) + L(p,2p+3)i)j + \\
&+ (L(p,3p) + L(p,3p+1)i) + (L(p,3p+2) + L(p,3p+3)i)j + \ldots + \\
&+ (L(p,np) + L(p,np+1)i) + (L(p,np+2) + L(p,np+3)i)j + \\
&= L(p,2p) + L(p,3p) + \ldots + L(p,np) + \\
&+ (L(p,2p+1) + L(p,3p+1) + \ldots + L(p,np+1))\, i + \\
&+ [(L(p,2p+2) + L(p,3p+2) + \ldots + L(p,np+2)) + \\
&+ (L(p,2p+3) + L(p,3p+3) + \ldots + L(p,np+3))\, i]\, j.
\end{aligned}$$

Writing (11) as $\sum_{l=2}^{n} L(p,pl) = L(p,np+1) - L(p,p+1)$ and using (12)–(14) we obtain (21). □

Theorem 12. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$BL^p_n = p \cdot BF^p_{n-(2p-1)} + BF^p_{n-p}. \tag{22}$$

Proof. Using (16) we have

$$\begin{aligned}
BF^p_{n-(2p-1)} &= (F(p, n-(2p-1)) + F(p, n-(2p-1)+1)i) + \\
&+ (F(p, n-(2p-1)+2) + F(p, n-(2p-1)+3)i)j
\end{aligned}$$

and

$$\begin{aligned}
BF^p_{n-p} &= (F(p, n-p) + F(p, n-p+1)i) + \\
&+ (F(p, n-p+2) + F(p, n-p+3)i)j,
\end{aligned}$$

consequently

$$\begin{aligned}
&p\cdot BF^{p}_{n-(2p-1)}+BF^{p}_{n-p}=\\
&=p\cdot F(p,n-(2p-1))+F(p,n-p)+\\
&+(p\cdot F(p,(n+1)-(2p-1))+F(p,(n+1)-p))\,i+\\
&+[(p\cdot F(p,(n+2)-(2p-1))+F(p,(n+2)-p))+\\
&+(p\cdot F(p,(n+3)-(2p-1))+F(p,(n+3)-p))\,i]\,j
\end{aligned}$$

Using (15) we have

$$\begin{aligned}
&p\cdot BF^{p}_{n-(2p-1)}+BF^{p}_{n-p}=\\
&=(L(p,n)+L(p,n+1)i)+(L(p,n+2)+L(p,n+3)i)j,
\end{aligned}$$

which ends the proof. □

For integers p, n, l, $p\geq 2$, $n\geq 2$, $0\leq l\leq n$ we have (see [9]) the direct formula for $F(p,n)$-Fibonacci number

$$F(p,n)=\sum_{l\geq 0}f(p,n,l),$$

where

$$f(p,n,l)=\binom{n-(p-1)(l-1)}{l}.$$

Using this direct formula other forms of given earlier identities can be given.

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