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Safety of some homogeneous series-„*k* out of *n*” systems

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large systems, safety of large systems, asymptotic approach, limit reliability function

Abstract
In the reliability investigation of large-scale systems, the problem of the complexity of their safety functions arises. This problem may be approximately solved by assuming that the number of system components tends to infinity and finding the limit safety function of the system. The problem of finding the limit reliability function of large-scale systems is well known for basic two-state systems (Gnedenko for series and parallel systems [1], Smirnov for “*k* out of *n*” systems [8]). The problem of possible limit reliability functions for series-parallel and parallel-series is also well recognized [3]. The solution for two-state series–“*k* out of *n*” systems, while $k/n \to 0$ and also while $k/n \to 1$ can be found in [5] and [6]. The paper is concerned with mathematical methods in multistate asymptotic approach to series–“*k* out of *n*” systems safety and shows the methods of establishing its possible limit safety functions, while $k/n \to \lambda, 0 < \lambda < 1$.

1. System safety – multistate approach

Like in previous paper, following [4] to define the system with degrading components we assume that

$$E_{ij},\ i = 1,2,...,k_n,\ j = 1,2,...,l_n;\ k_n, l_1, l_2,..., l_{k_n} \in N$$

are the system components and all of them have the safety state set {0, 1,…, *z*}, *z* ≥ 1, where safety states are ordered, the safety state 0 is the worst and the state *z* is the best. Moreover $T_{ij}(u)$ is a random variable representing the lifetimes of assets E_{ij} in the safety state subset {*u*, *u*+1,…, *z*}, *u* = 1,2,…,*z*, while they were in the safety state *z* at the moment *t* = 0 and $T(u)$ is a random variable representing the lifetime the system in the safety state subset {*u*, *u*+1,…, *z*}, *u* = 1,2,…,*z*, while it was in the safety state *z* at the moment $t = 0$.

We also assume that the safety states, the assets and critical system degrade with time *t*, $s_{ij}(t)$ is the asset's E_{ij} safety state and $s(t)$ is the system safety state at the moment t, $t \in <0, \infty)$, given that it was in the safety state z at the moment $t = 0$. The above assumptions mean that the safety state of the system with degrading assets may be changed in time only from better to worse.

We denote the safety function of the system by a vector

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2),..., \boldsymbol{S}(t,z)],$$

with the coordinates defined by safety functions

$$\boldsymbol{S}(t,u) = P(T(u) > t) \text{ for } t \in <0, \infty), u = 1, 2,..., z.$$

Obviously $T(u)$ is a random variable representing the lifetime the system in the safety state subset {*u*, *u*+1,…, *z*}, *u* = 1,2,…,*z*, while it was in the safety state *z* at the moment $t = 0$.

2. Limit safety functions - essential notions and definitions

We assume that the lifetime distributions do not necessarily have to be concentrated on the interval <0,∞). Then coordinates of safety function does not have to satisfy condition

$$S(t,u) = 1 \text{ for } t \in (-\infty, 0), u = 1,2,...z.$$

This generalization is convenient in the theoretical considerations. At the same time, the achieved results for the generalized safety functions, also hold for the usually used safety function. It is clear, that a safety function $S(t,u)$, $u = 1,2,..., z$, is non-increasing, right-continuous, $S(-\infty,u)=1$ and $S(+\infty,u)=0$.

Definition 1
A safety function $S(t,u),\ u\in\{1,2,...,z\}$ is called degenerate if there exists $t_0(u)\in(-\infty,\infty)$ such that

$$S(t,u)=\begin{cases}1, & t<t_0(u)\\ 0, & t\ge t_0(u).\end{cases}$$

Further we will deal with a safety function of the form

$$\tilde{\mathfrak{S}}(t,u)=1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-v(t,u)}\exp[-\frac{x^2}{2}]dx\,, \qquad (1)$$

Corollary.1
The safety function $\tilde{\mathfrak{S}}(t,u)$ given by (1) is a safety function if and only if $v(t,u)$ is a non-increasing, $v(-\infty,u)=\infty,\ v(\infty,u)=-\infty$ and besides $v(t,u)$ may be identically equal to ∞ or $-\infty$ in an interval.

Definition 2
A non-increasing function $v(t,u)$ defined for $t\in(-\infty,\infty),\ u=1,2,...z$ and such that $v(-\infty,u)=\infty$, $v(\infty,u)=-\infty$ is called degenerate if there exists $t_0(u)\in(-\infty,\infty)$ such that

$$v(t,u)=\begin{cases}\infty, & t<t_0(u)\\ -\infty, & t\ge t_0(u).\end{cases}$$

Now, the following corollary is clear.

Corollary 2
A safety function $\tilde{\mathfrak{S}}(t,u)$ given by (1) is degenerate if and only if a function $v(t,u)$ is degenerate.
Suppose that

$$E_{ij},\ i=1,2,...,k_n,\ j=1,2,...,l_n;\ k_n,\ l_1,\ l_2,...,\ l_{k_n}\in N$$

are components of a system having safety functions

$$S_{ij}(t,u)=P(T_{ij}(u)>t),\ t\in(-\infty,\infty),\ u=1,2,...,z$$

where $T_{ij}(u)$ are independent random variables representing the lifetimes of E_{ij}, having distribution functions

$$F_{ij}(t,\cdot)=[1,F_{ij}(t,1),...,F_{ij}(t,z)]$$

where

$$F_{ij}(t,u)=P(T_{ij}(u)\le t),\ u=1,2,...,z,\ t\in(-\infty,\infty).$$

Definition 3
A system is called regular series–“m_n out of k_n” if its lifetime $T(u)$ in the safety state subset $\{u,u+1,...,z\}$ is given by

$$T(u)=T_{(k_n-m_n+1)}(u),\ \ m_n=1,2,...,k_n,\ u=1,2,...,z,$$

where $T_{(k_n-m_n+1)}(u)$ is the m_n-th maximal order statistics in a sample of random variables

$$T_i(u)=\min_{1\le j\le l_n}\{T_{ij}(u)\},\ i=1,2,...,k_n$$

Definition 4
A regular series–“m_n out of k_n” system is called homogeneous if component lifetimes $T_{ij}(u)$ in the safety state subset $\{u,u+1,...,z\}$ have an identical distribution function

$$F(t,u)=P(T_{ij}(u)\le t),\ t\in(-\infty,\infty),$$
$$i=1,2,...,k_n,\ j=1,2,...,l_n,\ u=1,2,...,z.$$

i.e. if its components E_{ij} have the same safety function

$$S(t,u)=1-F(t,u),\ t\in(-\infty,\infty),\ u=1,2,...,z.$$

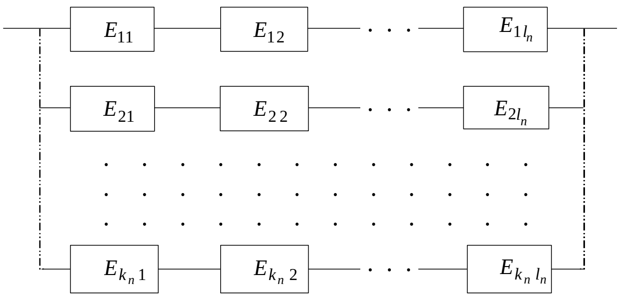

Figure 1. The scheme of a regular series– – “m_n out of k_n” system

The safety function of the homogeneous regular series– “m_n out of k_n” system with degrading/ageing components is given by:

$$\overline{\mathbf{S}}^{(m_n)}_{k_n,l_n}(t,\cdot)=\left[1,\overline{\mathbf{S}}^{(m_n)}_{k_n,l_n}(t,1),...,\overline{\mathbf{S}}^{(m_n)}_{k_n,l_n}(t,z)\right],\ t\in(-\infty,\infty),$$

with the coordinates defined by

$$\overline{\mathbf{S}}^{(m_n)}_{k_n,l_n}(t,u)=\sum_{i=0}^{k_n-m_n}\binom{k_n}{i}[1-S^{l_n}(t,u)]^i[S^{l_n}(t,u)]^{k_n-i} \qquad (2)$$

for $t\in(-\infty,\infty),\ u=1,2,...,z.$
We investigate limit distributions of a standardized random variable $\left(T(u)-b_n(u)/a_n(u)\right)$, where $u=1,2,...,z$ and $T(u)$ is the lifetime of the system in the safety state subset $\{u,u+1,...,z\}$ and $a_n(u),\ b_n(u)$

are some suitably chosen numbers such that $a_n(u) > 0$ and $b_n(u) \in (-\infty, \infty)$. And, since

$$P\big([(T(u) - b_n(u)) / a_n(u)] > t\big) =$$
$$= P\big(T(u) > (a_n(u)t + b_n(u))\big) =$$
$$= \boldsymbol{S}_{k_n,l_n}^{(m_n)}\big(a_n(u)t + b_n(u), u\big),\ u = 1,2,...,z,$$

then we assume the following definition.

Definition 5
A safety function

$$\mathfrak{I}(t,\cdot) = [1, \mathfrak{I}(t,1), \mathfrak{I}(t,2), ..., \mathfrak{I}(t,z)]$$

is called the limit safety function of the homogeneous regular series–"m_n out of k_n" multistate system if there exist normalising constants $a_n(u) > 0$, $b_n(u) \in (-\infty, \infty)$ such that for $u = 1,2,...,z$

$$\lim_{n\to\infty} \boldsymbol{S}_{k_n,l_n}^{(m_n)}\big(a_n(u)t + b_n(u), u\big) = \mathfrak{I}(t,u) \quad \text{for } t \in C_{\mathfrak{I}(u)},$$

where for $u \in \{1,2,...,z\}$, $C_{\mathfrak{I}(u)}$ is the set of continuity points of $\mathfrak{I}(t,u)$. Hence, for $u = 1,2,...,z$ and for sufficiently large n and $t \in (-\infty, \infty)$ we get the following approximate formula:

$$\boldsymbol{S}_{k_n,l_n}^{(m_n)}(t,u) \cong \mathfrak{I}\big((t - b_n(u)) / a_n(u), u\big) \qquad (3)$$

Definition 6
The safety functions $\mathfrak{I}(t,u)$ and $\mathfrak{I}_0(t,u)$, $u = 1,2,...,z,$ are said to be of the same type if there exist numbers $a(u) > 0$ and $b(u) \in (-\infty, \infty)$ such that

$$\mathfrak{I}_0(t,u) = \mathfrak{I}\big(a(u)t + b(u), u\big).$$

Definition 7
The functions $v(t,u)$ and $v_0(t,u)$, $u = 1,2,...,z,$ with properties included in *Corollary 1* are said to be of the same type if there exist numbers $a(u) > 0$ and $b(u) \in (-\infty, \infty)$ such that

$$v_0(t,u) = v\big(a(u)t + b(u), u\big).$$

We assume the following notations:
$x(n) << y(n)$, where $x(n)$ and $y(n)$ are positive functions, means that $x(n)$ is of order much less than $y(n)$ in the sense that

$$\lim_{n\to\infty} \frac{x(n)}{y(n)} = 0,$$

$x(n) \approx y(n)$, where $x(n)$ and $y(n)$ are positive or negative functions, means that $x(n)$ is of order $y(n)$ in the sense

$$\lim_{n\to\infty} \frac{x(n)}{y(n)} = 1,$$

$x(n) >> y(n)$, where $x(n)$ and $y(n)$ are positive functions, means that $x(n)$ is of order much greater than $y(n)$ in the sense

$$\lim_{n\to\infty} \frac{x(n)}{y(n)} = \infty,$$

3. Series– "m_n out of k_n" systems with ageing components while $m_n \approx k_n$ and their limit safety functions

Assumptions 1
We put all possible relations between k_n and l_n into the form

$$k_n = n,\ l_n = c \log^{\rho(n)} n,\ n \in (0,\infty),\ c > 0,$$

where

Case 1. $\rho(n) << \dfrac{1}{\sqrt{n}\log\log n}$,

Case 2. $\rho(n) >> \dfrac{1}{\sqrt{n}\log\log n}$,

Case 3. $\rho(n) \approx \dfrac{1}{\sqrt{n}\log\log n}$.

Basing on [8] and formulas (1) and (2) we obtain the following lemma.

Lemma 1
If for $u = 1,2,...,z$

(i) The family $\overline{\boldsymbol{S}}_{k_n,l_n}^{(m_n)}(t,u)$ is given by (2),

(ii) The non-degenerate safety function $\tilde{\mathfrak{I}}(t,u)$ is given by (1),

(iii) $\dfrac{m_n}{k_n} \to \lambda$, $\quad 0 < \lambda < 1$, $\quad \lim_{n\to\infty} k_n = \infty$.

Then for $u = 1,2,...,z$ and $t \in (-\infty, \infty)$ the assertion

$$\lim_{n\to\infty} \overline{\boldsymbol{S}}_{k_n,l_n}^{(m_n)}\big(a_n(u)t + b_n(u), u\big) =$$
$$= \tilde{\mathfrak{I}}(t,u) \text{ for } t \in C_{\tilde{\mathfrak{I}}}$$

is equivalent to the assertion

$$\lim_{n\to\infty} \frac{\sqrt{k_n+1}[S^{l_n}\big(a_n(u)t + b_n(u), u\big) - \lambda]}{\sqrt{\lambda(1-\lambda)}} = v(t,u)$$

The proof is based on results of [6].

Lemma 2
If

(i) $k_n = n, l_n = c\log^{\rho(n)} n, n \in (0,\infty), c > 0,$

$\lim_{n\to\infty} \frac{m_n}{k_n} = \lambda, \quad 0 < \lambda < 1,$

(ii) $\rho(n) << \frac{1}{\sqrt{n}\log\log n},$

(iii) $\tilde{\mathfrak{J}}(t,u), \quad u = 1,2,...,z$ are the non-degenerate coordinates of limit safety function of the homogeneous regular series–"m_n out of k_n" system given by (1),

then for every natural $\mu > 1$ there exist such $\alpha_\mu(u) > 0$ and $\beta_\mu(u) \in (-\infty,\infty)$ that for $t \in (-\infty,\infty)$

$$\sqrt{\mu} v\left(\alpha_\mu(u)t + \beta_\mu(u)\right) = v(t,u).$$

The proof is based on results of [7].

Lemma 3
If

(i) $k_n = n, l_n = c\log^{\rho(n)} n, n \in (0,\infty), c > 0$,

$\lim_{n\to\infty} \frac{m_n}{k_n} = \lambda, \quad 0 < \lambda < 1,$

(ii) $\rho(n) >> \frac{1}{\sqrt{n}\log\log n},$

(iii) $\tilde{\mathfrak{J}}(t,u), u = 1,2,...,z$ is the non-degenerate limit safety function of the homogeneous regular series-"m_n out of k_n" system with ageing components given by (1),

then for every natural $\mu > 1$ there exist such $\alpha_\mu(u) > 0$ and $\beta_\mu(u) \in (-\infty,\infty)$ that for $t \in (-\infty,\infty)$

$$v(\alpha_\mu t + \beta_\mu) \pm \mu\sqrt{\frac{\lambda}{(1-\lambda)}}\log\lambda = v(t,u).$$

The proof is based on results of [6].

Lemma 4
If

(i) $k_n = n, l_n = c\log^{\rho(n)} n, n \in (0,\infty), c > 0$,

$\lim_{n\to\infty} \frac{m_n}{k_n} = \lambda, \quad 0 < \lambda < 1,$

(ii) $\rho(n) \approx \frac{1}{\sqrt{n}\log\log n},$

(iii) $\tilde{\mathfrak{J}}(t,u), \quad u = 1,2,...,z$ is the non-degenerate limit safety function of the homogeneous regular series–"m_n out of k_n" system given by (1),

then for every natural $\mu > 1$ there exist such $\alpha_\mu(u) > 0$ and $\beta_\mu(u) \in (-\infty,\infty)$ that

$$\sqrt{\mu} v\left(\alpha_\mu(u)t + \beta_\mu(u)\right) - \sqrt{\frac{\lambda}{1-\lambda}}(\sqrt{\mu}-1)\log\lambda =$$
$$= v(t,u) \text{ for } t \in (-\infty,\infty).$$

The proof is based on results of [7].

In our further considerations the above lemmas will allow us to fix some closed classes of non-degenerate functions $v(t,u), u = 1,2,...,z$ for the considered cases of relations between k_n, l_n and m_n. Having these classes of functions, we shall obtain some classes of asymptotic safety functions for the homogeneous regular series–"m_n out of k_n" system with ageing components, we are looking for.

Lemma 5
The only possible types of non-degenerate function $v(t,u), u = 1,2,...,z$ as the solution of the equation

$$\sqrt{\mu} v\left(\alpha_\mu(u)t + \beta_\mu(u)\right) = v(t,u) \text{ for } t \in (-\infty,\infty),$$

where $\mu > 1$, $\alpha_\mu(u) > 0$ and $\beta_\mu(u) \in (-\infty,\infty)$ are some suitably chosen numbers, are

$$v_1(t,u) = \begin{cases} \infty, & t < 0, \\ -ct^{\alpha(u)}, & t \ge 0, c > 0, \alpha(u) > 0, \end{cases}$$

$$v_2(t,u) = \begin{cases} c|t|^{\alpha(u)}, & t < 0, c > 0, \alpha(u) > 0, \\ -\infty, & t \ge 0, \end{cases}$$

$$v_3(t,u) = \begin{cases} c_1|t|^{\alpha(u)}, & t < 0, c_1 > 0, \alpha(u) > 0, \\ -c_2 t^{\alpha(u)}, & t \ge 0, c_2 > 0, \alpha(u) > 0, \end{cases}$$

$$v_4(t,u) = \begin{cases} \infty, & t < -1, \\ 0, & -1 \le t < 1 \\ -\infty, & t \ge 1. \end{cases}$$

The proof is based on results of [1] and [8].

Lemma 6
The only possible types of non-degenerate function $v(t,u), \ u = 1,2,...,z,$ as the solution of the equation

$$v\left(\alpha_\mu(u)t + \beta_\mu(u)\right) \pm \mu\sqrt{\frac{\lambda}{(1-\lambda)}}\log\lambda = v(t,u),$$
$$t \in (-\infty,\infty),$$

where $\mu > 1$, $\alpha_\mu(u) > 0$ and $\beta_\mu(u) \in (-\infty,\infty)$ are some suitably chosen numbers, are

$$v_5(t,u) = \begin{cases} \infty, & t < 0, \\ -\alpha(u)\log t, & t > 0, \alpha(u) > 0, \end{cases}$$

$$v_6(t,u) = \begin{cases} \alpha(u)\log|t|, & t<0, \alpha(u)>0, \\ -\infty, & t>0, \end{cases}$$

$$v_7(t,u) = -t \ \text{ for } \ t \in (-\infty, \infty).$$

The proof is based on results of [6].

Lemma 7

The only possible types of non-degenerate function $v(t,u),\ u = 1,2,...,z,$ as the solution of the equation

$$\sqrt{\mu}\, v\left(\alpha_\mu(u)t + \beta_\mu(u)\right) - \sqrt{\frac{\lambda}{1-\lambda}}(\sqrt{\mu}-1)\log\lambda =$$
$$= v(t,u) \text{ for } t \in (-\infty,\infty),\ u = 1,2,...,\ z.$$

where $\mu > 1$, $\alpha_\mu(u) > 0$ and $\beta_\mu(u) \in (-\infty,\infty)$ are some suitably chosen numbers

$$\overset{+}{v}_1(t,u) = \begin{cases} \infty, & t<0, \\ -ct^{\alpha(u)} + \sqrt{\dfrac{\lambda}{1-\lambda}}\log\lambda, & t \ge 0, c>0, \\ & \alpha(u)>0, \end{cases}$$

$$\overset{+}{v}_2(t,u) = \begin{cases} c\,|t|^{\alpha(u)} + \sqrt{\dfrac{\lambda}{1-\lambda}}\log\lambda, & t<0, c>0, \\ & \alpha(u)>0, \\ -\infty, & t \ge 0, \end{cases}$$

$$\overset{+}{v}_3(t,u) = \begin{cases} c_1(u)\,|t|^{\alpha(u)} + \sqrt{\dfrac{\lambda}{1-\lambda}}\log\lambda\,, & t<0, \alpha(u)>0, \\ & c_1(u)>0, \\ -c_2(u)t^{\alpha(u)} + \sqrt{\dfrac{\lambda}{1-\lambda}}\log\lambda\,, & t \ge 0, \alpha(u)>0, \\ & c_2(u)>0, \end{cases}$$

$$\overset{+}{v}_4(t,u) = \begin{cases} \infty, & t<-1, \\ \sqrt{\dfrac{\lambda}{1-\lambda}}\log\lambda, & -1 \le t < 1 \\ -\infty, & t \ge 1. \end{cases}$$

The proof is based on results of [7].

The above lemmas lead us to formulate the result concerning the only possible asymptotic safety functions of the homogeneous regular series-"m_n out of k_n" system with ageing components, for the considered cases of relations between k_n, l_n and m_n , if

$$k_n = n,\ l_n = c\log^{\rho(n)} n,\ n \in (0,\infty),\ c>0,$$
$$\lim_{n\to\infty}\frac{m_n}{k_n} = \lambda,\ 0<\lambda<1,$$

It is given in *Theorem 2* in the next article.

4. Conclusion

The paper proposes an approach to the solution of practically very important problem of determining the safety functions of large scale multistate systems by assuming that the number of system component tends to infinity and finding the system limit safety function. This way, for sufficiently large systems their exact safety functions may be approximated by their limit safety functions. This approach gives practically important in everyday usage tool for safety evaluation of large systems that can be met for instance in piping transportation two-state systems considered in [9], where application of the proposed method is illustrated in the safety evaluation of the port oil pipeline transportation system.

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