Budownictwo i Architektura 22(3) 2023, 5-25
DOI: 10.35784/bud-arch.3325

Received: 28.11.2022; Revised: 05.07.2023;
Accepted: 18.08.2023; Available online: 29.09.2023

Orginal Article

Unconventional building forms roofed with innovative structures arranged on regular surfaces with the negative Gaussian curvature

Katarzyna Chrzanowska [1], Jacek Abramczyk [2]

[1] Doctoral School of the Rzeszow University of Technology; Rzeszow University of Technology; Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland; d565@stud.prz.edu.pl; ORCID: 0000-0002-6873-6984
[2] Department of Architectural Design and Engineering Graphics; Faculty of Civil Engineering Environment and Architecture; Rzeszow University of Technology; Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland; jacabram@prz.edu.pl; ORCID: 0000-0001-7016-6338

Abstract: A novel method for shaping innovative building forms, roofed with diversified complex continuous and discontinuous folded structures composed of many transformed corrugated shell units, is presented in the paper. The units are defined on the basis of specific reference polyhedral networks and arranged on arbitrary reference surfaces characterized by the negative Gaussian curvature. The method is presented using several computer models of complex building structures with folded plane-walled elevations. The proposed method significantly supplements the previous method developed for modelling building free forms, roofed with shell structures arranged in conformity with surfaces having the positive Gaussian curvature. Some basic rules using parameterization and governing the creation of the multi-plane elevations, ribbed continuous and discontinuous roof shell structures, arranged in different unconventional and visually attractive patterns, were developed. The elaborated specific sets of division coefficients are taken as parameters for the designed building structures. These sets determine unconventional polyhedral networks, which are composed of several specific sets that allow to define a polygonal eaves network, a reference surface and, finally, individual shell units of the roof structures. The developed method is presented using the example of three novel forms defined by means of the appropriately selected diversified sets of values of the division coefficients. The elaborated new forms confirm the innovative nature of the achieved results. By imposing appropriate proportions between the values of these division coefficients, the developed method enables the creation of two different groups continuous and discontinuous complex shell roof structures.

Keywords: novel building form, complex shell roof, parametric modelling, nominally flat thin-walled folded sheeting, multi-plane elevation.

1. Introduction

Nominally, flat folded metal sheets tend to take many different attractive ruled shell shapes as a result of their spatial elastic transformations. This property results from their relatively low transverse bending and longitudinal torsional stiffness [1], [2], Fig.1. Some significant geometric and material limitations in the formation of the transformed corrugated steel sheeting cause difficulties for their use by designers to shape transformed roofs or elevations. The limitations often result in significant values of stresses and strains for the cases of ill-considered transformations and their high degrees [3]. The intentional spatial shape deformations are called initial transformations and have to be effective, that is, transversal freedom with increments of the transformed sheeting must be ensured by imposing special boundary conditions [4], [5].

Fig. 1. Complete shell unit composed of two nominally plane folded sheets connected with themselves along their longitudinal edges and transformed with two skew straight directrices. *Source:* own study

When carrying out an analysis related to the geometric and mechanical properties of the plane corrugated sheeting transformed elastically into different roof shells, it is necessary to take into account the values of the initial bending, twisting, displacements, deformations, stresses, and strains diversified for all subsequent folds of all transformed sheeting [6]-[7]. The geometric and mechanical limitations of the sheets require composing multiple complete transformed shell sheeting units into a single edge structure to obtain medium- and large-spanned roofs [8].

The expected straightness of the longitudinal axes of all transformed shell folds and the longitudinal borders of all complete sheets, along with the whole shell sheeting after transforming, are other basic limitations in the use of the flat folded sheets. The developed models of the designed transformed shell units should be shaped in the form of various ruled surface sectors limited by spatial quadrangles with rectangular or similar corners [1], [8]. The more the corner angles differ from right angles, the greater the oblique cuts of the transverse edges of the transformed sheeting, passing parallel to the roof directrices supporting the sheeting.

The simplest way to model the complex building structures is to combine various polyhedrons into one folded, complex, polyhedral elevation structure roofed with special sums of many single parabolic-hyperbolic shell sectors. The edge lines of these sectors form closed spatial quadrangles and are contained in the planes of the polyhedra. Façade walls are also contained in these planes. To achieve a wide variety of the designed complex building forms, the façade elevation walls are folded and inclined to the vertical [8], Fig. 2.

Fig. 2. Computer model simulating a complex form with multi-plane folded elevations and multi-shell discontinuous roof. *Source:* own study

To model a novel complex building structure, a polyhedral reference network needs to be defined [8]. The network should be shaped based on many quadruples of vertices, defining regular tetrahedral meshes. They are joined with corresponding common triangular sides to form a spatial structure that fills the three-dimensional space tightly. The facade walls of each complex form are contained in the planes of the created tetrahedral meshes.

Complete segments of closed spatial quadrangles, which define the edge lines of all single transformed roof shells, are contained within the planes mentioned above. This results in a novel polygonal eaves network, which is formed by the sum of all employed quadrangles. By parameterizing the essential dimensions of the reference polyhedral network and the eaves polygonal network, as well as their elements, the geometric properties of these networks can be diversified. This leads to the creation of complex building free forms characterized by multi-wall folded facades and multi-shell ribbed continuous or discontinuous roofs [8].

2. Critical analysis

Winter [9], Parker [10], Gergely et al. [11], Egger et al. [12] and many other researchers have demonstrated great theoretical possibilities in the search for various ruled forms of transformed folded sheeting, as well as their simple structures composed of open thin-walled corrugated sheeting covered with flat smooth sheets on both the top and bottom sides. However, the tests and analysis conducted by Bryan and Davis [13] indicate that several significant material and technological limitations greatly restrict the diversification of the transformation types and degrees. As a result, the achieved corrugated shell forms are reduced to shallow hyperbolic paraboloids known as hypars.

Biswas and Iffland [14] expanded the range of the designed complex, ribbed structures consisting of multiple identical transformed shell corrugated units. They developed two simple systems of congruent shell units distributed on a sphere using bundles of planes. In these systems, the complete transformed shells are constructed from revolved hyperboloids or right hyperbolic paraboloids confined by spatial straight quadrangles.

Abdel [15] highlighted the potential to broaden the range of single transformed folded sheeting to include cylindrical and conical surfaces through transverse bending of flat folded sheets. Abdel and Mungan [16] present various unconventional building forms and structural systems designed for these forms. Some of these shapes can be incorporated into a design process of transformed roof shell sheeting. Similarly, certain types of the structural systems outlined in the comprehensive classification provided by Saitoh [17] can also be used for shaping the transformed shell sheeting.

A novel method developed by Reichhart [7] significantly increases the diversity of creating unconventional roof shells and their structures. These structures are composed of nominally flat sheeting that is transformed into various unconventional folded shell shapes using a few rigid skew directrices. The wide range of the shell roof forms achieved by this method is a result of the properties of the thin-walled single-layer sheeting. This type of sheeting allows for bending, twisting, and a combination of both to adapt to the supporting conditions determined by the roof skew directrices.

Abramczyk [1] conducted tests on the effective bending, twisting and bend-twist transformations of a number of single folded sheets with different profiles, as well as their single-layer strips. An important discovery was made regarding the contraction of each effectively transformed folded sheet. The effective transformation of a folded sheet aims to minimize stresses and strains, which requires ensuring freedom in the transverse width increments of the sheet during the initial design step. This property is achieved by introducing a line of striction of a ruled surface that models a sheet. The striction line must run half along the length of each fold of the transformed sheeting, where its shell folds have different directions. The developed method facilitates the use of different ruled surfaces in the rational design of transformed folded shell roofs.

Walentyński et al. [18] has tested curvilinear thin-walled folded sheets and curved folded panels for creating shell roof sheeting. It was elaborated a method for creating mechanical computer models to simulate the mechanical performance of these panels.

Prokopska and Abramczyk [19] have developed a system for creating simple sets of reference tetrahedrons that can be used to model complete free forms with multi-plane folded oblique elevations. These free forms are roofed with transformed folded shell units, which are modelled using various quarters of hyperbolic paraboloids arranged symmetrically. The system enables one to achieve a variety of diversified free form configurations.

Abramczyk and Prokopska proposed a general parametric sector of a warped surface, [20], which is bounded by a closed spatial quadrangle with sides defined with a set of four planes. This system of many quadrangles of planes determines and divides each designed shell ribbed roof structure into multiple complete shell segments, arranged regularly in accordance with a reference surface characterized by the positive Gaussian curvature in the three-dimensional space. However, there is a gap in the current knowledge because there is no analogous comprehensive method for shaping continuous and discontinuous shell roof structures, whose complete shell units are arranged on reference surfaces characterized by negative Gaussian curvature. There is a need for a detailed description of the procedure for shaping such complex building forms. This issue is analyzed in the present article.

Pottman [21] has described several methods for shaping various systems of smooth plane and shell sectors arranged according to the geometric properties of different regular surfaces. However, if we want to shape transformed folded shell sheeting using these methods, significant modifications must be made to account for the geometric and material constraints of the folded sheets.

Samyn [22] developed a method for creating transformed folded shells using aluminium or PVC. The method differs from previous methods, which involved joining adjacent sheets along their longitudinal borders using bolts or rivets. Instead, Samyn's method requires the sheets to be joined by welding.

Folded shell sheeting can also be used to construct supports for folded panels arranged according to some convex shells. Żwirek [23] performed forced plastic deformations on thin-walled folded sheets to obtain doubly-curved bases that support the convex shells composed of the panels.

3. Objective of the research

The aim is to analyse the possibilities of shaping unconventional folded building forms, specifically their complex multi-plane façades and multi-shell roofs. These shell sectors are arranged in the three-dimensional space, following regular surfaces with the negative Gaussian curvature. Through this analysis, an innovative method is presented, which involves search for computational models of such forms. To achieve these goals, a novel procedure was developed for creating specific types of auxiliary reference polyhedral network and polyhedral eaves networks.

The algorithm used in this method is different from other algorithms used so far because the characteristic vertices of the developed models are positioned between the vertices of the reference polyhedral network. The roof structures are grouped into basic continuous ribbed and discontinuous shell structures derived from the basic ones. The invented algorithm allows for arranging many façade walls and many roof shell units in relation to the vertices of the polyhedral reference network using respective discrete values of division coefficients. These coefficients define the adopted proportions between the distances of the appropriate vertices of the polygonal eaves network and the distances of the selected pairs of the vertices of the reference polyhedral networks Γ.

The novel procedure carried out is related to the differentiation of the values of the division coefficients and enables us to expect some differentiation of the obtained basic and derivative building forms. The proposed method is described in detail using a few specific examples. Several different forms, obtainable through the procedure, are provided to demonstrate a wide range of possibilities offered by the method.

The main relationships governing the position of each mesh B_{vij} and shell unit Ω_{ij} in a polygonal eaves network B_v and shell roof structure Ω deserve special emphasis. The specific sets of values of the partition coefficients presented, which define the required positions of the vertices of each developed reference polyhedral network, each reference surface, and each reference polygonal eaves network, play a distinctive role in shaping rational and attractive basic and derivative configurations of the proposed general building forms.

The algorithm of the presented method is supported by the authors' novel computer program written in the AutoLISP programming language, specifically designed for the AutoCAD graphics editor. Each CAD object built using this program can be used as a geometric model of a building, which is helpful in shaping its structural system.

4. Methodology

Creating each polyhedral network Γ composed of a number of appropriately arranged tetrahedra Γ_{ij} begins with the first central tetrahedron $\Gamma_{11,}$ defined by four arbitrary points W_{AB11}, W_{CD11}, W_{AD11} and W_{BC11} which serve as vertices of Γ, Fig. 3.a. The location of the vertices is determined using two arbitrary oblique straight lines u_{11} and v_{11} that are perpendicular to each other. These lines are referred to as the axes of Γ_{11}.

The arbitrary distance d_{uv11} between u_{11} and v_{11} measured along the straight line perpendicular to u_{11} and v_{11}, needs to be determined. To define the positions of four vertices W_{AB11}, W_{CD11}, W_{AD11} and W_{BC11} on u_{11} and v_{11}, the distance du_{11} between W_{BC11} and W_{AD11} and the distance dv_{11} between W_{AB11} and W_{CD11} also need to be determined.

The subsequent tetrahedral meshes Γ_{ij} of Γ are positioned in two orthogonal directions of $\Gamma_{11,}$ according to the directions of the axes v_{11}, v_{12} and v_{12L}, Fig. 3.b, and the axes u_{11}, u_{21} and u_{21L}, Fig. 3.c, as follows: For the tetrahedron Γ_{12}, its three vertices are set to be identical to the vertices W_{AB11}, W_{CD11} and W_{BC11} of Γ_{11}. The axis u_{12} of Γ_{12} is also identical to u_{11}. The location of the fourth vertex W_{BC12} of Γ_{12} (that does not belong to Γ_{11} and defines the axis v_{12}) is determined so that it is contained in the first plane of symmetry of Γ_{11}, distant from W_{BC11} by the adopted value of dv_{12} and distant from the axis u_{11} by the height d_{BC11} of the triangle $W_{BC11}W_{CD11}W_{AB11}$. The third tetrahedron Γ_{12L} is symmetrical to Γ_{12} with respect to the second plane of symmetry of Γ_{11}. The second plane of symmetry of Γ_{11} is defined by the points W_{AD11}, W_{BC11} and the midpoint of the segment $W_{AB11}W_{CD11}$. The first plane of symmetry of Γ_{11} is defined by the points W_{AB11}, W_{CD11} and the midpoint of the segment $W_{AD11}W_{BC11}$.

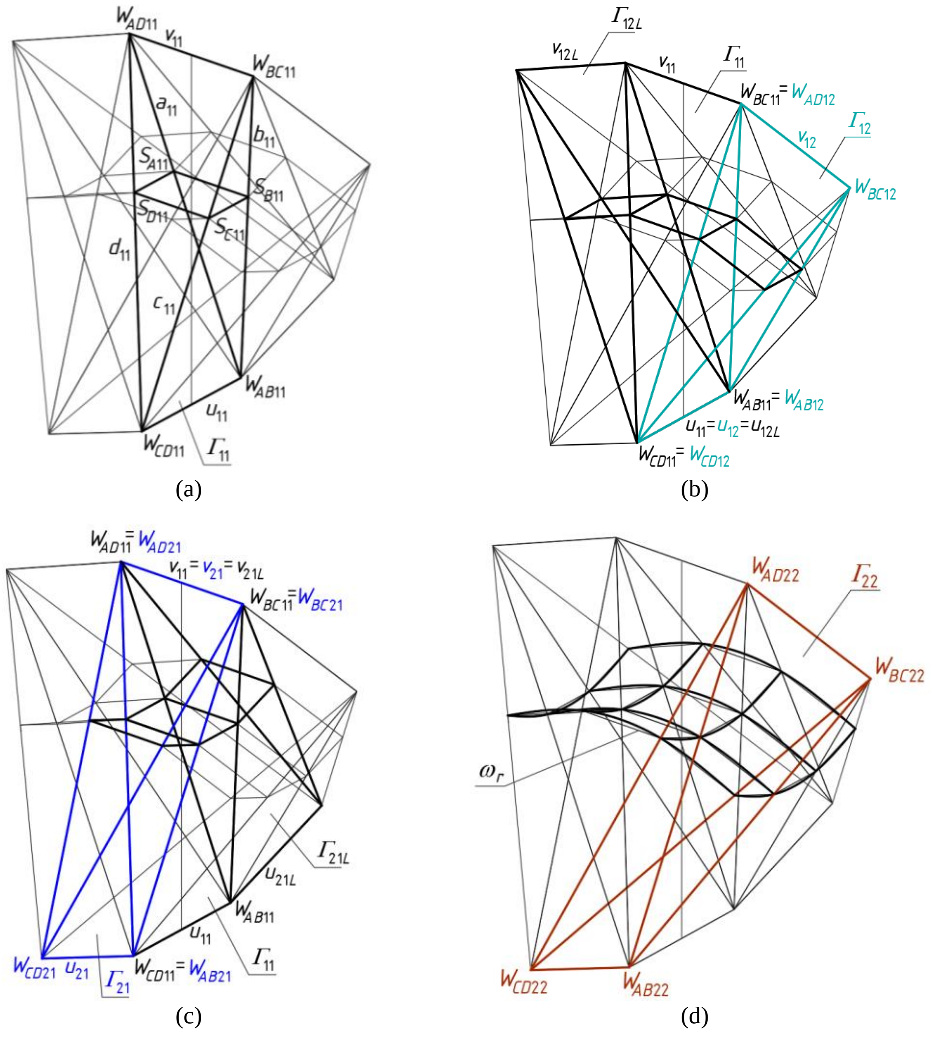


Fig. 3. The main steps in shaping a polyhedral reference network Γ and reference surface ω_r are as follows: (a) the first tetrahedron Γ_{11} of Γ, (b) the first orthogonal strip Γ_{1j} containing the second tetrahedron Γ_{12} of Γ, (c) the second orthogonal strip Γ_{i1} containing the subsequent tetrahedron Γ_{21} of Γ, and (d) reference surface ω_r and the diagonal tetrahedron Γ_{22} of Γ. *Source:* own study

All tetrahedra Γ_{1j} created so far belong to the first orthogonal strip. Similarly, all tetrahedra Γ_{i1} of the second searched orthogonal strip can be formed in the same manner as Γ_{1j} tetrahedra belonging to the first strip. Here's how it is done: For the tetrahedron Γ_{21}, its three vertices are set to be identical to the vertices W_{AD11}, W_{BC11} and W_{CD11} of Γ_{11}. The axis v_{21} of Γ_{21} is also identical to v_{11}. The location of the fourth vertex W_{CD21} of Γ_{21} (which is not part of Γ_{11} and defines the axis u_{21}) is determined in such a way that it lies in the first plane of symmetry of Γ_{11}, at a distance of du_{21} from W_{CD11}, and at a distance of $d_{CD21} = d_{CD11}$ (the height of the triangle $W_{BC11}W_{CD11}W_{AD11}$) from the axis v_{11}. The third tetrahedron Γ_{21L} of this orthogonal strip is symmetrical to Γ_{21} with respect to the second plane of symmetry of Γ_{11}'s, as defined above.

All tetrahedra are arranged in diagonal strips Γ_{ij} (where i, $j \neq 1$ and $i = j$) and are defined by the vertices of two aforementioned orthogonal strips composed of the tetrahedrons Γ_{i1} and Γ_{1j}, or the previously created diagonal tetrahedra. Figure 3.d presents an example of a diagonal tetrahedron Γ_{ij} for $i = j = 2$. In the presented method, selected proportions $dd_{v12} = dv_{12} / dv_{11}$ and $dd_{BC12} = d_{BC12} / d_{BC11}$ are used, which allows for better control of the shape of Γ, B_v and Σ. Here, d_{BC12} refers to the distance of W_{BC12} from u_{12} and d_{BC11} refers to the distance of W_{BC11} from u_{11}. Σ represents the model of the entire building, as shown in Fig. 4.c. By adopting such proportions for all orthogonal tetrahedral meshes, it becomes possible to parametrize and control the shapes of Γ, B_v and Σ. For the mesh Γ_{12} used in the following section, $dd_{v12} = 1$ and $dd_{BC12} =1$. The parametrization should result in dividing the desired forms Σ into several distinct groups with similar geometric properties.

Since the employed method for parameterizing Γ involves defining a set of division coefficients that express the proportions between the distances of the vertices of the subsequently created Γ_{ij}'s meshes, it is important to consider the corresponding proportions between the lengths of the axes of the subsequent meshes. For the created mesh Γ_{21}, $dd_{u21} = dd_{CD21} = 1$, see Figs 3-4, where: $dd_{u21} = d_{u21} / d_{u11}$ and $dd_{CD21} = d_{CD21} / d_{CD11}$.

In a general case, $dd_{v1j} = dd_{ui1} = 1$, $dd_{BC1j} = dd_{CDi1} = 1$ for Γ_{1j} and Γ_{i1}. The above activities should be repeated to ensure that all subsequent meshes Γ_{ij} of the orthogonal strips are congruent to each other.

For all diagonal tetrahedra, there are $W_{ADij} = W_{BCi\text{-}1j\text{-}1}$, $W_{BCij} = W_{AD\, i\text{-}1j}$, $W_{CDij} = W_{CDij\text{-}1}$, $W_{ABij} = W_{CDi\text{-}1j\text{-}1}$. The developed procedure allows one to locate W_{ADij}, W_{BCij}, W_{CDij} and W_{ABij} at any points of the respective side edges of Γ. Thus, they do not have to only be located at the vertices of the previously created tetrahedra. This modification related to the properties of Γ leads to fundamental changes in the proportions between the overall dimensions and the size of the elements of the building model shaped. The description of the modified procedure for shaping such networks Γ goes beyond the scope of this article.

The polyhedral network Γ presented in Figs 3-4 consists of four symmetrical parts Γ_k (k = 1 to 4), where the first quarter Γ_1 is composed of Γ_{ij} (i,j = 1 to 2). Based on the symmetrical reference network Γ, a specific polygonal eaves network B_v can be created to define a multi-shell roof structure Σ. To determine the eaves network B_v, appropriate relationships between the location of the vertices of Σ and B_v, in relation to the selected pairs of the Γ's vertices, need to be adopted. According to the method's assumptions, these relationships are defined using the division coefficients of the pairs $\{W_{ABij}, W_{ADij}\}$, $\{W_{ABij}, W_{BCij}\}$, $\{W_{ADij}, W_{CDij}\}$, $\{W_{ADij}, W_{BCij}\}$ of each Γ_{ij} by the vertices of each $B_{vij} = \langle A_{ij}B_{ij}C_{ij}D_{ij}, \rangle$, each base quadrangle $\langle P_{Aij}P_{Bij}P_{Cij}P_{Dij}, \rangle$, Fig. 4.a, and each plane quadrangle $\langle S_{Aij}S_{Bij}S_{Cij}S_{Dij} \rangle$ of the reference surface ω_r used, Figs 3.a, 4.

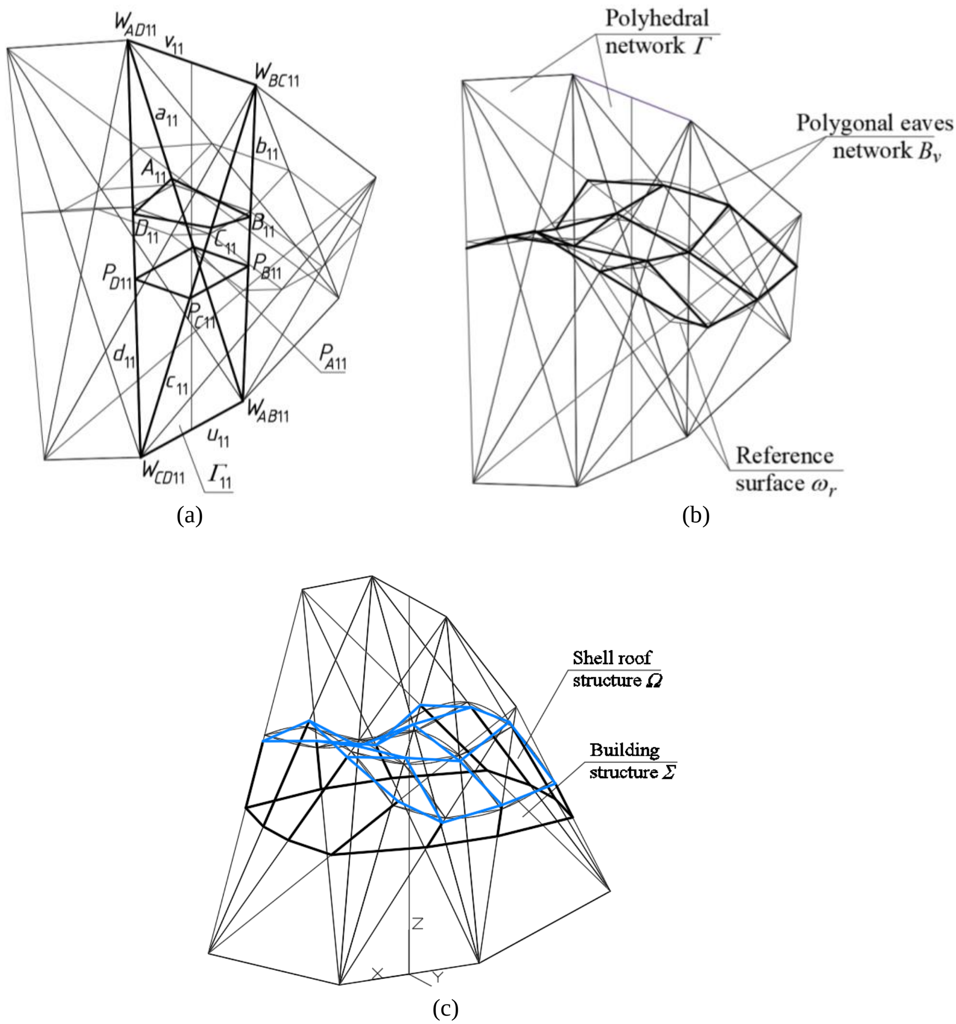


Fig. 4. The steps in shaping a whole building structure Σ: (a) the reference polyhedral network Γ, (b) network B_v and reference surface ω_r, and (c) the shell roof structure Ω. *Source:* own study

The set {dd_{SAij}, dd_{SBij}, dd_{SCij} and dd_{SDij}} of division coefficients is used to determine the positions of the points S_{Aij}, S_{Bij}, S_{Cij} and S_{Dij} located on the side edges of Γ and defining a reference surface ω_r. To search for the network B_v, another set {dd_{Aij}, dd_{Bij}, dd_{Cij} and dd_{Dij}} of the division coefficients is employed to define the positions of the points A_{ij}, B_{ij}, C_{ij} and D_{ij} on the side edges of Γ. These points constitute the vertices of B_v, Fig. 4.a-c. On the basis of the network B_v, the model Σ of the searched complex folded building form is created as follows. The walls of Σ should be included in the planes of Γ. The elevation edges of Σ should be included in the side edges of Γ. The eaves line B_{vij} of each single shell Ω_{ij} of the Ω's roof structure must be contained in the planes of Γ. The vertices of each eaves net B_v belong to the side edges of Γ. In the presented procedure, the considered vertices of ω_r and B_v have to be located between the vertices of Γ.

The bases of the created forms are flat and horizontal. The arbitrary level of each base plane P_b must belong to the interval <0, d_{uv11}>. The positions of all points P_{Aij}, P_{Bij}, P_{Cij} and P_{Dij} of each base, belonging to the edges of the facade walls, can be obtained as a result of the intersection of the plane P_b with all Γ's side edges. The methodology for studying the shaping of unconventional folded building forms is presented in Figure 5.

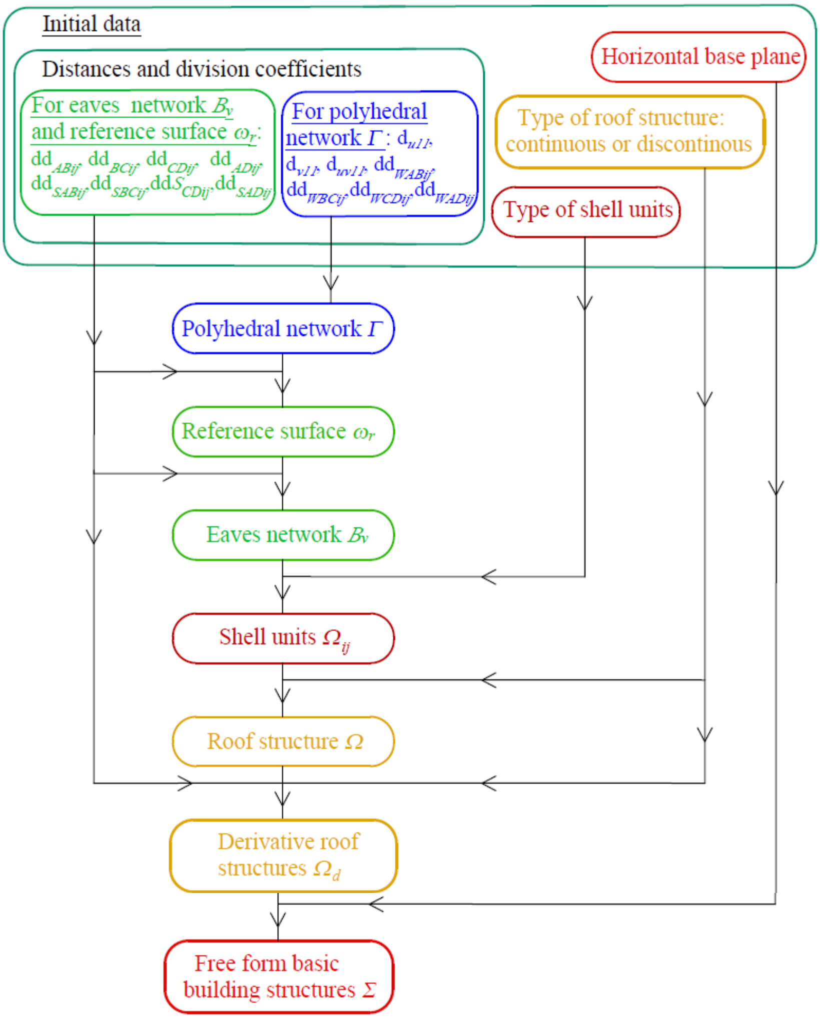


Fig. 5. Methodology of the research

For the examined vertices of ω_r, B_v and Σ, all acceptable values of the above-mentioned two types of the division coefficients must be taken from the range (0, 1). In addition, the arbitrary reference surface ω_r, defined by all tetrads of the points S_{Aij}, S_{Bij}, S_{Cij} and S_{dij}, has to have negative Gaussian curvature. This assumption results in the specific geometric properties of B_v and Σ, proving the innovative nature of the performed analysis and the developed procedures. In the case of other articles, the values of the division coefficients employed are within the range $(1, \infty)$ and the reference surface is convex and characterized by the positive Gaussian curvature.

Each set of the division coefficients taken into account should determine the mutual position of all vertices of the reference polyhedral Γ network, the polygonal B_v network, the base plane P_b and the reference surface ω_r. The adopted proportions define the values of the distances between the above-mentioned vertices, so the coordinates of the vertices can be calculated in the three-dimensional space. In order to illustrate the impact of adopting different sets of values of the above-mentioned coefficients on the shape of Γ, B_v and Σ, a few examples of complex folded base continuous building forms and their discontinuous derivatives with roof shell units arranged on surfaces characterized by the negative Gaussian curvature are presented in the next section.

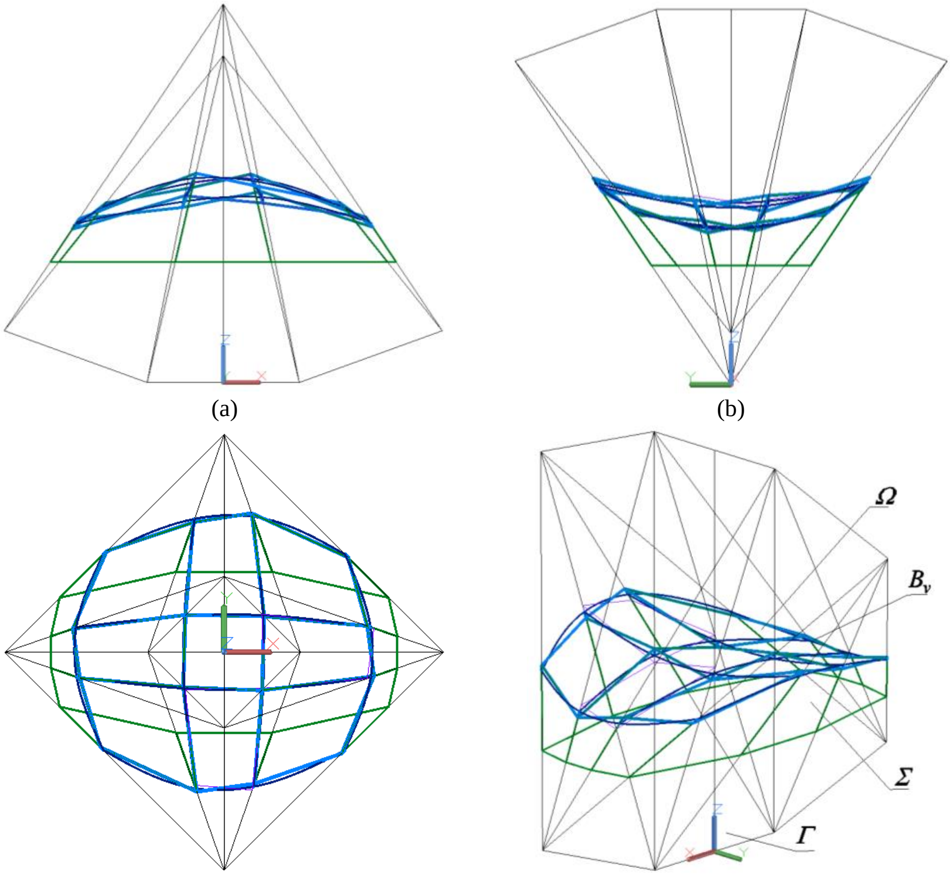
Two adjacent tetrahedra of each polyhedral network Γ, created in the above way, have one common face and three common vertices. Two adjacent closed spatial quadrangles of each polygonal eaves network B_v have one edge or vertex in common. Roof shell structures Σ constructed with the help of the method are called complex basic shells and are characterized by the continuity of the entire complex roof shell.

In general case, some roof structures are created so that their meshes do not have common sides or vertices. This property is characteristic of the roof shell structures derived from the aforementioned basic ones. These structures will be presented in the following section through a few specific examples. The roof structures created in this manner are referred to as complex derivative roof structures. They are characterized by multiple areas of discontinuity between their shell units. The empty spaces between the roof shell sectors can be designated for window openings.

5. Results

The algorithm of the method is presented using above-mentioned examples of parametric modelling of various complex roof forms covered with continuous and discontinuous shell structures using the same reference polyhedral network Γ. The parameters used to define Γ are as follows. The first parameter is the distance d_{v11} of two vertices of the axis v_{11}. The second one is the ratio dd_{uv11} of the distance d_{uv11} between the skew axes u_{11} and v_{11}, and d_{v11}. The third parameter is the ratio dd_{u11} of the distance d_{u11} of two vertices of u_{11} to d_{v11}. The values of all similar parameters that define all tetrahedral meshes Γ_{1j} and Γ_{i1} of Γ belonging to the two orthogonal strips of Γ are equal to 20,000 mm, 5 and 1, respectively.

Thus, the examined network Γ is composed of tetrahedrons Γ_{1j} and Γ_{i1}, which are congruent to each other and positioned in two orthogonal strips. Additionally, it is characterized by two mutually perpendicular planes of symmetry defined by the principal axes of $[x, y, z]$. The vertices of other meshes Γ_{ij} ($i \neq 1$ or $j \neq 1$) located diagonally in relation to Γ_{11} are taken at the corresponding vertices of the orthogonal tetrahedral meshes Γ_{1j} and Γ_{i1}.



(a) (b) (c) (d)

Fig. 6. The designed reference network Γ and the eaves network B_v, of the basic complex building structure Σ: (a) the front view, (b) the side view, (c) the top view, and (d) axonometric view. *Source:* own study

The coordinates of the vertices of the examined network Γ, shown in Fig. 6. in the orthogonal coordinate system [x, y, z] are given in Tab. 1. The published values refer to one quarter Γ_1 of the symmetrical network Γ, located between the principal planes (x, z) and (y, z) in the dihedral angle containing the positive senses of x and y.

Table 1. The coordinates of the vertices W_{ABij}, W_{CDij}, W_{ADij}, W_{BCij} (for i, j = 1, 2) of the polyhedral reference network Γ_1. *Source:* own study

Vertex	x-coordinate [mm]	y-coordinate [mm]	z-coordinate [mm]
W_{AB11}	-10,000	0	0
W_{CD11}	10,000	0	0
W_{AD11}	0	-10,000	50,000
W_{BC11}	0	10,000	50,000
W_{AB12}	-10,000	0	0
W_{CD12}	10,000	0	0
W_{AD12}	0	10,000	50,000
W_{BC12}	0	28,793	43,160
W_{AB21}	10,000	0	0
W_{CD21}	28,793	0	6,840
W_{AD21}	-10,000	0	50,000
W_{BC21}	10,000	0	50,000
W_{AB22}	10,000	0	0
W_{CD22}	28,793	0	6,840
W_{AD22}	0	10,000	50,000
W_{BC21}	0	28,793	43,160

During the second step of the algorithm, the positions of the subsequent tetrads of the vertices S_{Aij}, S_{Bij}, S_{Cij} and S_{Dij} of a reference surface ω_r, located on the side edges of Γ, are defined using the division coefficients dd_{SAij}, dd_{SBij}, dd_{SCij} and dd_{SDij}. The adopted values of these division coefficients, taken for the exemplary basic continuous structure, shown in Fig. 6.a-d, are given in Tab. 2.

Table 2. The initial data – division coefficients defining the positions of the points S_{Aij}, S_{Bij}, S_{Cij} and S_{Dij} of the reference surface ω_r for i, j = 1, 2. *Source:* own study

Ratio	Value
dd_{SA11}	0.478
dd_{SB11}	0.478
dd_{SC11}	0.489
dd_{SD11}	0.489
dd_{SA12}	0.478
dd_{SB12}	0.612
dd_{SC12}	0.623
dd_{SD12}	0.489
dd_{SA21}	0.489
dd_{SB21}	0.489
dd_{SC21}	0.332
dd_{SD21}	0.332
dd_{SA22}	0.489
dd_{SB22}	0.623
dd_{SC22}	0.332
dd_{SD22}	0.332

The above values are related to one symmetrical quarter of ω_r contained in the subspace limited by the planes (x, z) and (y, z) defined by positive senses of the axes x and y. The values of the distinguished points of this ω_r's quarter are published in Tab. 3.

Table 3. The coordinates of the selected points defining the reference surface ω_r for i, j = 1, 2. *Source:* own study

Points	x-coordinate [mm]	y-coordinate [mm]	z-coordinate [mm]
S_{D11}	5,107.3	-4,892.7	24,463.6
S_{C11}	5,107.3	4,892.7	24,463.6
S_{B11}	-5,221.5	4,778.5	23,892.6
S_{A11}	-5,221.5	-4,778.5	23,892.6
S_{D12}	5,107.3	4,892.7	24,463.6
S_{C12}	3,772.1	17,932.4	26,879.2
S_{B12}	-3,880.6	17,620.2	26,411.2
S_{A12}	-5,221.5	4,778.5	23,892.6
S_{D21}	19,237.5	-3,318.8	21,164.1
S_{C21}	19,224.6	3,323.3	21,183.4
S_{B21}	5,107.3	4,892.7	24,463.6
S_{A21}	5,107.3	-4,892.7	24,463.6
S_{D22}	19,224.6	3,323.3	21,183.4
S_{C22}	15,836.3	12,957.2	23,184.0
S_{B22}	3,772.1	17,932.4	26,879.2
S_{A22}	5,107.3	4,892.7	24,463.6

At the third step of the method's algorithm, the locations of the vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} of the subsequent meshes B_{vij} of B_v, which define the designed roof shell Ω are determined using the respective division coefficients dd$_{Aij}$, dd$_{Bij}$, dd$_{Cij}$ and dd$_{Dij}$. The division coefficients values used for the examined basic structure Σ roofed with Ω, as shown in Fig. 6.a-d, are listed in Tab. 4. The coordinates of the vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} of the developed B_v are provided in Tab. 5.

Table 4. The values of the division coefficients employed to calculate the coordinates of the vertices of the basic structure Ω_1 for i,j =1,2. *Source:* own study

Ratio	Value
dd$_{D11}$	0.505
dd$_{C11}$	0.474
dd$_{B11}$	0.493
dd$_{A11}$	0.462
dd$_{D12}$	0.474
dd$_{C12}$	0.638
dd$_{B12}$	0.597
dd$_{A12}$	0.493
dd$_{D21}$	0.317
dd$_{C21}$	0.347
dd$_{B21}$	0.474
dd$_{A21}$	0.505
dd$_{D22}$	0.347
dd$_{C22}$	0.435
dd$_{B22}$	0.638
dd$_{A22}$	0.474

Table 5. The coordinates of the vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} (for i, j = 1, 2) of the basic structure Ω_1 for i,j =1,2. *Source:* own study

Point	*x*-coordinate [mm]	*y*-coordinate [mm]	*z*-coordinate [mm]
D_{11}	4,953.3	-5,046.7	25,233.4
C_{11}	5,261.2	4,738.8	23,693.8
B_{11}	-5,067.5	4,932.5	24,662.4
A_{11}	-5,375.4	-4,624.6	23,122.8
D_{12}	5,261.2	4,738.8	23,693.8
C_{12}	3,620.7	18,368.4	27,532.7
B_{12}	-4,032.0	17,184.3	25,757.8
A_{12}	-5,067.5	4,932.5	24,662.4
D_{21}	19,673.4	-3,167.4	20,510.6
C_{21}	18.788.6	3,474.7	21,836.9
B_{21}	5,261.2	4,738.8	23,693.8
A_{21}	4,953.3	-5,046.7	25,233.4
D_{22}	18,788.6	3,474.7	21,836.9
C_{22}	16,258.5	12,535.1	22,651.5
B_{22}	3,620.7	18,368.4	27,532.7
A_{22}	5,261.2	4,738.8	23,693.8

At the fourth step of the algorithm, the vertices of the plane base of the developed basic continuous structure Σ are determined as the points of intersection of the base horizontal plane (parallel to (x, y)) and the side edges of the previously created mesh Γ. The p_z coordinate level of the base plane is equal to 15,920 mm in [x,y,z].

At the fifth step, two discontinuous roof structures Ω_{d1} and Ω_{d2} derivative to the obtained continuous structure Ω are created. The first discontinuous derivative structure Ω_{d1}, shown in Fig. 7.a-d, is created through a significant modification of Ω as follows.

Four vertices belonging to four adjacent meshes B_{vij}, B_{vij+1}, B_{vi+1j} and $B_{vi+1j+1}$ of the network B_v (structure Ω) are separated into pairs, where two vertices from opposite meshes B_{vij} and $B_{vi+1j+1}$ (shell sectors Ω_{ij} and $\Omega_{vi+1j+1}$) in a diagonal strip are shifted to the opposite side of the surface ω_r compared to the position of other two stationary vertices belonging to other meshes B_{vij+1} and B_{vi+1j} (sectors Ω_{vij+1} and Ω_{vi+1j}). These stationary vertices belong to the meshes contained in a different diagonal strip of B_v (Ω). This action is performed for all common vertices of B_v (Ω). As a result of this modification, new discontinuous structures Ω_{d1} and Σ_{d1}, shown in Fig. 7.a-d, are obtained.

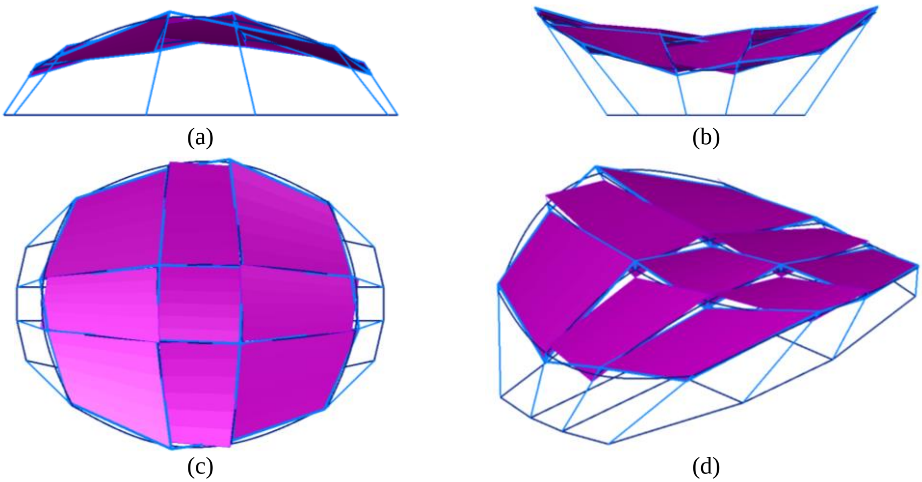


Fig. 7. The designed reference network Γ and the eaves network B_{vd1} of the first complex derivative discontinuous structure Σ_{d1}: (a) the front view, (b) the side view, (c) the top view, and (d) the axonometric view. *Source:* own study

As a result of the displacing all sets of vertices belonging to adjacent meshes of B_v along the respective side edge of Γ, triangular flat areas are formed in the discontinuous roof structure Ω_{d1} (the network B_{vd1}) allowing sunlight to enter the interior of the designed building. The differentiation of the vertex positions, belonging to adjacent meshes of B_{vd1} (Ω_{d1}) and located on the same side edge of Γ, can be achieved by appropriately adjusting the values of the division coefficients associated with the vertices of B_{vd1} and ω_r. The division coefficient values used are presented in Tab. 6.

Table 6. The values of the division coefficients defining the coordinates of the vertices of the first derivative structure Ω_{d1} for i,j =1,2. *Source:* own study

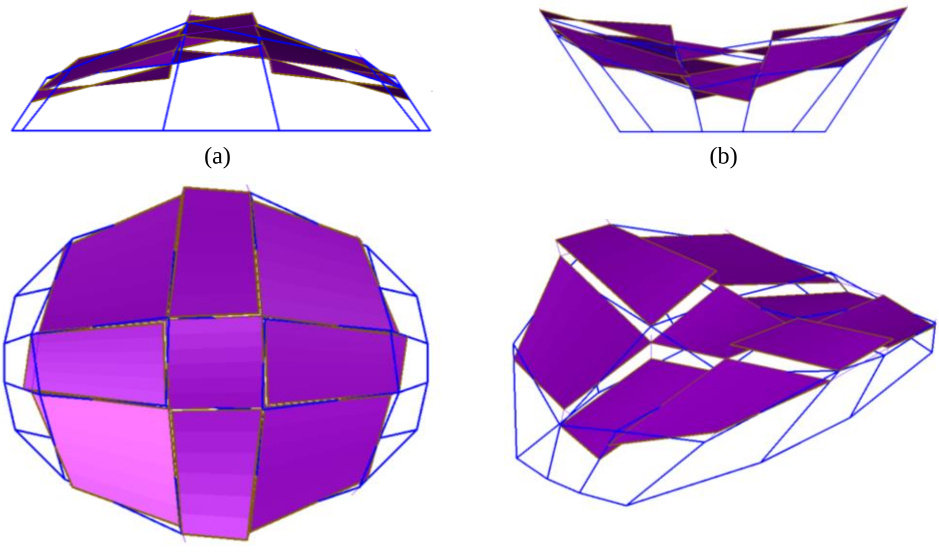
Ratio	Value
dd_{D11}	0.493
dd_{C11}	0.462
dd_{B11}	0.505
dd_{A11}	0.474
dd_{D12}	0.462
dd_{C12}	0.640
dd_{B12}	0.602
dd_{A12}	0.505
dd_{D21}	0.347
dd_{C21}	0.317
dd_{B21}	0.505
dd_{A21}	0.474
dd_{D22}	0.343
dd_{C22}	0.435
dd_{B22}	0.640
dd_{A22}	0.462

The coordinates of the vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} of Ω_{d1} obtained based on these coefficients are provided in Tab. 7.

Table 7. The coordinates of the vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} (for i, j = 1, 2) of the first derivative structure Ω_{d1}. *Source:* own study

Vertex	*x*-coordinate [mm]	*y*-coordinate [mm]	*z*-coordinate [mm]
A_{11}	-5,375.4	-4,624.6	23,122.8
B_{11}	-5,067.5	4,932.5	24,662.4
C_{11}	5,261.2	4,738.8	23,693.8
D_{11}	4,953.3	-5,046.7	25,233.4
A_{12}	-5,375.4	4,624.6	23,122.8
B_{12}	-3,729.2	18,056.2	27,064.7
C_{12}	3,923.5	17,496.5	26,225.8
D_{12}	4,953.3	5,046.7	25,233.4
A_{21}	5,261.2	-4,738.8	23,693.8
B_{21}	4,953.3	5,046.7	25,233.4
C_{21}	19,660.5	3,171.9	20,530.0
D_{21}	18,801.5	-3,470.2	21,817.5
A_{22}	5,261.2	4,738.8	23,693.8
B_{22}	3,620.7	18,368.4	27,532.7
C_{22}	16.,258.5	12,535.1	22,651.5
D_{22}	18,788.6	3,474.7	21,836.9

The second discontinuous structure Ω_{d2}, derived from the original continuous structure Ω, is obtained by displacing the vertices of all shell sectors Ω_{i1} and Ω_{1j} of Ω located in two orthogonal strips relative to the central sector Ω_{11} along the side edges of Γ, as shown in Figure 8.a-d. However, the vertices of the diagonally located shell sectors do not change their position.



(a) (b)

(c) (d)

Fig. 8. The reference network Γ and the eaves network B_{vd2} of the second complex derivative discontinuous structure Σ_{d2}: (a) the front view, (b) the side view, (c) the top view, and (d) the axonometric view. *Source:* own study

Another characteristic feature of the second discontinuous derivative structure Σ_{d2} is that the sectors Ω_{1j} of one orthogonal strips are shifted above the reference surface ω_r, while the sectors of the other orthogonal strip Ω_{i1} are displaced below the surface ω_r. The result of the displacement of the selected sectors Ω_{i1} and Ω_{1j} along the side edges of Γ comes from the assumption of the appropriate differences in the values of the division coefficients related to the vertices of B_v and the points of ω_r. The values of these coefficients are given in the Tab. 8.

Table 8. The values of the division coefficients defining the coordinates of the vertices of the second derivative structure Ω_{d2} for i,j =1,2. *Source:* own study

Ratio	Value
dd_{D11}	0.462
dd_{C11}	0.493
dd_{B11}	0.475
dd_{A11}	0.505
dd_{D12}	0.458
dd_{C12}	0.640
dd_{B12}	0.391
dd_{A12}	0.505
dd_{D21}	0.317
dd_{C21}	0.287
dd_{B21}	0.474
dd_{A21}	0.443
dd_{D22}	0.312
dd_{C22}	0.465
dd_{B22}	0.610
dd_{A22}	0.493

The coordinates of the vertices belonging to Ω_{d2} (B_{vd2}) are provided in the Tab. 9.

Table 9. The coordinates of the vertices A_{ij}, B_{ij}, C_{ij}, and D_{ij} (for i, j = 1, 2) belonging to the second derivative structure Ω_{d2}. *Source:* own study

Vertex	*x*-coordinate [mm]	*y*-coordinate [mm]	*z*-coordinate [mm]
A_{11}	-5,067.5	-4,932.5	24,662.4
B_{11}	-5,375.4	4,624.6	23,122.8
C_{11}	4,953.3	5,046.7	25,233.4
D_{11}	5,261.2	-4,738.8	23,693.8
A_{12}	-5,067.5	4,932.5	24,662.4
B_{12}	-3,426.3	18,928.1	28,371.6
C_{12}	3,620.7	18,368.4	27,532.7
D_{12}	4,645.4	5,354.6	26,773.0
A_{21}	5,569.2	-4,430.8	22,154.2
B_{21}	5,261.2	4,738.8	23,693.8
C_{21}	20,532.4	2,869.0	19,223.0
D_{21}	19,673.4	-3,167.4	20,510.6
A_{22}	4,953.3	5,046.7	25,233.4
B_{22}	3,923.5	17,496.5	26,225.8
C_{22}	15,414.2	13,379.4	23,716.5
D_{22}	19,660.5	3,171.9	20,530.0

6. Discussion

The specificity of the roof structures Ω, created with the help of the innovative method, is related to the fact that the individual shell segments Ω_{ij} are distributed on the basis of reference surfaces ω_r with negative Gaussian curvature. To construct a complex building structure having this property the points of intersection of the reference surface ω_r with all side edges of the reference network Γ have to lie between the vertices of Γ. For example, the point S_{C11} lies on the side edge c_{11} between the vertices W_{CD11} and W_{BC11}, Fig. 4.a. Therefore, the value of the division coefficient of the above pair of vertices by S_{C11} must belong to the range (0, 1). The vertex C_{11} of the mesh B_{v11} must lie also on the same straight line c_{11}. The value of the division coefficient of $\{W_{CD11}, W_{BC11}\}$ by C_{11} must also belong to (0, 1). The position of C_{11} in relation to S_{C11} is dependent on the difference between the values of the adopted division coefficients.

The characteristic feature of the created structures Ω and Γ is that the vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} of the subsequent meshes B_{vij} of B_v lie alternately above and below the arbitrary reference surface ω_r. This action is aimed at achieving specific properties of the ribbed roof structures Ω in accordance with the properties of the reference surface ω_r. Additionally, the size of folding of Ω depends mainly on the differences between the values of the division coefficients corresponding to all vertices of B_v and points of ω_r.

For the examined continuous roof structures Ω, the positions of the examined tetrads of vertices, e.g. C_{11}, D_{12}, A_{22} and B_{21} of four adjacent meshes, e.g. B_{v11}, B_{v12}, B_{v22} and B_{v21}, are identical. Therefore, their division coefficients are assumed to be equal. This is the property specific to some group of structures Ω (Σ) called basic. The structure presented in Fig. 6 in the previous section belongs to this group.

It should be summarized that the vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} of the subsequent meshes B_{vij} of B_v alternate above and below ω_r. Therefore, the division coefficients assigned to these vertices are divided into two different groups. The values of the division coefficients belonging to one group are greater than the values of the division coefficients corresponding to the points of ω_r. The values of the division coefficients belonging to other group are smaller than the values of the division coefficients corresponding to these points of ω_r. However, these values are not significantly different from those related to ω_r. Since the above-mentioned vertices of the four adjacent meshes B_{vij}, B_{vij+1}, B_{vi+1j} and $B_{vi+1j+1}$ have one common vertex C_{ij}, the respective division coefficients have to take equal values.

If a discontinuous structure is to be created, it is most convenient to build it as a derivative configuration of the base continuous structure. Each discontinuous structure is characterized by the fact that at least one of four common vertices C_{ij}, B_{ij+1}, D_{i+1j} and A_{i+1j+1} of four adjacent meshes B_{vij}, B_{vij+1}, B_{vi+1j} and $B_{vi+1j+1}$ is shifted along the respective side edge c_{ij} with respect to three remaining vertices. An analysis was carried out to explore the possibility of creating different groups of similar discontinuous structures derived from the base continuous structure Ω_c (Σ_c) presented in Fig. 6. The form Σ_c was chosen because it is characterized by an attractive general form and specific properties that allow for rational design of its structure. To create a few derivative structures Σ_d and Ω_d, an analysis was conducted to determine different mutual locations of the above-mentioned tetrads of vertices located on the same side edge of Γ, which was previously created for Σ_c. A comprehensive description of the detailed results of this analysis goes beyond the scope of this work.

This article presents the description of two different derivative structures, which have particularly attractive forms and represent two different groups of discontinuous structures, as the result of this analysis. The first of these structures is shown in Fig. 7. Its characteristic feature is that two specific pairs of identical vertices of four adjacent meshes of B_v are distinguished on each side edge of Γ. One pair consists of two identical vertices belonging to two adjacent meshes located diagonally in B_v. The second pair also consists of two identical vertices, but they belong to two other adjacent meshes located in a different diagonal strip of B_v. These two strips also run diagonally to the first central mesh. Thus, each mesh B_{vij} has two opposite vertices in common with the adjacent meshes of the first diagonal strip, and its other two vertices are shared with the corresponding vertices of other two adjacent meshes belonging to the second diagonal strip.

It should be added that the vertices belonging to the first diagonal strip, as discussed above, lie below the reference surface. However, the vertices belonging to the second diagonal strip lie above this surface. The position of the vertices of these two groups is generated using the respective different values of the division coefficients.

Depending on the differences in the values of the division coefficients associated with these quadruples of all B_v's vertices, the structure Ω can be more or less folded, resulting in the larger or smaller flat discontinuous areas for the window openings. In the case of the first discontinuous derivative structure (Fig. 6), these openings are triangular. In the second case, (Fig. 8), the openings are square. The proportions between the areas of flat elevation walls, flat roof windows and shell roof sectors, as well as the lengths of all edges of Σ and Ω, determine the attractiveness and rationality of the structures shaped by appropriately selected values of the division coefficients. However, a detailed discussion of these architectural issues is beyond the scope of the study.

Other characteristic features of the complex forms created with the algorithm of this method are: (1) different inclination of the adjacent facade walls to the vertical, and (2) various inclination of the facade walls with their bases shifted to the inside or outside of each considered building form, depending on the direction of the axis v_{ij} or u_{ij} of Γ.

7. Conclusions

The novel method for modelling unconventional building forms with complex folded elevation walls, inclined to the vertical and roofed with complex transformed shell structures composed of many shell units arranged on regular surfaces characterized by the negative Gaussian curvature, is presented. The algorithm of this method enables the determination of innovative basic ribbed continuous and derivative discontinuous shell roof structures. The method complements the existing methods related to shaping polyhedral free forms roofed with structures composed of many shells arranged on regular surfaces with the positive Gaussian curvature.

The procedure of this method is presented using three specific examples of parametric building forms, controlled with specific sets of division coefficients that express the proportions between the distances of the characteristic vertices of their multi-plane elevation walls and multi-shell roofs, as well as novel auxiliary polyhedral and polygonal networks. The ranges of variability for these coefficients are also discussed to create several specific types of novel building forms. Furthermore, the main relationships governing the positions of all vertices of the polyhedral network Γ, eaves network B_v (shell structure Ω) and the values assigned to these vertices as partition coefficients are presented.

The characteristic feature of the presented method is that the vertices of the designed shell roof units are generated using division coefficients that take values from the range (0, 1) in relation to the vertices of the reference polyhedral networks Γ. This limitation results in the fact that the innovative ribbed roof structures are limited to those that are composed of transformed roof shells arranged in conformity with the selected saddle surface, while the facades are limited to polyhedral structures. Additionally, all individual cells of Γ are tetrahedrons, and all shell roof units are hyperbolic-parabolic segments limited by spatial quadrangles.

The algorithm of this method, as presented in these examples, allows for further exploration of innovative and visually appealing types of the unconventional complex building forms, roofed with multiple complete transformed shells assembled into complex ribbed shell structures based on reference surfaces with almost any Gaussian curvature. Further research will focus on optimizing the forms and positions of these ribbed roof structures in relation to the smooth reference surfaces. Additionally, research has already begun on development of a parametric method for shaping structural systems intended for these unconventional building forms.

References

[1] Abramczyk J., *Shell Free Forms of Buildings Roofed with Transformed Corrugated Sheeting*, Monograph, Publishing House of Rzeszow University of Technology, Rzeszów, 2017.

[2] Wei-Wen Yu, P.E., *Cold Formed Steel Design*; John Wiley and Sons Inc., New York, NY, 2000.

[3] Reichhart A., "Corrugated Deformed Steel Sheets as Material for Shells", in Proceedings - Int. Conf. Lightweight Struct. Civ. Eng., Warsaw, 1995, pp. 625–636.

[4] Abramczyk J., "Folded Sheets as a Universal Material for Shaping Transformed Shell Roofs", in *Materials* 2021, 14(8), 2051. https://doi.org/10.3390/ma14082051

[5] Reichhart A., "Principles of designing shells of profiled steel sheets", in Proceedings - the X International Conference on Lightweight Structures in Civil Engineering, Rzeszow, 2004, pp. 138-145.

[6] Abramczyk J., "Shape transformations of folded sheets providing shell free forms for roofing", in Proceedings - the 11th Conference on Shell Structures Theory and Applications, Gdańsk, Wojciech Pietraszkiewicz, Wojciech Witkowski, CRC Press Taylor and Francis Group, 2017, pp. 409-412.

[7] Reichhart A., *Geometrical and Structural Shaping Building Shells Made up of Transformed Flat Folded Sheets*; Publishing House of Rzeszow University of Technology, Rzeszów, 2002. (in Polish)

[8] Abramczyk J., "Transformed Corrugated Shell Units Used as a Material Determining Unconventional Forms of Complex Building Structures", in *Materials* 2021, *14*(9), 2402. https://doi.org/10.3390/ma14092402

[9] Winter G., "Strength of thin steel compression flanges", in Trans. ASCE 1974, 112, 895–912

[10] Parker J.E., *Behavior of Light Gauge Steel Hyperbolic Paraboloid Shells*, Ph.D. Thesis, Cornell University. Ithaca, NY, 1969.

[11] Gergely P., Banavalkar P.V. and Parker J.E., "The analysis and behavior of thin-steel hyperbolic paraboloid shells", in A Research Project Sponsored by the America Iron and Steel Institute, Report No. 338, Dept. of Struct. Engng., 1971, Cornell University: Ithaca, New York, NY, 1971.

[12] Egger H., Fischer M. and Resinger F., "Hyperschale aus Profilblechen", Stahlbau 1971, 12, pp. 353–361.

[13] Davis J.M. and Bryan E.R., "Manual of Stressed Skin Diaphragm Design", Granada Publishing Ltd., London, UK, 1982.

[14] Abdel S.G., "Critical shear loading of curved panels of corrugated sheets.", in Proc. ASCE J. Struct. Div. 1970, 96, pp. 1279–1294.

[15] Biswas M. and Iffland J.S.B., "Metal decks used to form hypar-shell panels", in Proceedings of the 2nd Speciality Conference on Cold-Formed Steel Structures, Rolla, MO, USA, 1973.

[16] Abel J.F. and Mungan I., *Fifty Years of Progress for Shell and Spatial Structures. Monograph by the International Association for Shell and Spatial Structures*, Madrid, Spain, 2011.

[17] Saitoh M., Recent Spatial Structures in Japan, in JASS: Madrid, Spain, 2001.

[18] Walentyński R. and Cybulski R., "Stability analyses and experimental investigation of doubly corrugated steel arch panels", in ACEE, 5/4, 2012, pp. 79-82.

[19] Prokopska A. and Abramczyk J., "Responsive Parametric Building Free Forms Determined by Their Elastically Transformed Steel Shell Roofs", *Buildings*, 2019, 9(2), 46. https://doi.org/10.3390/buildings9020046

[20] Carmo M.P., Differential Geometry of Curves and Surfaces, Prentice-Hall, Inc.: Englewood Cliffs, NJ, USA, 1976.

[21] Pottmann,H., Asperi A., Kilian, A. and Hofer M., Architectural Geometry, Bentley, Institute Press, USA, 2007.

[22] Samyn P.P., "Structures isobarres et isonoeuds", in Proceedings - the 2nd International Conference on Space Structures – University of Surrey, Guilford, England, 1975, pp. 621-634.

[23] Żwirek P., "Selected issues of using nominally flat folded sheets and self-bearing panels for curved" in Proceedings - the 9th Scientific and Technical Conference on Shaping Structures - Thin-walled structures, Rzeszów, Poland, 2015. (In Polish)