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ANALYSIS OF THE TRANSIENT STATE IN A CIRCUIT WITH SUPERCAPACITOR

The paper presents an analysis of the transient state in a simple circuit of RC_α class with a supercapacitor. The behavior of supercapacitors differs from that of classic capacitors, which influences voltage and current waveforms in circuits containing them. The waveforms are described by relations of fractional-order integral-differential calculus. A simple fractional-order supercapacitor model, including its series internal resistance, has been assumed for the analysis. The obtained solution of the fractional-order differential equation describing the examined circuit is presented. The impact of different values of the parameter α on the solution has been analyzed too. The derived relations are illustrated by simulation examples for the circuit powered by a DC voltage source. This situation describes the supercapacitor charging process. Its charging time depends mainly on the value of fractional-order parameter α of the supercapacitor.

KEYWORDS: fractional-order differential equation, first-order RC_α circuit, supercapacitor

1. INTRODUCTION

From experimental studies it is known, that charging and discharging waveforms of supercapacitors differ from those of classic dielectric capacitors. It is due to their high capacity, up to even a few thousand Farads, their electrochemical structure and a relatively large internal series resistance ESR [1]. Therefore, their behaviour is more and more frequently accurately described using fractional-order integral-differential calculus [2]. It shows a good accuracy in describing these elements. It is also used to describe real, lossy coils, especially those with soft, ferromagnetic cores [3].

The analysis of transient states in circuits with fractional-order elements is the subject of several works [4-6]. They present numerical methods for solving linear fractional-order differential equations or analyze some particular cases of fractional-order parameters. This paper is devoted to the analysis of the transient state in a simple circuit with a resistor and a supercapacitor modeled as a fractional-order C_α element.

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2. THE MODEL OF THE SYSTEM CIRCUIT

The model of the analyzed (in time domain) simple RC_{α} circuit with supercapacitor is presented in Fig. 1.

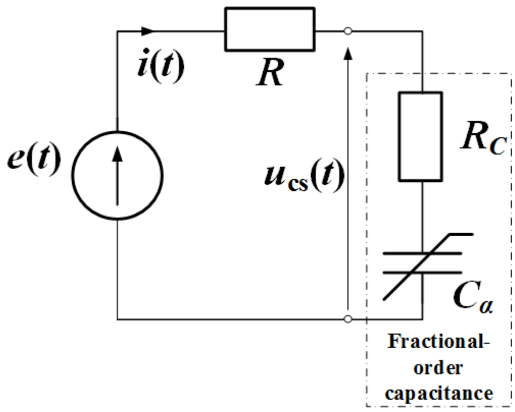

Fig. 1. Model of the analyzed simple RC_{α} circuit with supercapacitor

The model from Fig. 1 includes the voltage source $e(t)$, the series resistance R, limiting the charging current, the supercapacitor modeled as a fractional-order C_{α} capacitance and its internal series resistance R_C. In the analysis of charging and discharging of the fractional-order capacitor (and transient states with other kinds of sources), the voltage $u_{cs}(t)$ and current $i(t)$ waveforms are most interesting. Zero initial conditions for supercapacitor have been assumed $u_{c\alpha}(0) = u_{c\alpha}(0^-) = u_{c\alpha}(0^+)$. Starting from the simple impedance model $Z(\mathrm{j}\omega)$ of the fractional-order capacitor:

$$Z(\mathrm{j}\omega) = \frac{U_{C\alpha}(\mathrm{j}\omega)}{I(\mathrm{j}\omega)} = \frac{1}{(\mathrm{j}\omega)^{\alpha} C}, \tag{1}$$

and treating it as a voltage-current transmittance, the impedance can be written in the Laplace domain as:

$$Z(s) = \frac{U_{C\alpha}(s)}{I(s)} = \frac{1}{s^{\alpha} C}. \tag{2}$$

Transforming expression (2) and calculating the inverse transform, the current flowing in the analyzed circuit can be written in the form:

$$i(t) = C_{\alpha} \frac{d^{\alpha} u_{C\alpha}(t)}{dt^{\alpha}}. \tag{3}$$

It means that the current flowing in the circuit is a fractional-order derivative of the voltage on the supercapacitor. The next part of the paper is the analysis of the concerned circuit in the time domain. The relations describing voltage $u_{cs}(t)$ and current i(t) have been obtained by solving the fractional-order differential equation. The derived relations have been simulated and illustrated in Figs. 2 – 4.

3. ANALYSIS OF THE CIRCUIT STATE EQUATION

For the examined circuit, at any voltage source waveform, the state equations can be written as:

$$i(t) - C_\alpha \frac{d^\alpha u_{C\alpha}(t)}{dt^\alpha} = 0, \tag{4}$$

and:

$$(R + R_C)i(t) + u_{C\alpha}(t) = e(t). \tag{5}$$

By substituting equation (4) into (5) the fractional-order differential equation, describing the voltage $u_{c\alpha}(t)$ in time domain, has been obtained:

$$\frac{d^\alpha u_{C\alpha}(t)}{dt^\alpha} + \frac{1}{R_Z C} u_{C\alpha}(t) = \frac{1}{R_Z C} e(t), \tag{6}$$

where:

$$R_Z = R + R_C. \tag{7}$$

Solving the above fractional-order differential equation is possible using the Laplace transform method, since the analyzed system is linear. Using the Laplace of a fractional derivative defined by Caputo [7]:

$$\mathcal{L}\left\{{}_0^C D_t^\alpha f(t)\right\} = s^\alpha F(s) - \sum_{k=0}^{n} s^{\alpha-k-1} f^{(k)}(0), \tag{8}$$

equation (6) can be written in the s-domain as:

$$U_{C\alpha}(s)\left(s^\alpha + \frac{1}{R_Z C}\right) = \frac{1}{R_Z C} E(s), \tag{9}$$

hence:

$$U_{C\alpha}(s) = \frac{1}{R_Z C} E(s) \frac{1}{\left(s^\alpha + \frac{1}{R_Z C}\right)}. \tag{10}$$

The current $I(s)$ can be calculated from the transmittance:

$$I(s) = s^\alpha C\, U_{C\alpha}(s) = \frac{E(s)}{R_Z}\left(1 - \frac{\frac{1}{R_Z C}}{\left(s^\alpha + \frac{1}{R_Z C}\right)}\right), \tag{11}$$

For a constant voltage source e(t) = E = const. in the simple RC_α circuit charging the supercapacitor, equations describing the voltage across the supercapacitor $U_{cs}(s)$, containing the internal resistance R_C, based on formulas (10) and (11) can be defined as:

$$U_{cs}(s)=R_C I(s)+U_{C\alpha}(s)=\frac{R_C}{R_Z}\frac{E}{s}+\frac{1}{R_Z C}\frac{E}{s}\frac{1}{s^{\alpha}+\frac{1}{R_Z C}}\left(1-\frac{R_C}{R_Z}\right). \tag{12}$$

A relationship occurs [7]:

$$\mathrm{L}^{-1}\left\{\frac{k!\, s^{\nu-\mu}}{\left(s^{\nu}\pm a\right)^{k+1}}\right\}=t^{\nu k+\mu-1}E_{\nu,\mu}^{(k)}\left(\mp a t^{\nu}\right),\ \text{dla}\ k\in \mathbf{Z}_{+}\ \operatorname{Re}\{s\}>|a|^{\nu^{-1}}. \tag{13}$$

where: $E_{\nu,\mu}^{(k)}\left(\pm a t^{\nu}\right)$ is a classic *k*-th order derivative of a two-parameter Mittag-Leffler function. From the above relation and the convolution theorem we obtain:

$$u_{cs}(t)=\frac{R_C}{R_Z}E+\frac{E}{R_Z C}\left(1-\frac{R_C}{R_Z}\right)\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{R_Z C}\tau^{\alpha}\right)d\tau, \tag{14}$$

or in the form of a series:

$$u_{cs}(t)=\frac{R_C}{R_Z}E+\frac{E}{R_Z C}\left(1-\frac{R_C}{R_Z}\right)\sum_{k=0}^{\infty}\left(-\frac{1}{R_Z C}\right)^k\frac{1}{\Gamma(\alpha(k+1))}\int_0^t \tau^{\alpha(k+1)-1}d\tau. \tag{15}$$

Integral (15) can be solved analytically, so it finally takes the form:

$$u_{cs}(t)=\frac{R_C}{R_Z}E+\frac{E}{R_Z C}\left(1-\frac{R_C}{R_Z}\right)t^{\alpha}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{R_Z C}t^{\alpha}\right)^k}{\Gamma(\alpha(k+1)+1)}. \tag{16}$$

Calculating the inverse Laplace transform of the current $i(t)$ analogically, as in the case of the voltage u_{cs}(t), the relation has the form of integral:

$$i(t)=\frac{E}{R_Z}-\frac{E}{R_Z}\frac{1}{R_Z C}\sum_{k=0}^{\infty}\left(-\frac{1}{R_Z C}\right)^k\frac{1}{\Gamma(\alpha(k+1))}\int_0^t \tau^{\alpha(k+1)-1}d\tau, \tag{17}$$

and finally:

$$i(t)=\frac{E}{R_Z}-\frac{E}{R_Z}\frac{1}{R_Z C}t^{\alpha}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{R_Z C}t^{\alpha}\right)^k}{\Gamma(\alpha(k+1)+1)}. \tag{18}$$

In the next section an example of RC_α circuit with supercapacitor, powered by a DC voltage source, is presented. Illustrations of charging voltage and current are also included.

4. EXAMPLE

Based on the previous studies, simulations of the transient state in an exemplary simple RC_α circuit with supercapacitor, modeled as a fractional-order element were conducted. There were assumed the following parameters of the circuit elements: the supercapacitor of nominal capacitance $C = 0{,}1$ F and the resistance $R_C = 28\ \Omega$ [8], the charging current limiting resistor $R = 100\ \Omega$ and the DC voltage source $E = 5$ V. Simulations of the charging voltage $u_{cs}(t)$ and the current $i(t)$ in the circuit were made in Mathematica, PSpice and Maple programs. Illustrations of these waveforms are shown in Figs. 2-4. For practical reasons, $k = 2000$ elements were assumed in numerical computations (instead of ∞ in Mittag-Leffler function).

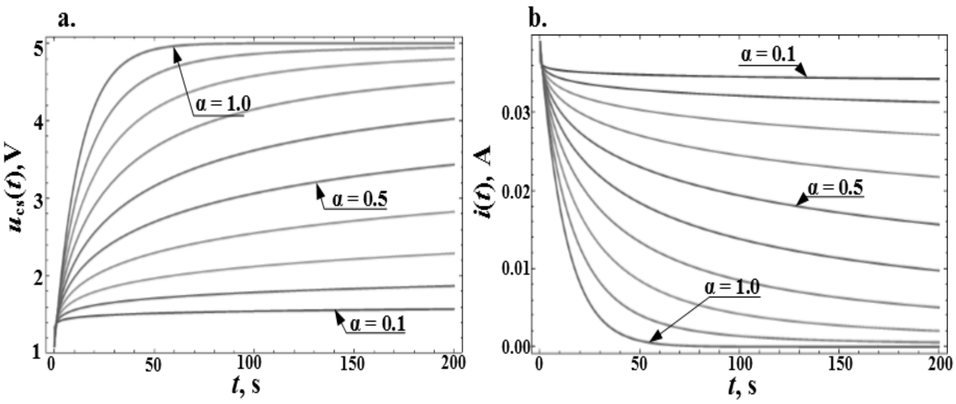

Fig. 2. Waveforms of a. voltage $u_{cs}(t)$ and b. current $i(t)$ based on fomulas (16) and (18) for $\alpha \in$ <0,1> and $k = 2000$, obtained in Mathematica

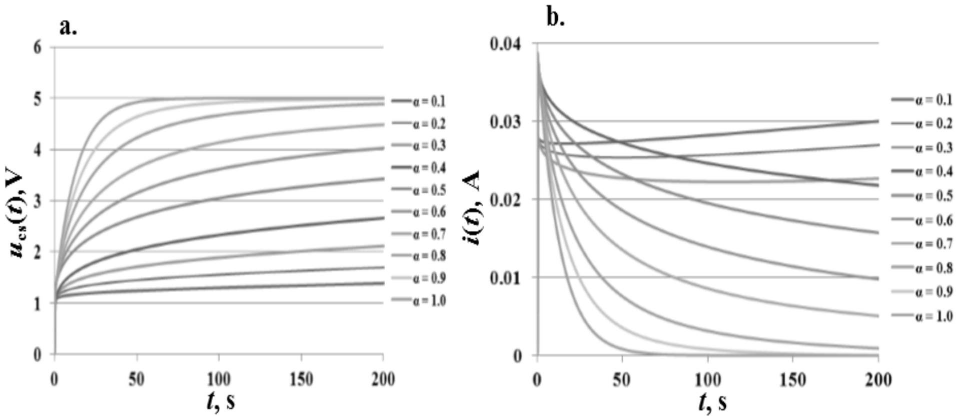

Fig. 3. Wavefoms of a. voltage $u_{cs}(t)$ and b. current $i(t)$ for $\alpha \in$ <0,1> obtained in PSpice program

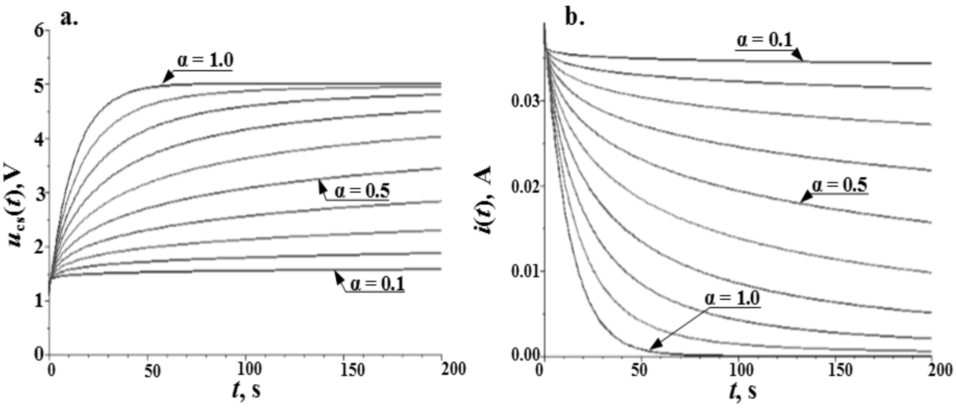

Fig. 4. Waveforms of a. voltage $u_{cs}(t)$ and b. current $i(t)$ based on formulas (21) and (24) for $\alpha \in$ <0,1> and $k = 2000$, obtained in Maple

The voltage waveforms obtained in Mathematica, PSpice and Maple programs have the same form for all the specified values of the coefficient α, compare with Figs. 2-4a, but the current waveforms look the same only for the simulations performed in Mathematica and Maple programs (see. Figs. 2-4b). Instantaneous current value from PSPice program for small values of α, e.g. up to $\alpha \approx 0.3$ does not decrease, but begins to grow (see Fig. 3b). This means that the PSpice algorithms do not give reliable numerical results for small values of fractional-order coefficients.

5. SUMMARY

The paper analyzes the transient state in a simple RC_α circuit with a supercapacitor. Voltage and current waveforms in circuits with supercapacitors are described by relations using fractional-order integral or differential equations. A simple fractional-order supercapacitor model has been assumed for the analysis. It takes into account the supercapacitor internal equivalent series resistance ESR (R_C) too. The solution of the fractional-order differential equation which describes the analyzed circuit has been derived and presented. Various cases of the fractional-order parameter α have been examined. The derived relations have been illustrated by simulation examples for DC power supply of the circuit. The supercapacitor charging time depends largely on the value of its fractional-order parameter α. The smaller the value of α is, the longer its charging lasts. For α = 1 the transient state is described by a standard first-order differential equation.

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