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REMARKS ON THE SOBOLEV TYPE SPACES OF MULTIFUNCTIONS

Summary. In this paper we introduce the spaces of multifunctions $\mathbf{S}_{X,pq}$ and $\mathbf{X}_{pq}$ which correspond with the Sobolev space W_{pq} and the space of multifunctions $\mathbf{X}_{mkc,\varphi,k,Y}$ which correspond with the Orlicz-Sobolev space W^k_φ. We study completeness of them. Also we give some theorems.

UWAGI O PRZESTRZENIACH MULTIFUNKCJI TYPU SOBOLEVA

Streszczenie. W artykule wprowadzamy przestrzenie multifunkcji $\mathbf{S}_{X_{pq}}$ and $\mathbf{X}_{pq}$, które odpowiadają przestrzeni Soboleva W_{pq}, oraz przestrzeń multifunkcji $\mathbf{X}_{mkc,\varphi,k,Y}$, która odpowiada przestrzeni Orlicza-Soboleva W^k_φ. Badamy zupełność tych przestrzeni. Podajemy także pewne twierdzenia dotyczące tych przestrzeni.

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1. Introduction

The notion of differential of multifunction was introduced in many papers (see [3, Chapter 6, section 7]). In this paper we apply the De Blasi definition of differential of multifunction from [1], and the Martelli-Vignoli definition from [9]. In the Definition 1 we join the definitions of a derivative of multifunction from [2, 3, 5, 9]. We introduce the multiderivatives F', $D^\alpha F$ and DF. We introduce also the spaces of multifunctions $\mathbf{S}_{X,pq}$, $\mathbf{X}_{pq}$ and $\mathbf{X}_{mkc,\varphi,k,Y}$ and we prove completeness of them. In the Section 3 we generalize some results from [6, 8]. Additionally we give some theorems. The space W_{pq} and its applications was presented in [4]. The aim of this note is to obtain the generalization of the Sobolev space W_{pq} on the multifunctions.

We use the definitions and theorems connected with multifunctions from [3].

Let Y be the real Banach space with the norm $\|\cdot\|$ and θ be the zero in Y. Let $T \subset R$, let 2^Y denote the set all subsets of Y and let

$$\mathbf{X} = \{F : T \to 2^Y : F(t) \text{ is nonempty for every } t \in T\}.$$

For all nonempty and compact $A, B \subset Y$ we introduce the famous Hausdorff distance by

$$\operatorname{dist}(A, B) = \max(\max_{x\in A} \min_{y\in B} \|x - y\|, \max_{y\in B} \min_{x\in A} \|x - y\|).$$

Denote

$$P_c(Y) = \{A \subset Y : A \text{ is nonempty and compact}\},$$
$$P_{kc}(Y) = \{A \subset Y : A \text{ is nonempty and convex and compact}\}.$$

We define

$$\mathbf{X}_{kc} = \{F \in \mathbf{X} : F(t) \in P_{kc}(Y) \text{ for a.e. } t \in T\},$$
$$\mathbf{X}_{mkc} = \{F \in \mathbf{X}_{kc} : F \text{ is graph measurable}\}.$$

(See [3, Chapter 2: Definition 1.1, Theorem 2.4, Proposition 5.3]).

Let $B \in P_c(Y)$. Denote $|B| = \operatorname{dist}(B, \{\theta\})$. Let $F \in \mathbf{X}_{mkc}$. Now we introduce the function $|F|$ by the formula

$$|F|(t) = |F(t)| \quad \text{for every } t \in T.$$

Let $F, G \in \mathbf{X}$, $a \in R$. We define $F + G$ and aF by the formulae

$$(F + G)(t) = \{x + y : x \in F(t), y \in G(t)\},$$
$$(aF)(t) = \{ax : x \in F(t)\}$$

for every $t \in T$.

2. On the spaces of differentiable multifunctions

Let now T be open.

Definition 1. *We say that $F \in \mathbf{X}_{kc}$ is differentiable if there is $H_F \in \mathbf{X}_{kc}$ such that for a.e. $t \in T$ there is $\delta > 0$ such that*

$$\operatorname{dist}(F(t+h) - hH_F(t), F(t)) \leqslant |h| A_t^1(h),$$

or

$$\operatorname{dist}(F(t+h), F(t) + hH_F(t)) \leqslant |h| A_t^2(h)$$

for every $h \in (-\delta, \delta)$, where

$$\lim_{h \to 0} A_t^1(h) = \lim_{h \to 0} A_t^2(h) = 0.$$

If F is differentiable then we write $F' = H_F$ and F' should be called the multiderivative of F.

Let $F(t) = [0, t]$ for every $t \geqslant 0$ and $F(t) = [t, 0]$ for every $t < 0$. We have $F'(t) = [0, 1]$ for every $t \in R$.

Let $p \geqslant 1$, $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$\mathbf{X}_p = \{F \in \mathbf{X}_{mkc} : |F| \in L^p(T, R)\},$$

$$\mathbf{S}_{X,pq} = \{F \in \mathbf{X}_{mkc} : F \in \mathbf{X}_p, F \text{ is differentiable and } F' \in \mathbf{X}_q\}.$$

It is easy to see that $\mathbf{X}_p$ is a linear subset of $\mathbf{X}$ and $\mathbf{S}_{X,pq}$ is a linear subset of $\mathbf{X}_p$.

Let now $\mu(T) < \infty$. For $F, G \in \mathbf{X}_p$ we define

$$D_p(F, G) = (\int_T (\operatorname{dist}(F(t), G(t)))^p dt)^{\frac{1}{p}}.$$

We easily obtain (see [8, Theorem 4.1 and the proof of Theorem 4.3]).

Theorem 2. *The set $\mathbf{X}_p$ with the metric D_p is a complete metric space.*

For $F, G \in \mathbf{S}_{X,pq}$ we define

$$d_{S_{X,pq}}(F, G) = D_p(F, G) + D_q(F', G').$$

Theorem 3. *The set $\mathbf{S}_{X,pq}$ with metric $d_{S_{X,pq}}$ is a complete metric space.*

Proof. Let $\{F_n\}$ be the Cauchy sequence in $(\mathbf{S}_{X,pq}, d_{S_{X,pq}})$. So $\{F_n\}$ is the Cauchy sequence in $(\mathbf{X}_p, D_p)$, $\{F_n'\}$ is the Cauchy sequences in $(\mathbf{X}_q, D_q)$.

So there are $F \in \mathbf{X}_p$, $G \in \mathbf{X}_q$ such that $F_n \to F$ and $F_n' \to G$, as $n \to \infty$. We must prove that G is a multiderivatives of F. We have for a.e. $t \in T$:
if

$$\operatorname{dist}(F_n(t+h) - hF_n', F_n(t)) \leqslant |h| A^1_{n,t}(h),$$

we have

$$\begin{aligned}\operatorname{dist}(F(t+h) - hG(t), F(t)) &\leqslant \operatorname{dist}(F(t+h) - hG(t), F_n(t+h) - hF_n'(t)) \\ &+ \operatorname{dist}(F_n(t+h) - hF_n', F_n(t)) + \operatorname{dist}(F_n(t), F(t)) \\ &\leqslant \operatorname{dist}(F(t+h), F_n(t+h)) + |h| \operatorname{dist}(F_n'(t), G(t)) \\ &+ \operatorname{dist}(F_n(t+h) - hF_n'(t), F_n(t)) + \operatorname{dist}(F_n(t), F(t)) \\ &\leqslant \operatorname{dist}(F(t+h), F_n(t+h)) + |h| \operatorname{dist}(G(t), F_n'(t)) \\ &\quad + |h| A^1_{n,t}(h) + \operatorname{dist}(F_n(t), F(t)) = |h| A^1_t(h),\end{aligned}$$

where

$$\lim_{h \to 0} A^1_t(h) = 0.$$

The proof in the second case is analogous. □

Let now Y be Hilbert space, $T = [0, b]$. Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. We define

$$W_{pq}(T, Y) = \{x \in L^p(T, Y) : x' \in L^q(T, Y)\},$$

where x' is understood in the sense of vector-valued distribution,

$$\|x\|_{W_{pq}(T,Y)} = (\|x\|^2_{L^p(T,Y)} + \|x'\|^2_{L^q(T,Y)})^{\frac{1}{2}}$$

for every $x \in W_{pq}(T, Y)$.

Let $F \in \mathbf{X}_p$, we define

$$K_{F,pq} = \{f_F : f_F(t) \in F(t), \|f_F(t)\| = |F(t)| \text{ a.e. and } f_F \in W_{pq}(T, Y)\},$$

$$\mathbf{X}_{pq} = \{F \in X_p : K_{F,pq} \neq \emptyset\}.$$

For $F, G \in \mathbf{X}_{pq}$ we define

$$\rho(F, G) = D_p(F, G) + \operatorname{dist}(K_{F,pq}, K_{G,pq}) + \||F| - |G|\|_{L^p(T,R)},$$

where

$$\operatorname{dist}(K_{F,pq}, K_{G,pq}) =$$

$$= \max\Big(\sup_{a \in K_{F,pq}} \inf_{b \in K_{G,pq}} \|a-b\|_{W_{pq}(T,Y)}, \sup_{b \in K_{G,pq}} \inf_{a \in K_{F,pq}} \|a-b\|_{W_{pq}(T,Y)}\Big).$$

We obtain

Theorem 4. *The set $\mathbf{X}_{pq}$ with metric ρ is a linear complete metric space.*

Proof. Let $\{F_n\}$ be a Cauchy sequence in $(\mathbf{X}_{pq}, \rho)$. So $\{F_n\}$ is a Cauchy sequence in $(\mathbf{X}_p, D_p)$ hence there is $F \in \mathbf{X}_p$ such that $D_p(F_n, F) \to 0$ as $n \to \infty$. Also $\{|F_n|\}$ is a Cauchy sequence in $L^p(T,R)$, so there is $a \in L^p(T,R)$ such that $\||F_n| - a\|_{L^p(T,R)} \to 0$ as $n \to \infty$. Next there are $f_{F_n} \in K_{F_n,pq}$ such that $\{f_{F_n}\}$ is the Cauchy sequence in $W_{pq}(T,Y)$, so there is $h \in W_{pq}(T,Y)$ such that $\|f_{F_n} - h\|_{W_{pq}(T,Y)} \to 0$ as $n \to \infty$. Then $f_{F_n} \to h$ in measure, hence $h(t) \in F(t)$ and $\|h(t)\| = |F(t)|$ a.e. □

3. Generalized Orlicz-Sobolev spaces of multifunctions

Let now φ be a locally integrable, convex φ-function, let φ fulfils the Δ_2 condition and let

$$\inf_{t \in T} \varphi(t,1) > 0.$$

Let $W^k_\varphi(T)$ denotes the generalized Orlicz-Sobolev space (see [10, p. 66–68]), let $\|\cdot\|^k_\varphi$ denotes the norm in $W^k_\varphi(T)$, $\|\cdot\|_\varphi$ denotes the Luksemburg norm in $L^\varphi(T)$ and $Y = R$. Let $\mathcal{D}^a x$ denotes the generalized derivatives of orders $a \leqslant k$ of $x \in W^k_\varphi(T)$. Let

$$\mathbf{X}_{mkc,\varphi} = \{F \in \mathbf{X}_{mkc} : F(t) = s(t) + r(t)[-1,1] \text{ for every } t \in T, \quad s, r \in L^\varphi(T)\},$$

$$\mathbf{X}_{mkc,\varphi,k} = \{F \in \mathbf{X}_{mkc} : F(t) = s(t) + r(t)[-1,1] \text{ for every } t \in T, \quad s, r \in W^k_\varphi(T)\}.$$

It is easy to see that $\mathbf{X}_{mkc,\varphi}$ and $\mathbf{X}_{mkc,\varphi,k}$ are the linear subsets of $\mathbf{X}$ and we will be call $\mathbf{X}_{mkc,\varphi,k}$ the generalized Orlicz-Sobolev space of multifunctions.

If $F \in \mathbf{X}_{mkc,\varphi,k}$, then we define the generalized derivatives of order $a \leqslant k$ of F by

$$D^a F(t) = \mathcal{D}^a s(t) + \mathcal{D}^a r(t)[-1,1] \quad \text{for every } t \in T.$$

Let $F_1, F_2 \in \mathbf{X}_{mkc,\varphi,k}$ and

$$F_1(t) = f_1(t) + g_1(t)[-1,1], \qquad F_2(t) = f_2(t) + g_2(t)[-1,1]$$

for every $t \in T$. We define

$$\rho_1(F_1, F_2) = \|f_1 - f_2\|_\varphi^k + \|g_1 - g_2\|_\varphi^k.$$

It is easy to see that ρ_1 is the metric in $\mathbf{X}_{mkc,\varphi,k}$ and $(\mathbf{X}_{mkc,\varphi,k}, \rho_1)$ is a complete linear metric space.

Let now $Y = R^n$. We define

$$\mathbf{X}_{mkc,\varphi,Y} = \{F \in \mathbf{X}_{mkc} : |F| \in L^\varphi(T,R)\}.$$

It is easy to see that $\mathbf{X}_{mkc,\varphi,Y}$ is a linear space. Let $F \in \mathbf{X}_{mkc,\varphi,Y}$ we define

$$K_{F,\varphi} = \{f_F : f_F(t) \in F(t) \text{ and } \|f(t)\| = |F(t)| \text{ a.e.}\}.$$

It is easy to see that if $g \in K_{F,\varphi}$, then $g \in L^\varphi(T,Y)$.

We define

$$\mathbf{X}_{mkc,\varphi,k,Y} = \{F \in \mathbf{X}_{mkc,\varphi,Y} : |F| \in W_\varphi^k(T)\}.$$

Let $F, G \in \mathbf{X}_{mkc,\varphi,k,Y}$, we define

$$\rho_2(F,G) = \|\operatorname{dist}(F(\cdot), G(\cdot))\|_\varphi + \||F| - |G|\|_\varphi^k + \operatorname{dist}(K_{F,\varphi}, K_{G,\varphi}),$$

where

$$\operatorname{dist}(K_{F,\varphi}, K_{G,\varphi}) = \max\Big(\sup_{a \in K_{F,\varphi}} \inf_{b \in K_{G,\varphi}} \|a-b\|_{L^\varphi(T,Y)}, \sup_{b \in K_{G,\varphi}} \inf_{a \in K_{F,\varphi}} \|a-b\|_{L^\varphi(T,Y)}\Big).$$

Theorem 5. $(\mathbf{X}_{mkc,\varphi,k,Y}, \rho_2)$ *is a complete metric space.*

Proof. Let $\{F_n\}$ be a Cauchy sequence in $(\mathbf{X}_{mkc,\varphi,k,Y}, \rho_2)$, then (see [7, Corollary 1]) there is $F \in \mathbf{X}_{mkc,\varphi}$ such that

$$\|\operatorname{dist}(F_n(t), F(t))\|_\varphi \to 0 \quad \text{as } n \to \infty.$$

Also

$$\operatorname{dist}(F_n(t), F(t)) \to 0 \quad \text{as } n \to \infty$$

in measure. So there is subsequence $\{F_{n_k}\}$ of the sequence $\{F_n\}$ such that

$$\operatorname{dist}(F_{n_k}(t), F(t)) \to 0 \ \text{ a.e.}$$

Also there are $f_{F_n} \in K_{F_n,\varphi}$ such that $\{f_{F_n}\}$ is a Cauchy sequence in $L^\varphi(T,Y)$, so there is $h \in L^\varphi(T,Y)$ such that

$$\|f_{F_n} - h\|_\varphi \to 0 \quad \text{as } n \to \infty.$$

We must prove that $h \in K_{F,\varphi}$ and $h \in W^k_\varphi(T)$. It is easy to see that $h(t) \in F(t)$ a.e. because $F_n(t)$ and $F(t)$ are convex and compact. Also we have

$$\operatorname{dist}(F(t),\{\theta\}) \leqslant \operatorname{dist}(F(t),F_n(t)) + \operatorname{dist}(F_n(t),\{\theta\}),$$

and

$$\operatorname{dist}((F_n(t),\{\theta\}) \leqslant \operatorname{dist}(F_n(t),F(t)) + \operatorname{dist}(F(t),\{\theta\}),$$

so we have $h \in K_{F,\varphi}$. It is easy to see that $|F| \in W^k_\varphi(T)$. □

We define

$$S^\varphi_F = \{f \in L^\varphi(T,Y) : f(t) \in F(t) \text{ a.e.}\}.$$

Let $F \in \mathbf{X}_{mkc,\varphi,1,Y}$. By Theorem 3 and Remark 1 from [7] we define the generalized derivative of F by the formula

$$DF = \{\mathcal{D}x : x \in W^1_\varphi(T), x \in S^\varphi_F\}.$$

Let $F_1, F_2 \in \mathbf{X}_{mkc,\varphi,1,Y}$, let $S^\varphi_{F_1}, S^\varphi_{F_2} \neq \emptyset$ and let $F(t) = F_1(t) + F_2(t)$ for a.e. $t \in T$. By Theorem 4 and Remark 1 from [7] $S^\varphi_{F_1} + S^\varphi_{F_2} \subset S^\varphi_F$, so if $DF_1, DF_2 \neq \emptyset$, then

$$DF_1 + DF_2 \subset DF.$$

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