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ONE-DIMENSIONAL DIFFUSIONS IN BOUNDED DOMAINS WITH A POSSIBLE JUMP-LIKE EXIT FROM A STICKY BOUNDARY

***Bohdan Kopytko*[1]*, Roman Shevchuk*[2]**

[1] *Institute of Mathematics, Czestochowa University of Technology, Częstochowa, Poland*
[2] *Ivan Franko National University of Lviv, Lviv, Ukraine*
[1] *bohdan.kopytko@im.pcz.pl,* [2] *r.v.shevchuk@gmail.com*

Abstract. Using analytical methods we obtain the integral representation of a two-parameter Feller semigroup on a closed interval $[r_1, r_2]$ corresponding to such a diffusion phenomenon that sticking, partial reflection, absorption and jump phenomena occur at the endpoints r_1, r_2 and at some interior point $r \in (r_1, r_2)$.

Keywords: *diffusion process, parabolic potential, conjugation problem*

Introduction

Let $C(\overline{D})$ be the Banach space of all real-valued continuous functions on a closed interval $\overline{D} = [r_1, r_2]$. Denote by $D_i,\ i = 1,2$, the two intervals (r_1, r) and (r, r_2), respectively, where $-\infty < r_1 < r < r_2 < \infty$ and by φ_i the restriction of any function φ defined on $\overline{D}$ to the closure $\overline{D}_i$.

Assume that the inhomogeneous diffusion process is given on $D_i,\ i = 1,2$, and it is generated by the second-order differential operator $A_s^{(i)}, s \in [0, T]$ ($T > 0$ fixed), with the domain of definition $C^2(\overline{D}_i)$:

$$A_s^{(i)}\varphi_i(x) = \frac{1}{2} b_i(s,x)\frac{d^2\varphi_i(x)}{dx^2} + a_i(s,x)\frac{d\varphi_i(x)}{dx}, \quad i = 1,2,$$

where the diffusion coefficient $b_i(s,x)$ and the drift coefficient $a_i(s,x)$ satisfy the conditions:

1) there exist the constants b and B such that $0 < b \le b_i(s,x) \le B$ for all $(s,x) \in [0,T] \times \overline{D}_i$;
2) for all $s, s' \in [0,T],\ x, x' \in \overline{D}_i$ the next inequalities hold:

$$|b_i(s,x) - b_i(s',x')| \le c\left(|s - s'|^{\frac{\alpha}{2}} + |x - x'|^{\alpha}\right),$$
$$|a_i(s,x) - a_i(s',x')| \le c\left(|s - s'|^{\frac{\alpha}{2}} + |x - x'|^{\alpha}\right),$$

where c and α are the positive constants, $0 < \alpha < 1$.

Define the differential operator $A_s,\ s \in [0,T]$, as follows:

$$\vartheta(A_s) = \left\{\varphi \in C(\overline{D}):\ \varphi_i \in \vartheta\left(A_s^{(i)}\right) \text{ for } i = 1,2,\ A_s^{(1)}\varphi(r) = A_s^{(2)}\varphi(r)\right\},$$

$$A_s\varphi(x) = \begin{cases} A_s^{(1)}\varphi_1(x), & x \in \overline{D}_1, \\ A_s^{(2)}\varphi_2(x), & x \in \overline{D}_2. \end{cases} \tag{1}$$

Consider also the conjugation operator L_s and the two boundary operators $L_s^{(1)}$, $L_s^{(2)}$ of Feller-Wentzell defined at points $r,\ r_1,\ r_2$, respectively,

$$L_s\varphi(r) = \sigma(s)A_s\varphi(r) + q_1(s)\frac{d\varphi(r-)}{dx} - q_2(s)\frac{d\varphi(r+)}{dx} + \gamma(s)\varphi(r) + \\ + \int\limits_{D_1 \cup D_2} [\varphi(r) - \varphi(y)]\mu(s,dy),$$

$$L_s^{(i)}\varphi(r_i) = \sigma_i(s)A_s^{(i)}\varphi(r_i) + (-1)^i p_i(s)\frac{d\varphi(r_i)}{dx} + \gamma_i(s)\varphi(r_i) + \\ + \int\limits_{D_i} [\varphi(r_i) - \varphi(y)]\pi_i(s,dy), i = 1,2,$$

where:

a) the functions $\sigma(s),\ \sigma_i(s),\ i = 1,2$, are positive and Hölder continuous with exponent $\frac{\alpha}{2}$ (α is the constant from 2)) on $[0,T]$;

b) the functions $q_1(s),\ q_2(s),\ \gamma(s),\ p_i(s),\ \gamma_i(s)$ are nonnegative and continuous on $[0,T]$;

c) for a fixed s, $\mu(s,\cdot)$ and $\pi_i(s,\cdot)$, $i = 1,2$, are the nonnegative measures on $D_1 \cup D_2$ and D_i, respectively, such that $\mu(s, D_1 \cup D_2) > 0$, $\pi_i(s, D_i) > 0$ and for all $f \in C(\overline{D})$ the integrals

$$\int\limits_{D_1 \cup D_2} |y - r| f(y)\mu(s,dy), \quad \int\limits_{D_i} |y - r_i| f_i(y)\pi_i(s,dy)$$

exist and are Hölder continuous with exponent $\frac{\alpha}{2}$ on $[0,T]$ as functions of variable s.

It is known (see [1-3]) that the conjugation condition and the boundary conditions

$$L_s\varphi(r) = 0, \quad L_s^{(i)}\varphi(r_i) = 0, \quad i = 1,2, \tag{2}$$

restrict A_s to the infinitesimal generator of some Feller semigroup in the space of continuous functions. Such a semigroup is constructed in the present paper. Thus, we are interested in the following problem:

Problem. Construct the two-parameter Feller semigroup T_{st}, $0 \le s < t \le T$, on $\overline{D}$ whose infinitesimal generator is the restriction of A_s in (1) to the set of all functions $\varphi \in \vartheta(A_s)$ satisfying the conjugation condition and the boundary conditions of Feller-Wentzell in (2).

This problem is often called the problem of pasting together two one-dimensional diffusion processes (see [4-8]). A process that is a result of pasting together two diffusions generated by $A_s^{(1)}$ and $A_s^{(2)}$, respectively, coincides with them in $D_1 = (r_1, r)$ and $D_2 = (r, r_2)$ and its behaviour at each point r, r_1, r_2 is determined by the corresponding condition in (2). The coefficients σ, q_i, γ and the measure μ are supposed to correspond to the sticking phenomenon, the partial reflection phenomenon, the absorption phenomenon and the jump phenomenon, respectively (see [9, 10]).

The study of the problem is performed by analytical methods. With such an approach the question on existence and construction of the operator family describing the required process in fact is being reduced to the investigation of the corresponding problem of conjugation for a linear parabolic equation of the second order with variable coefficients, discontinuous at the point r. This problem is to find the function $u(s,x,t) = T_{st}\varphi(x)$ satisfying the following conditions:

$$\frac{\partial u(s,x,t)}{\partial s} + A_s^{(i)} u(s,x,t) = 0, \quad 0 \le s < t \le T, \ x \in D_i, \ i = 1,2, \tag{3}$$

$$\lim_{s \uparrow t} u\,(s,x,t) = \varphi(x), \quad x \in \overline{D}, \tag{4}$$

$$u(s, r-, t) = u(s, r+, t), \quad 0 \le s < t \le T, \tag{5}$$

$$L_s u(s,r,t) = 0, \quad 0 \le s < t \le T, \tag{6}$$

$$L_s^{(i)} u(s, r_i, t) = 0, \quad 0 \le s < t \le T, \ i = 1,2. \tag{7}$$

If $\varphi \in \vartheta(A_s)$, it is clear that the desired function $T_{st}\varphi$ is to satisfy the equation (3) and the "initial" condition (4). The condition (5) is the consequence of the Feller property of the desired semigroup T_{st}. Since $T_{st}\varphi \in \vartheta(A_s)$ when $\varphi \in \vartheta(A_s)$, the boundary conditions (6) and (7) are also to be satisfied. Taking into account that the semigroup T_{st} is to be defined in $C(\overline{D})$, we shall solve the problem (3)-(7) under the assumption that $\varphi \in C(\overline{D})$.

The classical solvability of the problem (3)-(7) is established by the boundary integral equations method with the use of the ordinary fundamental solution of equation (3) and associated parabolic potentials. Application of this method permits us to obtain the integral representation of the solution of the problem (3)-(7), which can be useful in studying additional properties of the constructed process (see [5, 6]).

It is necessary to note that in the present paper we generalize the result obtained in [6] where the similar problem was analyzed in case two inhomogeneous diffusion processes are given in $D_i = \{x \in \mathbb{R}\colon\ (-1)^i x > 0\},\ i = 1,2$, with general Feller-Wentzell type conjugation condition imposed at the beginning. In [7] the problem (3)-(7) was solved in a special case $\sigma \equiv \sigma_1 \equiv \sigma_2 \equiv 0$ and in [8] it was solved in case $\equiv 1,\ \sigma_1 \equiv \sigma_2 \equiv 0$. We should also mention works [11, 12], where the related problems were studied by the methods of stochastic analysis.

1. Preliminaries

Without loss of generality we may suppose that the coefficients $a_i(s,x)$ and $b_i(s,x)$ in (3) are defined on $[0,T] \times \mathbb{R}$ and the conditions 1), 2) hold for all $(s,x) \in [0,T] \times \mathbb{R}$. We may also suppose that the function φ in (4) belongs to $C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ is the Banach space of real-valued bounded continuous functions on $\mathbb{R}$ with norm

$$\| \varphi \| = \sup_{x \in \mathbb{R}} |\varphi(x)|.$$

Denote by $G_i(s,x,t,y),\ i = 1,2$, the fundamental solution of the equation (3) in $[0,T] \times \mathbb{R}$ (its existence is assured by 1), 2)). Recall that the function G_i is nonnegative, continuously differentiable with respect to s, twice continuously differentiable with respect to x and can be represented as (see [13-15])

$$G_i(s,x,t,y) = Z_i(s,x,t,y) + Z_i'(s,x,t,y), \tag{8}$$

where

$$Z_i(s,x,t,y) = [2\pi b_i(t,y)(t-s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y-x)^2}{2b_i(t,y)(t-s)}\right\},$$

and the function Z_i' satisfies the inequality

$$\left|D_s^r D_x^p Z_i'(s,x,t,y)\right| \le c(t-s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-h\frac{(y-x)^2}{t-s}\right\} \tag{9}$$

for all $0 \le s < t \le T,\ x,y \in \mathbb{R}$, where r and p are the nonnegative integers so that $2r + p \le 2$; D_s^r is the partial derivative with respect to s of order r; D_x^p is the partial derivative with respect to x of order p; c, h are positive constants[1]; α is the constant in 2). In addition,

$$\left|D_s^r D_x^p G_i(s,x,t,y)\right| \le c(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-h\frac{(y-x)^2}{t-s}\right\}, \tag{10}$$

where $0 \le s < t \le T,\ x,y \in \mathbb{R},\ 2r + p \le 2$.

[1] We will subsequently denote various positive constants by the same symbol c (or h).

Given fundamental solution G_i, we define the parabolic potentials that will be used to solve the problem (3)-(7), namely the Poisson potential

$$u_{i0}(s,x,t) = \int_{\mathbb{R}} G_i\,(s,x,t,y)\varphi(y)dy,$$

and the simple-layer potentials

$$u_{i1}(s,x,t) = \int_s^t G_i\,(s,x,\tau,r)V_i(\tau,t)d\tau,$$

$$u_{i2}(s,x,t) = \int_s^t G_i\,(s,x,\tau,r_i)V_{i+2}(\tau,t)d\tau,$$

where $0 \le s < t \le T,\ x \in \overline{D}_i$; φ is the function in (4); $V_k,\ k = \overline{1,4}$, are continuous functions in $s \in [0,t)$ satisfying the inequality

$$|V_k(s,t)| \le c(t-s)^{-1+\varepsilon}$$

for any $\varepsilon > 0$.

Note that the functions u_{i0}, u_{i1}, u_{i2} satisfy the equation (3) in the domains $[0,t) \times \overline{D}_i$, $[0,t) \times (\overline{D}_i \setminus \{r\})$, $[0,t) \times (\overline{D}_i \setminus \{r_i\})$, respectively, and the initial conditions

$$\lim_{s \uparrow t} u_{i0}\,(s,x,t) = \varphi(x), \quad x \in \overline{D}_i,$$
$$\lim_{s \uparrow t} u_{i1}\,(s,x,t) = 0, \quad x \in \overline{D}_i \setminus \{r\}, \qquad \lim_{s \uparrow t} u_{i2}\,(s,x,t) = 0, \quad x \in \overline{D}_i \setminus \{r_i\}.$$

In addition, the relations

$$\left|D_s^r D_x^p u_{i0}(s,x,t)\right| \le c \parallel \varphi \parallel (t-s)^{-\frac{2r+p}{2}}, \quad 2r+p \le 2, \tag{11}$$

$$\frac{\partial u_{i1}(s,r\mp,t)}{\partial x} = \pm\frac{V_i(s,t)}{b_i(s,r)} + \int_s^t \frac{\partial Z_i'(s,r,\tau,r)}{\partial x} V_i(\tau,t)d\tau, \tag{12}$$

$$\frac{\partial u_{i2}(s,r_i,t)}{\partial x} = (-1)^i\frac{V_{i+2}(s,t)}{b_i(s,r_i)} + \int_s^t \frac{\partial Z_i'(s,r_i,\tau,r_i)}{\partial x} V_{i+2}(\tau,t)d\tau, \tag{13}$$

hold.

Note also that the last two relations follow from the theorem on the jump of the conormal derivative of a simple-layer potential (see [14, Ch. V, §§2-4]).

2. Solution of the parabolic conjugation problem

The aim of this section is to establish the classical solvability of the conjugation problem (3)-(7).

We find the solution of (3)-(7) of the form ($x \in \overline{D}_i, 0 \le s < t \le T$)

$$u(s,x,t) = u_{i0}(s,x,t) + u_{i1}(s,x,t) + u_{i2}(s,x,t) \tag{14}$$

with the unknown functions V_k, $k = \overline{1,4}$, to be determined. First we note that in view of relations (3)-(5) the conditions (6), (7) reduce to

$$u(s,r,t) = \varphi(r) - \int_s^t g(\tau,t)d\tau\,, \tag{15}$$

$$u(s,r_i,t) = \varphi(r_i) - \int_s^t h_i(\tau,t)d\tau\,, \tag{16}$$

where

$$g(\tau,t) = \frac{1}{\sigma(\tau)}\Bigg(q_1(\tau)\frac{\partial u(\tau,r-,t)}{\partial x} - q_2(\tau)\frac{\partial u(\tau,r+,t)}{\partial x} + \gamma(\tau)u(\tau,r,t)$$
$$+ \int_{D_1\cup D_2} [u(\tau,r,t) - u(\tau,y,t)]\mu(\tau,dy)\Bigg),$$
$$h_i(\tau,t) = \frac{1}{\sigma_i(\tau)}\Bigg((-1)^i p_i(\tau)\frac{\partial u(\tau,r_i,t)}{\partial x} + \gamma_i(\tau)u(\tau,r_i,t) + \int_{D_i}[u(\tau,r_i,t)$$
$$- u(\tau,y,t)]\pi_i(\tau,dy)\Bigg).$$

Then, substituting (14) into (15) and (16), we get, upon using the relations (12), (13), the system of Volterra integral equations of the first kind

$$\sum_{j=1}^{4}\int_s^t N_{ij}(s,\tau)\,V_j(\tau,t)d\tau = \Phi_i(s,t), \quad 0 \le s < t \le T, i = \overline{1,4}, \tag{17}$$

where

$$N_{ii}(s,\tau) = G_i(s,r,\tau,r) + \frac{q_i(\tau)}{\sigma(\tau)b_i(\tau,r)} + \int_s^\tau \frac{\gamma(\rho)}{\sigma(\rho)}G_i(\rho,r,\tau,r)d\rho +$$
$$+P_{s\tau}^{(i)}G_i(s,x,\tau,r)|_{x=r}\,, \quad i = 1,2,$$

$$N_{i\,3-i}(s,\tau)=\frac{q_{3-i}(\tau)}{\sigma(\tau)b_{3-i}(\tau,r)}+P_{s\tau}^{(3-i)}G_{3-i}(s,x,\tau,r)|_{x=r}\,,\qquad i=1,2,$$

$$N_{i\,i+2}(s,\tau)=G_i(s,r,\tau,r_i)+\int_s^\tau\frac{\gamma(\rho)}{\sigma(\rho)}G_i(\rho,r,\tau,r_i)d\rho+$$
$$+P_{s\tau}^{(i)}G_i(s,x,\tau,r_i)|_{x=r}\,,\qquad i=1,2,$$

$$N_{i\,5-i}(s,\tau)=P_{s\tau}^{(3-i)}G_{3-i}(s,x,\tau,r_{3-i})|_{x=r}\,,\qquad i=1,2,$$

$$P_{st}^{(i)}f(s,x,t)=\int_s^t(-1)^{i+1}\frac{q_i(\rho)}{\sigma(\rho)}\frac{\partial f(\rho,x,t)}{\partial x}d\rho+$$
$$+\int_s^t d\rho\int_{D_i}[f(\rho,x,t)-f(\rho,y,t)]\frac{\mu(\rho,dy)}{\sigma(\rho)},\qquad i=1,2,$$

$$N_{ii}(s,\tau)=G_{i-2}(s,r_{i-2},\tau,r_{i-2})+\frac{p_{i-2}(\tau)}{\sigma_{i-2}(\tau)b_{i-2}(\tau,r_{i-2})}+$$
$$+Q_{s\tau}^{(i-2)}G_{i-2}(s,x,\tau,r_{i-2})|_{x=r_{i-2}}\,,\qquad i=3,4,$$

$$N_{i\,i-2}(s,\tau)=G_{i-2}(s,r_{i-2},\tau,r)+Q_{s\tau}^{(i-2)}G_{i-2}(s,x,\tau,r)|_{x=r_{i-2}}\,,\qquad i=3,4,$$

$$N_{i\,5-i}(s,\tau)=N_{i\,7-i}(s,\tau)=0,\qquad i=3,4,$$

$$Q_{st}^{(i)}f(s,x,t)=\int_s^t\left((-1)^i\frac{p_i(\rho)}{\sigma_i(\rho)}\frac{\partial f(\rho,x,t)}{\partial x}+\frac{\gamma_i(\rho)}{\sigma_i(\rho)}f(\rho,x,t)\right)d\rho+$$
$$+\int_s^t d\rho\int_{D_i}[f(\rho,x,t)-f(\rho,y,t)]\frac{\pi_i(\rho,dy)}{\sigma_i(\rho)},\qquad i=1,2,$$

$$\Phi_i(s,t)=\varphi(r)-u_{i0}(s,r,t)-\int_s^t\frac{\gamma(\rho)}{\sigma(\rho)}u_{i0}(\rho,r,t)d\rho-\sum_{j=1}^2P_{st}^{(j)}u_{j0}(s,x,t)|_{x=r}\,,$$
$$i=1,2,$$

$$\Phi_i(s,t)=\varphi(r_{i-2})-u_{i-2\,0}(s,r_{i-2},t)-Q_{st}^{(i-2)}u_{i-2\,0}(s,x,t)|_{x=r_{i-2}}\,,\ i=3,4.$$

Now we have to reduce (17) to the system of Volterra integral equations of the second kind. For this purpose we consider the Holmgren transform

$$\mathcal{E}_{st}\Delta(s,t)=\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_s^t(\rho-s)^{-\frac{1}{2}}\Delta(\rho,t)d\rho,\quad 0\le s<t\le T,$$

and apply it to both sides of each equation in (17). We get ($i=\overline{1,4}$)

$$\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\sum_{j=1}^{4}\int_{s}^{t}(\rho-s)^{-\frac{1}{2}}d\rho\int_{\rho}^{t}N_{ij}(\rho,\tau)V_j(\tau,t)d\tau=\mathcal{E}_{st}\Phi_i(s,t). \tag{18}$$

Changing the order of integration in the right side of (18) and using the fact that

$$\frac{\partial}{\partial s}\int_{s}^{t}f(s,\rho)d\rho=\int_{s}^{t}\frac{\partial}{\partial s}f(s,\rho)d\rho,$$

when $\lim_{\rho\to s}f(s,\rho)=0$, we can write

$$\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_{s}^{t}V_i(\tau,t)d\tau\int_{s}^{\tau}(\rho-s)^{-\frac{1}{2}}N_{ii}(\rho,\tau)d\rho+$$
$$+\sum_{j\neq i}\int_{s}^{t}V_j(\tau,t)\mathcal{E}_{s\tau}N_{ij}(s,\tau)d\tau=\mathcal{E}_{st}\Phi_i(s,t),\qquad i=\overline{1,4}.$$

Denote by $N_{ii}^{(1)}$ the principal part of the fundamental solution which is the first term in the expression for N_{ii}, i.e.,

$$N_{ii}^{(1)}(s,\tau)=\begin{cases}Z_i(s,r,\tau,r), & i=1,2,\\ Z_{i-2}(s,r_{i-2},\tau,r_{i-2}), & i=3,4,\end{cases}$$

and by $N_{ii}^{(2)}$ all the rest of terms in the corresponding expression, so that $N_{ii}=N_{ii}^{(1)}+N_{ii}^{(2)}, i=\overline{1,4}$. It is easy to verify that

$$\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_{s}^{t}V_i(\tau,t)d\tau\int_{s}^{\tau}(\rho-s)^{-\frac{1}{2}}N_{ii}^{(1)}(\rho,\tau)d\rho=-\frac{V_i(s,t)}{d_i(s)},\qquad i=\overline{1,4},$$

where

$$d_i(s)=\begin{cases}\sqrt{b_i(s,r)}, & i=1,2,\\ \sqrt{b_{i-2}(s,r_{i-2})}, & i=3,4.\end{cases}$$

We therefore obtain the following system of Volterra integral equations of the second kind, which is equivalent to (17):

$$V_i(s,t)=\sum_{j=1}^{4}\int_{s}^{t}K_{ij}\,(s,\tau)V_j(\tau,t)+\Psi_i(s,t),\quad i=\overline{1,4}, \tag{19}$$

where

$$K_{ij}(s,\tau)=\begin{cases} d_i(s)\mathcal{E}_{s\tau}N_{ii}^{(2)}(s,\tau), & i=j,\\ d_i(s)\mathcal{E}_{s\tau}N_{ij}(s,\tau), & i\neq j,\end{cases} \qquad \Psi_i(s,t)=-d_i(s)\mathcal{E}_{st}\Phi_i(s,t).$$

Note that for kernels K_{ij} and functions Ψ_i in (19) the inequalities

$$\left|K_{ij}(s,\tau)\right|\le c(\tau-s)^{-1+\frac{\alpha}{2}}, \tag{20}$$

$$|\Psi_i(s,t)|\le c\|\varphi\|(t-s)^{-\frac{1}{2}}, \tag{21}$$

hold. To show how we estimate K_{ij} consider in detail the case $i=j\in\{1,2\}$. Thus, we have to estimate $K_{ii}(s,\tau)=\sqrt{b_i(s,r)}\mathcal{E}_{s\tau}N_{ii}^{(2)}(s,\tau), i=1,2$. Applying the transform $\mathcal{E}_{s\tau}$ to $N_{ii}^{(2)}$, we can write K_{ii} in the form

$$K_{ii}(s,\tau)=\sqrt{\frac{2b_i(s,r)}{\pi}}\left(R_i^{(1)}(s,\tau)+R_i^{(2)}(s,\tau)+R_i^{(3)}(s,\tau)\right), \qquad i=1,2,$$

where

$$R_i^{(1)}(s,\tau)=\frac{1}{2}\int_s^\tau(\rho-s)^{-\frac{3}{2}}\left(Z_i'(s,r,\tau,r)-Z_i'(\rho,r,\tau,r)\right)d\rho,$$

$$R_i^{(2)}(s,\tau)=\int_s^\tau(\rho-s)^{-\frac{1}{2}}\left[(-1)^i\frac{q_i(\rho)}{\sigma(\rho)}\frac{\partial Z_i'(\rho,r,\tau,r)}{\partial x}-\frac{\gamma(\rho)}{\sigma(\rho)}G_i(\rho,r,\tau,r)\right]d\rho+$$
$$+\left(Z_i'(s,r,\tau,r)-\frac{q_i(\tau)}{\sigma(\tau)b_i(\tau,r)}\right)(\tau-s)^{-\frac{1}{2}},$$

$$R_i^{(3)}(s,\tau)=-\int_s^\tau(\rho-s)^{-\frac{1}{2}}d\rho\int_{D_i}[G_i(\rho,r,\tau,r)-G_i(\rho,y,\tau,r)]\frac{\mu(\rho,dy)}{\sigma(\rho)}.$$

If we apply the Lagrange formula to increment $Z_i'(s,r,\tau,r)-Z_i'(\rho,r,\tau,r)$ in the expression for $R_i^{(1)}$ and use the inequality (9), we deduce that the estimate (20) is valid for $R_i^{(1)}$. The estimate (20) for $R_i^{(2)}$ follows easily from inequalities (9) and (10). In order to verify (20) for $R_i^{(3)}$ it suffices to consider the integral

$$J_i(s,\tau)=-\int_s^\tau(\rho-s)^{-\frac{1}{2}}d\rho\int_{D_i}[Z_i(\rho,r,\tau,r)-Z_i(\rho,y,\tau,r)]\frac{\mu(\rho,dy)}{\sigma(\rho)},$$

which differs from $R_i^{(3)}$ in that it contains Z_i instead of G_i. Let us write J_i in the form

$$J_i(s,\tau) = J_{i1}(s,\tau) + J_{i2}(s,\tau),$$

where

$$J_{i1}(s,\tau) = -\int_s^\tau (\rho - s)^{-\frac{1}{2}} d\rho \int_{D_i} [Z_i(\rho,r,\tau,r) - Z_i(\rho,y,\tau,r)] \frac{\mu(\tau,dy)}{\sigma(\tau)},$$

$$J_{i2}(s,\tau) = \int_s^\tau (\rho - s)^{-\frac{1}{2}} d\rho \int_{D_i} [Z_i(\rho,r,\tau,r) - Z_i(\rho,y,\tau,r)] \left(\frac{\mu(\tau,dy)}{\sigma(\tau)} - \frac{\mu(\rho,dy)}{\sigma(\rho)}\right).$$

For $J_{i1}(s,\tau)$ we have

$$|J_{i1}(s,\tau)| \le \frac{1}{\sqrt{2\pi b}} \int_s^\tau (\rho - s)^{-\frac{1}{2}} (\tau - \rho)^{-\frac{1}{2}} d\rho \int_{D_i} \left(1 - e^{\frac{-(y-r)^2}{2b\cdot(\tau-\rho)}}\right) \frac{\mu(\tau,dy)}{\sigma(\tau)} =$$

$$= -\frac{1}{\sqrt{2\pi b}} \int_s^\tau (\rho - s)^{-\frac{1}{2}} (\tau - \rho)^{-\frac{1}{2}} d\rho \int_{D_i} \frac{\mu(\tau,dy)}{\sigma(\tau)} \int_0^1 \frac{\partial}{\partial\theta} e^{\frac{-\theta(y-r)^2}{2b\cdot(\tau-\rho)}} d\theta =$$

$$= \frac{1}{2b\sqrt{2\pi b}} \int_{D_i} |y - r| \frac{\mu(\tau,dy)}{\sigma(\tau)} \int_0^1 |y - r| e^{\frac{-\theta(y-r)^2}{2b\cdot(\tau-s)}} d\theta \int_s^t \frac{e^{\frac{-\theta(y-r)^2}{2b\cdot(\tau-s)}\cdot\frac{\rho-s}{\tau-\rho}}}{(\rho - s)^{\frac{1}{2}} (\tau - \rho)^{\frac{3}{2}}} d\tau,$$

where b is the constant in 1). The change of variables $z = \frac{\rho-s}{\tau-\rho}$ in the inner integral in the last relation leads to

$$|J_{i1}| \le \frac{1}{2b\sqrt{2\pi b}(\tau - s)} \int_{D_i} |y - r| \frac{\mu(\tau,dy)}{\sigma(\tau)} \int_0^1 |y - r| e^{\frac{-\theta(y-r)^2}{2b\cdot(t-s)}} d\theta \int_0^\infty z^{-\frac{1}{2}} e^{\frac{-\theta(y-r)^2}{2b\cdot(t-s)}\cdot z} dz \le$$
$$\le c(\tau - s)^{-\frac{1}{2}}. \tag{22}$$

In view of property c) of measure μ, we can estimate $J_{i2}(s,\tau)$. We deduce that

$$|J_{i2}| \le c(\tau - s)^{-\frac{1}{2}+\frac{\alpha}{2}}. \tag{23}$$

By combining inequalities (22) and (23) we obtain that

$$J_i(s,\tau) \le c(\tau - s)^{-\frac{1}{2}}.$$

It is clear that the same estimate is also valid for $R_i^{(3)}(s,\tau)$.

Having estimated each function $R_i^{(1)}(s,\tau)$, $R_i^{(2)}(s,\tau)$ and $R_i^{(3)}(s,\tau)$, we conclude that for $K_{ij}(s,\tau)$ in case $i = j \in \{1,2\}$ the inequality (20) holds. Similarly, the inequality (20) is valid for kernels $K_{ij}(s,\tau)$, when $i,j \in \{1,\dots,4\}$.

Proceeding as in proof of the estimate (20), one can also prove the estimate (21) for functions $\Psi_i, i = \overline{1,4}$.

From (20) and (21) it follows that there exists a solution of the system of integral equations (19) which can be obtained by the method of successive approximations

$$V_i(s,t) = \sum_{k=0}^{\infty} V_i^{(k)}(s,t), \quad 0 \le s < t \le T, \quad i = \overline{1,4}, \tag{24}$$

where

$$V_i^{(0)}(s,t) = \Psi_i(s,t),$$

$$V_i^{(k)}(s,t) = \sum_{j=1}^{4} \int_s^t K_{ij}(s,\tau) V_j^{(k-1)}(\tau,t)d\tau, \quad k = 1,2,\ldots$$

Furthermore, functions V_i satisfy the inequality

$$|V_i(s,t)| \le c \parallel \varphi \parallel (t-s)^{-\frac{1}{2}}, \quad 0 \le s < t \le T. \tag{25}$$

We have thus constructed a solution $u(s,x,t)$ of the problem (3)-(7) which is of the form (14). Using the relations (8)-(11) and the estimate (25) it is easy to verify that

$$u(s,x,t) \in C^{1,2}([0,t) \times (D_1 \cup D_2)) \cap C([0,t] \times \overline{D}).$$

Concerning the uniqueness of the solution of (3)-(7), note that it follows from the maximum principle (see [14, Ch. II]).

We have proved the following theorem:

Theorem 1. Let the conditions 1), 2) and a)-c) hold, and let $\varphi \in C(\overline{D})$. Then the problem (3)-(7) has a unique solution

$$u(s,x,t) \in C^{1,2}([0,t) \times (D_1 \cup D_2)) \cap C([0,t] \times \overline{D}).$$

Furthermore, this solution can be represented as

$$u(s,x,t) = \int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy + \int_s^t G_i(s,x,\tau,r)V_i(\tau,t)$$

$$+ \int_s^t G_i(s,x,\tau,r_i)V_{i+2}(\tau,t)d\tau, \quad 0 \le s \le t \le T, \quad x \in \overline{D}_i, \quad i = 1,2,$$

where the collection $(V_k)_{k=\overline{1,4}}$ is the solution of the system of Volterra integral equations of the second kind (19).

3. Construction of Feller semigroup

We introduce the two-parameter family of linear operators

$$T_{st}\varphi(x) = u(s,x,t,\varphi), \quad 0 \le s \le t \le T, \quad x \in \overline{D}, \quad \varphi \in C_b(\mathbb{R}), \tag{26}$$

where $u(s,x,t,\varphi)$ is the solution of problem (3)-(7) with function φ in (4), and proceed to study its properties in space $C_b(\mathbb{R})$.

First we note that if $\varphi_n \in C_b(\mathbb{R})$ is a sequence of functions such that

$$\sup_n \parallel \varphi_n \parallel < \infty \quad \text{and} \quad \lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad x \in \overline{D},$$

then

$$\lim_{n\to\infty} T_{st}\,\varphi_n(x) = T_{st}\varphi(x), \quad 0 \le s \le t \le T, \quad x \in \overline{D}.$$

This property easily follows from Lebesgue bounded convergence theorem.

We next prove that the operators $T_{st},\ 0 \le s \le t \le T$, are positivity preserving.

Lemma 1. If $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \ge 0$ for all $x \in \overline{D}$, then $T_{st}\varphi(x) \ge 0$ for all $0 \le s \le t \le T,\ x \in \overline{D}$.

Suppose that $T_{st}\varphi(x)$ takes negative values in $[0,t] \times \overline{D}$ and we denote by m its minimum in $[0,t] \times \overline{D}$. Then, by the minimum principle, value m may be attained only on $(0,t) \times \{r_1, r, r_2\}$. Let $T_{s_0t}\varphi(x_0) = m$, $(s_0, x_0) \in (0,t) \times \{r_1, r, r_2\}$.

In case $x_0 = r$ the inequalities

$$q_1(s_0)\frac{\partial T_{s_0t}\varphi(r-)}{\partial x} \le 0, \quad q_2(s_0)\frac{\partial T_{s_0t}\varphi(r+)}{\partial x} \ge 0,$$
$$\gamma(s_0)T_{s_0t}\varphi(r) \le 0, \qquad \int_{D_1\cup D_2} [T_{s_0t}\varphi(r) - T_{s_0t}\varphi(y)]\mu(s_0, dy) < 0$$

hold and therefore $L_{s_0}T_{s_0t}\varphi(r) < 0$. This contradicts (6). Similarly, the case $x_0 = r_i,\ i \in \{1,2\}$ leads us to the inequality $L_{s_0}T_{s_0t}\varphi(r_i) < 0$ which contradicts (7). The contradiction at which we arrived indicates that $m \ge 0$. This completes the proof of the lemma.

By similar considerations to those in the proof of Lemma 1, one can easily verify that the operators T_{st} are contractive, i.e.,

$$\parallel T_{st} \parallel \le 1, \quad 0 \le s \le t \le T.$$

Note also that the operator family T_{st} has a semigroup property

$$T_{st} = T_{s\tau}T_{\tau t}, \quad 0 \le s \le \tau \le t \le T, \tag{27}$$

This property is a consequence of the assertion of the uniqueness of the solution of the problem (3)-(7). Indeed, considering the problem (3)-(7) in the time interval $[s,\tau]$ with the function $T_{\tau t}\varphi,\ \tau \leq t \leq T$, taken as the "initial" function, we deduce that $T_{s\tau}(T_{\tau t}\varphi),\ 0 \leq s \leq \tau \leq t \leq T$, is the solution of (3)-(7) with the function φ in (4), and hence (31) follows.

By combining the above properties we conclude (see [15], Ch. II) that T_{st}, $0 \leq s \leq t \leq T$, is a Feller semigroup on $\overline{D}$ for which there exists a unique transition function $P(s,x,t,\cdot)$ on $\overline{D}$ such that

$$T_{st}\varphi(x) = \int_{\overline{D}} P\,(s,x,t,dy)\varphi(y), \quad 0 \leq s \leq t \leq T, \quad x \in \overline{D}, \quad \varphi \in C_b(\mathbb{R}).$$

Thus, we have proved the following theorem:

Theorem 2. Let the conditions of Theorem 1 hold. Then the two-parameter semigroup of operators $T_{st},\ 0 \leq s \leq t \leq T$, defined by formula (26) describes the inhomogeneous Feller process on $\overline{D}$ which coincides on D_1 and D_2 with the diffusion processes generated by $A_s^{(1)}$ and $A_s^{(2)}$, respectively, and its behavior at each point r, r_1, r_2 is determined by corresponding conjugation condition or boundary condition of Feller-Wentzell in (2).

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